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PSEUDO-UMBILICAL SURFACES WITH NONZERO PARALLEL MEAN CURVATURE VECTOR IN CP^4

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Abstract. In this paper, we investigate pseudo-umbilical surfaces in a complex projective space under some additional condition.

1. Introduction

Let $CP^m(\widetilde{c})$ be a complex m-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $\widetilde{c}$.

Recently, Maeda [4] investigated the skew-Segre imbedding of $CP^n(1)$ into $CP^{n(n+2)}(2)$ and showed that $CP^n(1)$ is imbedded into $CP^{n(n+2)}(2)$ as a totally real and pseudo-umbilical submanifold with parallel mean curvature vector.

Now, the author proved that any pseudo-umbilical submanifold with nonzero parallel mean curvature vector in $CP^m(\widetilde{c})$ is a totally real submanifold (see [9]).

The class of totally umbilical submanifolds in $CP^m(\widetilde{c})$ was completely classified by Chen and Ogiue [2]. However, it is well known that the class of pseudo-umbilical submanifolds in $CP^m(\widetilde{c})$ is too wide to classify. Thus, it is reasonable to study pseudo-umbilical submanifolds in $CP^m(\widetilde{c})$ under some additional condition. The aim of this paper is to prove the following result.

Theorem A. *Let M be a complete pseudo-umbilical surface with nonzero parallel mean curvature vector in $CP^4(\widetilde{c})$. If M is isotropic and of $P(\boldsymbol{R})$-type, then M is an extrinsic hypersphere in a 3-dimensional real projective space $RP^3(\widetilde{c}/4)$ of $CP^3(\widetilde{c})$.*

Remark 1.1. The map $f: M^2 \to CP^4(\widetilde{c})$ in Theorem A is given as

$$f: M^2 \underset{\substack{\text{totally umbilical} \\ \text{non-totally geodesic}}}{\longrightarrow} RP^3(\widetilde{c}/4) \underset{\text{totally geodesic}}{\longrightarrow} CP^3(\widetilde{c}) \underset{\text{totally geodesic}}{\longrightarrow} CP^4(\widetilde{c}).$$

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2. Preliminaries

Let M be an n-dimensional submanifold of a complex m-dimensional Kaehler manifold $\widetilde{M}$ with complex structure J and Kaehler metric g. A submanifold M of a Kaehler manifold $\widetilde{M}$ is called *totally real* if each tangent space of M is mapped into the normal space by the complex structure of $\widetilde{M}$ (see [1]).

Let ∇ (resp. $\widetilde{\nabla}$) be the covariant differentiation on M (resp. $\widetilde{M}$). We denote by σ the second fundamental form of M in $\widetilde{M}$. Then the Gauss formula and the Weingarten formula are given respectively by

$$\sigma(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y, \quad \widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for vector fields X, Y tangent to M and a normal vector field ξ normal to M, where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. normal) component of $\widetilde{\nabla}_X \xi$. A normal vector field ξ is said to be *parallel* if $D_X\xi = 0$ for any vector field X tangent to M.

The covariant derivative $\overline{\nabla}\sigma$ of the second fundamental form σ is defined by

$$(\overline{\nabla}_X \sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

for all vector fields X, Y and Z tangent to M. The second fundamental form σ is said to be *parallel* if $\overline{\nabla}_X \sigma = 0$.

Let $\zeta = 1/n$ trace σ and $H = \|\zeta\|$ denote the mean curvature vector and the mean curvature of M in $\widetilde{M}$, respectively. If the second fundamental form σ satisfies $\sigma(X,Y) = g(X,Y)\zeta$, then M is said to be *totally umbilical* submanifold of $\widetilde{M}$. In particular, if σ vanishes identically, M is said to be *totally geodesic* submanifold of $\widetilde{M}$. If the second fundamental form σ satisfies $g(\sigma(X,Y),\zeta) = g(X,Y)g(\zeta,\zeta)$, then M is said to be *pseudo-umbilical* submanifold of $\widetilde{M}$.

Now, we recall the notion of an extrinsic sphere. By *extrinsic sphere* we mean a totally umbilical submanifold with nonzero parallel mean curvature vector (see [7]).

The submanifold M of $\widetilde{M}$ is called to be a $\lambda-$ *isotropic* submanifold if $\|\sigma(X,X)\| = \lambda$ for all unit tangent vectors X at each point.

The first normal space at x, $N_x^1(M)$ is defined to be the vector space spanned by all vectors $\sigma(X,Y)$. The first osculating space at x, $O_x^1(M)$ is defined by

$$O_x^1(M) = T_x(M) + N_x^1(M)$$

The submanifold M of $\widetilde{M}$ is called to be a submanifold of $P(\boldsymbol{R})$*-type* if $JT_x(M) \subset \{N_x^1(M)\}^\perp$ for every point $x \in M$.

Let R (resp. $\widetilde{R}$) be the Riemannian curvature for ∇ (resp. $\widetilde{\nabla}$). Then the Gauss and Codazzi equations respectively are given by

$$(2.1)\quad g(\widetilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(Y,Z),\sigma(X,W))$$

$$(2.2)\quad \{\widetilde{R}(X,Y)Z\}^{\perp} = (\overline{\nabla}_X\sigma)(Y,Z) - (\overline{\nabla}_Y\sigma)(X,Z),$$

for all vector fields X, Y, Z and W tangent to M.

3. Proof of Theorem A

Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $CP^4(\widetilde{c})$.

We recall the following result.

Theorem 3.1. [9]. *Let M be an n-dimensional pseudo-umbilical submanifold with nonzero parallel mean curvature vector in $CP^m(\widetilde{c})$. Then $2m > n$ and M^n is immersed in $CP^m(\widetilde{c})$ as a totally real submanifold.*

Since M is a totally real submanifold in $CP^m(\widetilde{c})$, the normal space $T_x^{\perp}M$ is decomposed in the following way; $T_x^{\perp}M = JT_xM \oplus \nu_x$ at each point x of M, where ν_x denotes the orthogonal complement of JT_xM in $T_x^{\perp}M$. We prepare the following.

Lemma 3.1. [9]. *Let M be a pseudo-umbilical submanifold with nonzero parallel mean curvature vector ζ in $CP^m(\widetilde{c})$. Then we have $\zeta \in \nu_x$.*

Lemma 3.2. *Let M be a pseudo-umbilical submanifold with nonzero parallel mean curvature vector ζ in $CP^m(\widetilde{c})$. Then we have*

$$g(\sigma(X,Y),J\zeta) = 0$$

for all vector fields X, Y tangent to M.

Proof. By the Gauss formula, we get

$$\begin{aligned} g(\sigma(X,Y),J\zeta) &= g(\widetilde{\nabla}_X Y, J\zeta) \\ &= -g(\widetilde{\nabla}_X(JY),\zeta) = g(JY,\widetilde{\nabla}_X\zeta) \\ &= g(JY, D_X\zeta) = 0, \end{aligned}$$

for all vector fields X, Y tangent to M. Q.E.D.

We choose a local orthonormal frame field

$$e_1,\ e_2,\ e_3,\ e_4,\ e_5 = Je_1,\ e_6 = Je_2,\ e_7 = Je_3,\ e_8 = Je_4$$

of $CP^4(\widetilde{c})$ such that e_1, e_2 are tangent to M. We choose e_4 in such a way that its direction coincides with that of the mean curvature vector ζ. Then it is easily seen that we have

$$tr\ H_4 = 2H, \quad tr\ H_\alpha = 0, \quad \alpha \neq 4,$$

where H_α denotes an 2×2 symmetric matrix (h^α_{ij}), $h^\alpha_{ij} = g(\sigma(e_i, e_j), e_\alpha)$ for $i, j = 1, 2$, $\alpha = 3, 4, 5, 6, 7, 8$.

Since M is a pseudo-umbilical surface, it is umbilic with respect to the direction of the mean curvature vector ζ. Thus, by Lemma 3.1 and Lemma 3.2, the surface of $P(\boldsymbol{R})$-type satisfies

$$\begin{cases} \sigma(e_1, e_1) = & ae_3 + He_4 + be_7 \\ \sigma(e_1, e_2) = & ce_3 \qquad\quad + de_7 \\ \sigma(e_2, e_2) = & -ae_3 + He_4 - be_7 \end{cases} \tag{3.1}$$

for some functions a, b, c, d with respect to the orthonormal local frame field.

We get the following.

Lemma 3.3. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $CP^m(\widetilde{c})$. If the surface is of $P(\boldsymbol{R})$-type, we have*

$$g((\overline{\nabla}_X \sigma)(Y, Z), JW) = g(J\sigma(Y, Z), \sigma(X, W))$$

for all vector fields X, Y, Z, W tangent to M.

Proof.

$$\begin{aligned} g((\overline{\nabla}_X \sigma)(Y, Z), JW) &= g(D_X(\sigma(Y, Z)), JW) \\ &= g(\widetilde{\nabla}_X(\sigma(Y, Z)), JW) \\ &= -g(\sigma(Y, Z), \widetilde{\nabla}_X(JW)) \\ &= g(J\sigma(Y, Z), \widetilde{\nabla}_X W) \\ &= g(J\sigma(Y, Z), \sigma(X, W)) \end{aligned}$$

for all vector fields X, Y, Z, W tangent to M. Q.E.D.

Lemma 3.4. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $CP^4(\widetilde{c})$. If the surface is of $P(\boldsymbol{R})$-type, we have*

$$g((\overline{\nabla}_{e_i} \sigma)(e_j, e_k), e_l) = 0, \ \textit{for } i, j, k = 1, 2 \textit{ and } l = 5, 6, 8.$$

Proof. Since the surface is immersed in $CP^4(\widetilde{c})$ as a totally real submanifold by Theorem 3.1, the equation (2.2) is reduced to (3.2).

$$(\overline{\nabla}_X\sigma)(Y,Z) = (\overline{\nabla}_Y\sigma)(X,Z), \tag{3.2}$$

for all vector fields X, Y, Z tangent to M.

By Lemma 3.3 and (3.1), we have

$$g((\overline{\nabla}_{e_1}\sigma)(e_1,e_2),Je_1) = g(J\sigma(e_1,e_2),\sigma(e_1,e_1)) = bc - ad \tag{3.3}$$

$$g((\overline{\nabla}_{e_2}\sigma)(e_1,e_1),Je_1) = g(J\sigma(e_1,e_1),\sigma(e_2,e_1)) = ad - bc \tag{3.4}$$

By (3.3), (3.4) and (3.2), we get

$$ad - bc = 0. \tag{3.5}$$

By Lemma 3.3 and (3.5), we obtain

$$g((\overline{\nabla}_{e_i}\sigma)(e_j,e_k),Je_l) = 0, \quad \text{for} \quad i,j,k,l = 1,2.$$

Moreover, by Lemma 3.2, we get

$$\begin{aligned} g((\overline{\nabla}_X\sigma)(Y,Z),J\zeta) &= g(D_X(\sigma(Y,Z)),J\zeta) \\ &= g(\widetilde{\nabla}_X(\sigma(Y,Z)),J\zeta) \\ &= -g(\sigma(Y,Z),\widetilde{\nabla}_X(J\zeta)) \\ &= g(J\sigma(Y,Z),\widetilde{\nabla}_X\zeta) \\ &= g(J\sigma(Y,Z),D_X\zeta) \\ &= 0 \end{aligned}$$

for all vector fields X, Y, Z tangent to M. Q.E.D.

Lemma 3.5. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $CP^m(\widetilde{c})$. Then we have*

$$g((\overline{\nabla}_X\sigma)(Y,Z),\zeta) = 0$$

for all vector fields X, Y, Z tangent to M.

Proof. Since M is a pseudo-umbilical surface, we get

$$g(\sigma(Y,Z),\zeta)=g(Y,Z)g(\zeta,\zeta), \tag{3.6}$$

for all vector fields Y, Z tangent to M. Differentiating (3.6) with respect to X, we get

$$\begin{aligned} g((\overline{\nabla}_X\sigma)(Y,Z)+\sigma(\nabla_XY,Z)+\sigma(Y,\nabla_XZ),\zeta)+g(\sigma(Y,Z),D_X\zeta) \\ =g(\nabla_XY,Z)g(\zeta,\zeta)+g(Y,\nabla_XZ)g(\zeta,\zeta)+2g(Y,Z)g(D_X\zeta,\zeta) \end{aligned} \tag{3.7}$$

Since M is a pseudo-umbilical surface, we get

$$\begin{aligned} g(\sigma(\nabla_XY,Z)+\sigma(Y,\nabla_XZ),\zeta) \\ =g(\nabla_XY,Z)g(\zeta,\zeta)+g(Y,\nabla_XZ)g(\zeta,\zeta) \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\begin{aligned} g((\overline{\nabla}_X\sigma)(Y,Z),\zeta)+g(\sigma(Y,Z),D_X\zeta) \\ =2g(Y,Z)g(D_X\zeta,\zeta) \end{aligned}$$

Since M has nonzero parallel mean curvature vector ζ, we obtain Lemma 3.5.
Q.E.D.

Immediately by Lemma 3.4 and Lemma 3.5, we obtain the following.

Proposition 3.1. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $\boldsymbol{C}P^4(\tilde{c})$. If the surface is of $P(\boldsymbol{R})$-type, then the second fundamental form σ satisfies $(\overline{\nabla}_X\sigma)(Y,Z)=xe_3+ye_7$ for all vector fields X, Y, Z tangent to M and for some functions x, y with respect to the orthonormal local frame field.*

Now, for an isotropic surface, we get (see [8], [5])

$$g(\sigma(e_1,e_1),\sigma(e_1,e_2))=g(\sigma(e_2,e_2),\sigma(e_1,e_2))=0. \tag{3.9}$$

By (3.1) and (3.9) we obtain

$$ac+bd=0. \tag{3.10}$$

For a unit tangent vector $(e_1+e_2)/\sqrt{2}$, we get

$$\left\|\sigma\left((e_1+e_2)/\sqrt{2},(e_1+e_2)/\sqrt{2}\right)\right\|^2=H^2+c^2+d^2 \tag{3.11}$$

On the other hand, we get

$$\|\sigma(e_1, e_1)\|^2 = H^2 + a^2 + b^2 \tag{3.12}$$

Since the surface is isotropic, by (3.11) and (3.12) we have

$$a^2 + b^2 = c^2 + d^2 \tag{3.13}$$

By (3.5), (3.10) and (3.13) we have

$$(a^2 + b^2)(c^2 + d^2) = (a^2 + b^2)^2 = 0.$$

Hence we see that the surface is immersed in $\boldsymbol{C}P^4(\widetilde{c})$ as a totally real, totally umbilical surface.

We recall the following Theorem 3.2.

Theorem 3.2. [3]. *Let M be an n-dimensional totally real, totally umbilical submanifold $(n \geq 2)$ of a complex m-dimensional complex space form $\widetilde{M}^m(\widetilde{c})$, $\widetilde{c} \neq 0$.*

(i) *If the mean curvature vector $\zeta = 0$, then M is contained in an n-dimensional totally geodesic complex submanifold $\overline{M}^n(\widetilde{c})$ of $\widetilde{M}^m(\widetilde{c})$.*

(ii) *If the mean curvature vector $\zeta \neq 0$, then M is contained in an $(n+1)$-dimensional totally geodesic complex submanifold $\overline{M}^{n+1}(\widetilde{c})$ of $\widetilde{M}^m(\widetilde{c})$.*

By Theorem 3.1 and 3.2, we see that the surface M is immersed in a totally geodesic submanifold $\boldsymbol{C}P^3(\widetilde{c})$ of $\boldsymbol{C}P^4(\widetilde{c})$ as a totally umbilical submanifold. Since the surface M is immersed in $\boldsymbol{C}P^3(\widetilde{c})$, we obtain $\overline{\nabla}\sigma \equiv 0$ by Proposition 3.1.

The assertion of Theorem A follows immediately from the following.

Theorem 3.3. [6] *If M^n is an $n(n \geq 2)$-dimensional complete nonzero isotropic $P(\boldsymbol{R})$-totally real submanifold with parallel second fundamental form in $\boldsymbol{C}P^m(\widetilde{c})$, there exists a unique totally geodesic submanifold $\boldsymbol{R}P^r(c)$ such that*

$$M^n \quad \textit{is a submanifold in} \quad \boldsymbol{R}P^r(c)$$

and that $O^1_x(M) = T_x(\boldsymbol{R}P^r(c))$ for every point $x \in M$.

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