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ON AN EXAMPLE OF SUMS OF PAIRWISE INDEPENDENT RANDOM VARIABLES FOR WHICH THE CENTRAL LIMIT THEOREM HOLDS

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Abstract. Let $\{X_n,\ n\geq 1\}$ be a sequence of symmetric pairwise independent and identically distributed (piid) random variables. If $EX_1=0$, $EX_1^2=1$, then the Central Limit Theorem (CLT) is proved by Dug Hun Hong [5]. In this paper we show that under the above assumptions the sequence so defined is a sequence of martingale differences and CLT follows from McLeish's result. The class of pairwise independent random variables for which CLT holds, described in [5], is in consequence the known class of martingale differences. Furthermore, the assumption of pairwise independency is not crucial there and may be weakened. It seems that the assumption of pairwise independency is not essential in CLT, but we give an interesting result in this direction. Furthermore, an example is provided to illustrate this result.

1. The Dug Hun Hong Example

The strong asymptotic behaviour of pairwise independent random variables, described for the first time by Etemadi [6] and developed in subsequent papers by Martinkainen [11], [12], seems very good. On the other hand, there is a lot of examples of sequences of pairwise independent random variables for which the central limit theorem (CLT) fails (cf. [3], [8]). The largest bibliography on this subject may be found in [4]. The problem arises: *What do we have to add to pairwise independency in order to obtain CLT ?* An answer was given in Dug Hun Hong [5]. He proved, that the sequence of symmetric and pairwise independent identically distributed random variables with variance 1 satisfies CLT. In this result the symmetry was defined as follows:

Definition 1. [2],[5] The sequence of random variables $\{X_n,\ n\geq 1\}$ is symmetrically distributed if every finite dimensional distribution function of the sequence is invariant under any changes in the signs of the $\{X_n,\ n\geq 1\}$.

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It is easy to check that the sequence of symmetric random variables $\{X_n,\ n \geq 1\}$ is conditionally independent under given σ-field $\sigma(|X_n|, n \geq 1)$ (cf. [2] or [5]). Now the proof of CLT follows from the conditional version of CLT and SLLN. On the other hand, we have:

Theorem 1. *Let $\{X_n,\ n \geq 1\}$ be a sequence of symmetric random variables such that $E|X_n| < \infty, n \geq 1$, then $\{X_n,\ n \geq 1\}$ is the martingale differences sequence.*

Proof. See Corollary 1.1 [2, p.220]. □

Corollary 1. *Assume that $\{X_n,\ n \geq 1\}$ is a sequence of symmetric random variables such that $EX_n = 0$, $EX_n^2 = \sigma_n^2 < \infty$, $n \geq 1$,*

$$\limsup_{n\to\infty} \sum_{1\leq i\leq j\leq n} EX_i^2 X_j^2 / s_n^4 \leq \frac{1}{2} \tag{1.1}$$

and for every $\varepsilon > 0$

$$\frac{1}{s_n^2} \sum_{i=1}^{n} EX_i^2 I\left[|X_i| > \varepsilon s_n\right] \longrightarrow 0, \quad \text{as } n \to \infty \tag{1.2}$$

then CLT holds, where

$$s_n^2 = \sum_{i=1}^{n} \sigma_i^2,\ n \geq 1.$$

Proof. See Corollary 2.13 [13, p.624]. □

Note that for pairwise independent random variables we have

$$\sum_{1\leq i<j\leq n} EX_i^2 X_j^2 / s_n^4 = \frac{1}{2}\left(1 - \sum_{i=1}^{n} (EX_i^2)^2 / s_n^4\right) \leq \frac{1}{2}.$$

On the other hand, if $\{X_n,\ n \geq 1\}$ are identically distributed random variables then (1.2) holds.

2. The CLT for pairwise independent random variables

The following result gives another class of pairwise independent random variables for which CLT holds.

Theorem 2. *Let $\{X_n,\ n \geq 1\}$ be a sequence of pairwise independent identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$, and let us put*

(2.1) $$T_n(t) = \Pi_{j=1}^{n}\left(1 + \frac{itX_j}{\sqrt{n}}\right),\ i^2 = -1,\ n \geq 1.$$

If, for every $t \in \boldsymbol{R}$, $T_n(t)$ is uniformly integrable and

(2.2) $$ET_n(t) \longrightarrow 1, \text{ as } n \to \infty,$$

then

(2.3) $$\sum_{i=1}^{n} X_i/\sqrt{n} \xrightarrow{\mathcal{D}} \Phi(.), \text{ as } n \to \infty.$$

Proof. By Theorem 2.1 [13] for (2.3) we must prove only

(2.4) $$\sum_{j=1}^{n} X_j^2/n \xrightarrow{\mathcal{P}} 1, \text{ as } n \to \infty,$$

and

(2.5) $$\max_{j\leq n}|X_j|/\sqrt{n} \xrightarrow{\mathcal{P}} 0, \text{ as } n \to \infty.$$

By the strong law of large numbers applied to the sequence $\{X_n^2,\ n \geq 1\}$ and $\{X_n^2 I[|X_n| > \varepsilon\sqrt{K}],\ n \geq 1\}$, for every ε, $K > 0$, (cf. Theorem 1 [6]), we have

(2.6) $$\sum_{j=1}^{n} X_j^2/n \xrightarrow{a.s.} 1, \text{ as } n \to \infty,$$

and

(2.7) $$\sum_{j=1}^{n} X_j^2 I\left[|X_j| > \varepsilon\sqrt{K}\right]/n \xrightarrow{a.s.} EX_1^2 I\left[|X_1| > \varepsilon\sqrt{K}\right], \text{ as } n \to \infty.$$

Now, (2.6) immediately implies (2.4).

Assume that $\{X_n,\ n \geq 1\}$ are not bounded (if they are then (2.5) is obvious). Let $\varepsilon > 0$ be arbitrarily chosen and let us put $K > 0$ such that

$$EX_1^2 I[|X_1| > \varepsilon\sqrt{K}] < \frac{\varepsilon^2}{2}$$

(this is possible as X_1 is unbounded), then, for $n > K$

(2.8) $$0 \leq P\left[\max_{j\leq n}|X_j| > \varepsilon\sqrt{n}\right] = P\left[\sum_{j=1}^{n} X_j^2 I\left[|X_j| > \varepsilon\sqrt{n}\right]/n > \varepsilon^2\right]$$

$$\leq P\left[\sum_{j=1}^{n} X_j^2 I\left[|X_j| > \varepsilon\sqrt{K}\right]/n > EX_1^2 I\left[|X_1| > \varepsilon\sqrt{K}\right] + \frac{\varepsilon^2}{2}\right]$$

$$\leq P\left[|\sum_{j=1}^{n} X_j^2 I\left[|X_j| > \varepsilon\sqrt{K}\right]/n - EX_1^2 I\left[|X_1| > \varepsilon\sqrt{K}\right]| > \frac{\varepsilon^2}{2}\right]$$

which from (2.7) proves (2.5). □

Remark 1.

(i) For pairwise independent random variables with $EX_n = 0$ assumption (2.2) may be replaced by

$$E\sum_{k=3}^{n}(it)^k \sum_{i\leq j_1<j_2<\ldots<j_k\leq n} X_{j_i}X_{j_2}\ldots X_{j_k}/n^{k/2} \longrightarrow 0, \text{ as } n\to\infty,$$

(ii) If

(2.9) $$\sup_n \|X_n\|_\infty < C < \infty$$

or for every $t \in \boldsymbol{R}$

(2.10) $$\sup_n E\exp(tX_n^2) = M(t) < \infty$$

then the sequence $\{T_n(t),\ n \geq 1\}$, defined in Theorem 2, is uniformly integrable.

Proof. Since $EX_{j_1} = 0$ and $EX_{j_1}X_{j_2} = EX_{j_1}EX_{j_2} = 0$, for $1 \leq j_1 < j_2 \leq n$, we have

$$ET_n(t) = 1 + \sum_{k=3}^{n}(it)^k \sum_{1\leq j_1<j_2<\ldots<j_k\leq n} EX_{j_1}X_{j_2}\ldots X_{j_k}/n^{k/2}.$$

From (2.9) we obtain

$$\sup_n |T_n(t)| \leq e^{C^2t^2/2} \text{ a.s.}$$

whereas if (2.10) holds then we have

$$E|T_n(t)|^2 = E\Pi_{j=1}^{n}\left(1 + \frac{t^2X_j^2}{n}\right) \leq E\Pi_{j=1}^{n}\exp\left(\frac{t^2X_j^2}{n}\right)$$

$$\leq \Pi_{j=1}^{n}\left(E\exp(t^2X_j^2)\right)^{\frac{1}{n}} \leq M(t^2).$$

□

For a sequence of nonindentically distributed random variables we have the following version of Theorem 2:

Theorem 3. *Let $\{X_n,\ n \geq 1\}$ be a sequence of pairwise independent random variables with $EX_j = 0$, $EX_j^4 < \infty$, $j \geq 1$ and for some divergent to infinity sequence $\{b_j,\ j \geq 1\}$ of positive numbers let us put*

$$T_n(t) = \Pi_{j=1}^n \left(1 + \frac{itX_j}{\sqrt{b_n}}\right),\quad n \geq 1. \tag{2.11}$$

Assume, for every $t \in \boldsymbol{R}$, $T_n(t)$ is uniformly integrable,

$$\sum_{i=1}^n EX_i^4/b_n^2 \longrightarrow 0 \text{ as } n \to \infty \tag{2.12}$$

$$\sum_{i=1}^n EX_i^2/b_n \longrightarrow 1 \text{ as } n \to \infty \tag{2.13}$$

and

$$ET_n(t) \longrightarrow 1,\ \text{ as } n \to \infty, \tag{2.14}$$

then

$$\sum_{i=1}^n X_i/\sqrt{b_n} \xrightarrow{\mathcal{D}} \Phi(.),\ \text{ as } n \to \infty. \tag{2.15}$$

Proof. Let us first prove

$$E\left(\sum_{i=1}^n X_i^2/b_n\right)^2 \longrightarrow 1,\ \text{as } n \to \infty. \tag{2.16}$$

By using $E(X_i^2X_j^2) = EX_i^2EX_j^2$ for $i \neq j$, we have

$$E\left(\sum_{i=1}^n X_i^2/b_n\right)^2 = \sum_{i=1}^n EX_i^4/b_n^2 + \sum_{\substack{1\leq i,\ j\leq n\\ i\neq j}} EX_i^2EX_j^2/b_n^2$$
$$= \sum_{i=1}^n \left(EX_i^4 - (EX_i^2)^2\right)/b_n^2 + \left(\sum_{i=1}^n EX_i^2/b_n\right)^2.$$

The first term goes to 0 since it is non-negative and is bounded from above by $\sum_{j=1}^n EX_j^4/b_n^2 \to 0$ (by (2.12)). The second term goes to 1 by (2.13), and thereby we have (2.16).

From (2.16) and (2.13) we get

$$E\left(\sum_{j=1}^{n} X_j^2/b_n - 1\right)^2 = E\left(\sum_{j=1}^{n} X_j^2/b_n\right)^2 - 2\sum_{j=1}^{n} X_j^2/b_n + 1 \to 0.$$

Since L^2-convergence implies convergence in probability, we have

$$\sum_{j=1}^{n} X_j^2/b_n \xrightarrow{\mathcal{P}} 1 \text{ as } n \longrightarrow \infty. \tag{2.17}$$

Let $\varepsilon > 0$ be arbitrarily chosen, then by (2.12)

$$0 \le P\left[\max_{j\le n} |X_j| > \varepsilon\sqrt{b_n}\right] = P\left[\sum_{j=1}^{n} X_j^2 I\left[|X_j| > \varepsilon\sqrt{b_n}\right]/b_n > \varepsilon^2\right]$$

$$\le \sum_{j=1}^{n} EX_j^2 I\left[|X_j| > \varepsilon\sqrt{b_n}\right]/(b_n\varepsilon^2)$$

$$\le \frac{1}{\varepsilon^4}\sum_{j=1}^{n} EX_j^4/b_j^2 \longrightarrow 0$$

which with (2.17) ends the proof. □

Remark 2.

(i) For pairwise independent random variables with $EX_n = 0$ assumptions (2.14) may be replaced by

$$E\sum_{k=3}^{n} (it)^k \sum_{1\le j_1<j_2<\ldots<j_k\le n} X_{j_1}X_{j_2}\ldots X_{j_k}/b_n^{k/2} \longrightarrow 0, \text{ as } n \to \infty.$$

(ii) If

$$\sup_n \max\left\{\|X_n\|_\infty, \frac{n}{b_n}\right\} < C < \infty$$

or

$$\sup_n E\exp\left(t\sum_{j=1}^{n} X_j^2/b_n\right) = M(t) < \infty$$

then the sequence $\{T_n(t),\ n \ge 1\}$ defined in Theorem 3 is uniformly integrable.

The proof of Remark 2 runs similarly as this for Remark 1.

3. The Example

Now we describe the construction of sequence of pairwise independent random variables satisfying the assumptions of Theorem 3, but at first we recall some definitions:

Definition 2. The sequence $\{\delta_n,\ n \geq 1\}$ is called k-algebraically independent if for every subset of natural numbers $0 < i_1 < i_2 < \ldots < i_k < \infty$ and numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in \boldsymbol{Q}$ such that $\sum_{i=1}^{n} \varepsilon_i^2 \neq 0$, we have

$$\sum_{i=1}^{k} \varepsilon_j \delta_j \neq 0, \tag{3.1}$$

where $\boldsymbol{Q}$ denotes the set of rational numbers (it is easy to check that we may consider the integer numbers only).

The finite sequence $\{\delta_k,\ 1 \leq k \leq n\}$ is algebraically independent if it is n-algebraically independent.

The sequence is called algebraically independent if every finite subset is algebraically independent.

Contrarily, the sequence is k algebraically dependent if for some choice of numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in \boldsymbol{Q}$ such that $\sum_{i=1}^{k} \varepsilon_i^2 \neq 0$ the inequality (3.1) isn't true.

We say that the sequence $\{\delta_n,\ n \geq 1\}$ satisties the signed sum condition (SS-condition) if (3.1) holds for every k, every subset $0 < i_1 < i_2 < \ldots < i_k < \infty$ and numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in \{-1, 1\}$.

It is easy to check that an algebraically independent sequence satisfies SS-condition, but not contrary. Now we give the construction:

The construction. Let $\{\alpha_k,\ k \geq 1\}$ be the sequence of algebraically independent numbers. For example, we may choose $\{e^j,\ j \geq 1\}$ or $\{\sqrt{\beta_j},\ j \geq 1\}$ where $\{\beta_j,\ j \geq 1\}$ are sequential prime numbers. Define

$$\lambda_i = \begin{cases} \alpha_1 & \text{if } \ i = 1, \\ \alpha_2 & \text{if } \ i = 2, \\ \alpha_1 + \alpha_2 & \text{if } \ i = 3 \\ \alpha_{2l+1} & \text{if } \ i = 4l,\ l \geq 1, \\ \alpha_{2l+2} & \text{if } \ i = 4l+1,\ l \geq 1, \\ \alpha_{2l+1} + \alpha_{2l+2} & \text{if } \ i = 4l+2,\ l \geq 1, \\ 4(\alpha_1 + \alpha_3 + \alpha_5 + \ldots + \alpha_{2l+1}) & \text{if } \ i = 4l+3,\ l \geq 1, \end{cases}$$

We note that such defined sequence $\{\lambda_j,\ j \geq 1\}$ is pairwise algebraically independent but triplewise algebraically dependent. Really, to this sequence belong the numbers α_1, α_2 and $\alpha_1 + \alpha_2$ but the linear combination of these numbers with the coefficients -1, -1, 1 gives zero. It is easy to check that this sequence is m-algebraically dependent for every $m > 2$ as the sebsequence $\{\alpha_1, \alpha_3, \ldots, \alpha_{2m-1}, 4(\alpha_1 + \alpha_3 + \alpha_5 + \ldots + \alpha_{2m-1})\}$ is algebraically dependent. Furthermore, the sequence $\{\lambda_j,\ j \geq 1\}$ does not satisfy the SS-condition.

Now let us define the probabilistic measure $\mu_R(.)$ as follows

$$\mu_R(E) = \lim_{T\to\infty} \frac{1}{2T}\mu(E \cap [-T, T]),$$

where $\mu(.)$ is the Lebesgue measure.

By Kolmogorov existence theorem there exists the probability space and the family of random variables $\{\xi_{\lambda_i},\ i \geq 1\}$ such that the subset $\{\xi_{\lambda_{i_r}},\ r = 1, 2, \ldots, k\}$ has law

$$(3.2)\ P\left[\xi_{\lambda_{i_r}} \in B_r,\ r = 1, 2, \ldots, k\right] = \mu_R\left(\bigcap_{r=1}^{k}\left\{x : \sqrt{2}\cos(\lambda_{i_r}, x) \in B_r\right\}\right).$$

By Lemma 2 [7, p.560] the sequence defined above is identically distributed, bounded, pairwise independent, but not m-wise independent for every $m > 2$. From the proof of this Lemma we have:

(3.3)

$$E\left(\xi_{\lambda_{i_1}}^{r_1}\xi_{\lambda_{i_2}}^{r_2}\cdots\xi_{\lambda_{i_k}}^{r_k}\right) = \sum_{p_1=0}^{r_1}\sum_{p_2=0}^{r_2}\cdots\sum_{p_k=0}^{r_k}\binom{r_1}{p_1}\binom{r_2}{p_2}\cdots\binom{r_k}{p_k}2^{-(r_1+r_2+\ldots+r_k)/2}\delta_{0,\sum_{j=1}^{k}(2p_j-r_j)\lambda_{i_j}}$$

where $\delta_{i,j}$ is Kronecker's delta.

Now we prove that the sequence defined in (3.2) is not the multiplicative system considered in [7] and is not symmetrically distributed but satisfied the assumptions of Theorem 2 and in consequence CLT holds. By (3.3) we have

$$E\left(\xi_{\lambda_{i_1}}\xi_{\lambda_{i_2}}\cdots\xi_{\lambda_{i_k}}\right) = \sum_{p_1=0}^{1}\sum_{p_2=0}^{1}\cdots\sum_{p_k=0}^{1}2^{-k/2}\delta_{0,\sum_{j=1}^{k}(2p_j-1)\lambda_{i_j}},$$

and it is easy to check (as $2p_j - 1 = -1$ or $2p_j - 1 = 1$, only) that

$$\sum_{j=1}^{k}(2p_j - 1)\lambda_{i_j} = 0$$

if and only if k divides $3(k = 3l$, say) and for some sequence $j_1, j_2, \ldots, j_l$ we have

$$\{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}\} = \left\{\alpha_{2j_1+1}, \alpha_{2j_1+2}, \alpha_{2j_1+1} + \alpha_{2j_1+2}, \alpha_{2j_2+1}, \alpha_{2j_2+2}, \alpha_{2j_2+1} + \alpha_{2j_2+2}, \ldots, \alpha_{2j_l+1}, \alpha_{2j_l+2}, \alpha_{2j_l+1} + \alpha_{2j_l+2}\right\}.$$

In this case we have

$$E\left(\xi_{\lambda_{i_1}}\xi_{\lambda_{i_2}}\dots\xi_{\lambda_{i_k}}\right) = 2^{-k/6}$$

so that $\{\xi_{\lambda_i},\ i \geq 1\}$ is not a multiplicative system.

Assume contrary that $\{\xi_{\lambda_i},\ i \geq 1\}$ is symmetrically distributed sequence. By Theorem 1 it is martingale differences, but every martingale differences sequence is multiplicative system, which is a contradiction.

Because $\{\xi_{\lambda_i},\ i \geq 1\}$ are bounded, $T_n(t)$ is unformly integrated. Furthermore, we have

$$\begin{aligned}
E\sum_{k=3}^{n}(it)^k & \sum_{1\leq j_1<j_2<\dots<j_k\leq n} \xi_{\lambda_{j_1}}\xi_{\lambda_{j_2}}\dots\xi_{\lambda_{j_k}}/n^{k/2} \\
&= \sum_{k=1}^{[(n+1)/4]} (it)^{3k} \sum_{1\leq j_1<j_2<\dots<j_k\leq[(n+1)/4]} 2^{-3k/6}n^{-3k/2} \\
&= \sum_{k=1}^{[(n+1)/4]} \binom{[(n+1)/4]}{k} (it)^{3k}2^{-k/2}n^{-3k/2} \\
&= \sum_{k=1}^{[(n+1)/4]} \binom{[(n+1)/4]}{k} \left((it)^3 2^{-1/2}n^{-3/2}\right)^k 1^{[(n+1)/4]-k} - 1 \\
&= \left(1 + \frac{(it)^3}{2^{1/2}n^{3/2}}\right)^{[(n+1)/4]} - 1 \longrightarrow 0, \text{ as } n \to \infty,
\end{aligned}$$

where $[k]$ denotes the integer part of k. Thus, the assumptions of Theorem 2 are satisfied, and in consequence CLT holds.

Remark 3.

(i) Since $E(\xi_{\lambda_{4l}}\xi_{\lambda_{4l+1}}\xi_{\lambda_{4l+2}}) = 2^{-1/2} \not\to 0$, as $l \to \infty$, it follows that

$$\sum_{1\leq i_1<i_2<\dots<i_k}^{\infty} |E(\xi_{\lambda_{i_1}}\xi_{\lambda_{i_2}}\dots\xi_{\lambda_{i_k}})| = \infty$$

for k divided 3. The sequence defined in our example is not, in consequence, strong multiplicative system in a sense of Kratz W. and Trautner R. (cf. [9]).

(ii) The sequence $\{\xi_{\lambda_i},\ i \geq 1\}$ is not stationary.

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