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ORIENTATION REVERSING DIHEDRAL GROUP ACTIONS ON 3-MANIFOLDS

By

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Abstract. In this paper, we consider an orientable closed 3-manifold M which admits a dihedral group $D_{2,p}$ $(p>1)$ action such that $D_{2,p}$ contains orientation reversing involutions, and the fixed point set consists of a finite number of points. For such a pair $(M, D_{2,p})$, we study the problem that which integer can occur as the first Betti number $q=\beta_{1}(M)$ of M. For a pair $(M, D_{2,p})$ as above we have (1) q is odd, or (2) p is odd and q is even integer greater than or equal to $p-1$. Furthermore, for any pair of integers (p, q) with condition (1) or (2), there is a pair $(M, D_{2,p})$ as above with $\beta_{1}(M)=q$.

1. Introduction

Throughout this paper we work in the piecewise-linear category.

Suppose a finite group G acts on a space X. The fixed point set of an action of G on M is the set $\{x|x\in X, g(x)=x$ for some $g\in G, g\neq id\}$.

In 1961, D.B.A.Epstein [3] proved that a finite group acting on a homotopy 3-sphere with 0-dimensional fixed point set must be Z_{2} (see [3] or [4]). In 1988, Mess observed that "homotopy 3-phere" can be replaced by "integral homology sphere". A proof of Mess's observation can be found in [11]. In [6] the followings are proved.

Theorem A. [6] *A finite group acting on a rational homology* 3*-sphere with* 0*-dimensional fixed point set must be* Z_{2}.

Theorem A immediately follows from the following two results.

Theorem B. [6] *If a finite group* $G\neq Z_{2}$ *acts on a rational homology* 3*-sphere with* 0*-dimensional fixed point set, then* G *must contain a dihedral group* $D_{2,n}=\langle g, h \mid g^{2}=h^{n}=(gh)^{2}=1\rangle$ *with* $n>1$ *odd as a subgroup where* g *is orientation reversing and* h *is orientation preserving.*

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Theorem C. [6] *A dihedral group $D_{2,n}$ with odd $n > 1$ can not act on a rational homology 3-sphere with 0-dimensional fixed point set.*

At first, we give an extension of Theorem C for general closed orientable 3-manifolds.

Let $\beta_1(M)$ be the first Betti number of M. We prove the following.

Theorem 1. *Suppose that a dihedral group $D_{2,p}$ $(p > 1)$ acts on a closed orientable 3-manifold M so that*

(a) *$D_{2,p} = \langle g, h \mid g^2 = h^p = (gh)^2 = 1\rangle$, g reverses the orientation of M, and h preserves the orientation of M,*

(b) *the fixed point set of the action of $D_{2,p}$ consists of a finite number of points (possibly empty).*

Then one of the following statements is hold:

(1) *$\beta_1(M)$ is odd.*

(2) *p is odd and $\beta_1(M)$ is even integer greater than or equal to $p-1$.*

Note that Theorem 1 implies that if M satisfy the conditions (a) and (b), then $\beta_1(M)$ can not be 0. This is just the assertion of Theorem C.

Then we show that the conditions (1) and (2) in Theorem 1 are sufficient for the existence of M and $D_{2,p}$ with the conditions (a), (b) and the prescribed first Betti number.

Theorem 2. *For any pair of integers $(p, b)(p > 1)$ such that*

(1) *b is odd, or*

(2) *p is odd and b is an even number greater than or equal to $p-1$,*

there exists an orientable 3-manifold M such that

(i) *$\beta_1(M)=b$, and*

(ii) *M admits an action of a dihedral group $D_{2,p} = \langle g, h \mid g^2 = h^p = (gh)^2 = 1\rangle$ such that*

  (a) *g reverses the orientation of M and h preserves the orientation of M,*

  (b) *the fixed point set of the action of $D_{2,p}$ on M consists of a finite number of points.*

Remark. *In case of orientation preserving group actions, F.Davis and R.J. Milgram [2] noted that for any finite group G there is a rational homology 3-sphere admitting an orientation preserving free G action.*

2. Proof of Theorem 1

For the proof of Theorem 1, we use Heegaard splittings of 3-manifolds. We say that a triple $(M_1, M_2 : F)$ is a *Heegaard splitting* of a closed 3-manifold M if $M_1 \cup M_2 = M$, $\partial M_1 = \partial M_2 = M_1 \cap M_2 = F$ and M_1 and M_2 are handlebodies.

Proposition 1. *Suppose that a dihedral group $D_{2,p} = \langle g, h \mid g^2 = h^p = (gh)^2 = 1\rangle$ acts on an orientable closed 3-manifold M so that*

(1) *g reverses the orientation of M,*
(2) *h preserves the orientation of M, and*
(3) *the fixed point set of the action of $D_{2,p}$ on M consists of a finite number of points (possibly empty).*

Then there exists a Heegaard splitting $(M_1, M_2 : F)$ of M such that $g(M_i) = M_{3-i}$ and $h(M_i) = M_i$ $(i = 1, 2)$.

To prove Proposition 1, we use the following lemma (cf. Proposition 2.2 [6], see also [9]).

Lemma. *Let M be a closed orientable 3-manifold admitting an orientation reversing involution g (i.e. $g^2 = \mathrm{id.}$) such that the fixed point set of g on M consists of a finite number of points. Then there is a Heegaard splitting $(M_1, M_2 : F)$ of M such that $g(M_1) = M_2$.*

Proof. We show that there are two (possibly disconnected) submanifolds M_1^*, M_2^* of M and an embedded 2-manifold F^* such that $M = M_1^* \cup M_2^*$, $M_1^* \cap M_2^* = \partial M_1^* = \partial M_2^* = F^*$, and $g(M_1^*) = M_2^*$, $g(F^*) = F^*$. Then by trading 1-handles of M_1^* and M_2^* g-equivariantly as in the proof of Proposition 2.4 in [8] or in the proof of Theorem 1 in [10], we can obtain a Heegaard splitting $(M_1, M_2 : F)$ of M such that $g(M_i) = M_{3-i}$ $(i = 1, 2)$.

Hence in the rest of the proof of Lemma we give the existence of M_1^*, M_2^* as above.

For a triangulation K of M, K^i denotes the i-skeleton of K, N_x denotes the simplicial star neighborhood of x in K, and K' denotes the barycentric subdivision of K. For an involution g of M, $\mathrm{Fix}(g, M)$ denotes the set $\{x \in M \mid g(x) = x\}$.

It is easy to see that there exists a triangulation K of M such that $g : K \to K$ is a simplicial isomorphism and in particular if $\mathrm{Fix}(g, M) \neq \emptyset$, K satisfies;

(K1) $\mathrm{Fix}(g, M) \subset K^0$,
(K2) for $x_1, x_2 \in \mathrm{Fix}(g, M)$, $N_{x_1} \cap N_{x_2} = \emptyset$.

For the proof of Lemma, we analyze the set ∂N_x for $x \in \mathrm{Fix}(g, M)$.

Claim. *For each fixed point x of g, there exists a simplicial closed curve ℓ_x on ∂N_x such that $g(\ell_x) = \ell_x$, $\ell_x \subset (K')^1$ and $\ell_x \cap K^0 = \emptyset$.*

Proof of Claim. By conditions (K1) and (K2), we can take a subset V of $(\partial N_x)^0$ such that $V \cup g(V) = (\partial N_x)^0$ and $V \cap g(V) = \emptyset$. Let U be a star neighborhood of V in $(\partial N_x)'$. Since $(\partial N_x)'$ is a barycentric subdivision of ∂N_x, U is a union of planar surfaces such that $U \cup g(U) = \partial N_x$, $U \cap g(U) = \partial U = \partial g(U)$ and $g(U) = cl(\partial N_x - U)$. Hence $g(\partial U) = \partial U$, $\partial U \subset (K')^1$, $\partial U \cap K^0 = \emptyset$.

Assume that for any component $\ell \subset \partial U$, $g(\ell) \neq \ell$. Let $p : \partial N_x \to \partial N_x/g$ be the standard projection. Since $p(\partial N_x) = \partial N_x/g$ is a projective plane, for a sufficiently small annulus neighborhood N_ℓ of ℓ in ∂N_x, $p(N_\ell)$ is an annulus. Hence a small regular neighborhood of $p(\partial U)$ in $\partial N_x/g$ is a union of annulus and $\partial N_x/g - p(\partial U)$ contains a nonorientable region W. Since ∂N_x is orientable, $p^{-1}(W)$ is connected. Then $p^{-1}(W) \subset \partial N_x - \partial U$ and $g(p^{-1}(W)) = p^{-1}(W)$ contradicting the above assertion $g(U) = cl(\partial N_x - U)$.

Therefore there exists a simplicial closed curve $\ell_x \subset \partial U$ such that $g(\ell_x) = \ell_x$, $\ell_x \subset (K')^1$ and $\ell \cap K^0 = \emptyset$.

This completes the proof of Claim. □

Since $g(\ell_x) = \ell_x$, there exists a properly embedded disk D_x in N_x such that $\partial D_x = \ell_x$ and $g(D_x) = D_x$ (Hence $x \in D_x$). Note that the curve ℓ_x divides ∂N_x into two 2-cells B_{x1} and B_{x2} with $g(B_{x1}) = B_{x2}$. Let $\mathcal{V}_{xi}$ be the vertices of ∂N_x^0 that lie in B_{xi}, $i = 1, 2$. Then $\partial N_x^0 = \mathcal{V}_{x1} \cup \mathcal{V}_{x2}$ and $g(\mathcal{V}_{x1}) = \mathcal{V}_{x2}$.

Since g is an involution and does not fix any element in $K^0-\mathrm{Fix}(g, M)$, we can easily see that there is a subset $\mathcal{V}$ of $K^0-\mathrm{Fix}(g, M)$ such that

(1) $\mathcal{V} \cup g(\mathcal{V}) = K^0-\mathrm{Fix}(g, M)$,
(2) $\mathcal{V} \cap g(\mathcal{V}) = \emptyset$,
(3) $\mathcal{V}_{x1} \subset \mathcal{V}$, $\mathcal{V}_{x2} \subset g(\mathcal{V})$ for each element $x \in \mathrm{Fix}(g, M)$.

Now we return to the proof of Lemma. Let $e_1, e_2, \cdots, e_n$ be the 1-simplices of K which intersect both $\mathcal{V}$ and $g(\mathcal{V})$. Let D_i be the dual 2-cell of e_i with respect to K, $i = 1, 2, \cdots, n$. Note that $g(e_i) = e_j$ for some j and that $g(D_i) = D_j$. Let $\mathcal{D} = \{D_1, D_2, \cdots, D_n\}$. Then the elements of $\mathcal{D}$ intersect a 3-simplex of K as indicated in Figure 1.

Thus $\mathcal{D}$ forms a (punctured) surface F_0^* (not necessarily connected) in M with $g(F_0^*) = F_0^*$. Note that for each x of $\mathrm{Fix}(g, M)$, $F_0^* \cap \partial N_x = \ell_x$. Hence $F^* = (F_0^* \cup_{x \in \mathrm{Fix}(g,M)} (F_0^* \cap N_x)) \cup_{x \in \mathrm{Fix}(g,M)} D_x$ is a closed surface with $g(F^*) = F^*$ and $\mathrm{Fix}(g, M) \subset F^*$. Let M_1^* be the closure of all components of $M - F^*$ intersecting $\mathcal{V}$. If there is a vertex v in a component M' of M_1^* with $v \in g(\mathcal{V})$, then there is a vertex $v' \in \mathcal{V} \cap M'$ and a path $\alpha \subset K^1 \cap M'$ connecting v and v', but such a path must meet $\mathcal{D}$, a contradiction. So M_1^* misses $g(\mathcal{V})$. Now let M_2^* be the

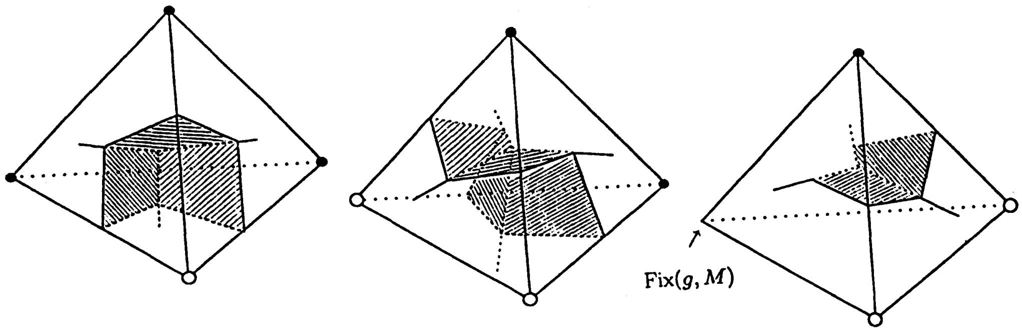

Figure 1

closure of $M - M_1^*$. Then $M_1^* \cap M_2^* = \partial M_1^* = \partial M_2^* = F^*$, and since $\mathcal{V} \subset M_1^*$ and $g(\mathcal{V}) \subset M_2^*$, $g(M_1^*) = M_2^*$.

This completes the proof of Lemma. □

Proof of Proposition 1. Let $\bar{M} = M/h$ and $q : M \to \bar{M}$ be the quotient map. Since h is a free action, $q : M \to \bar{M}$ is a regular covering. The orientation reversing involution $g : M \to M$ induces a unique orientation reversing involution $\bar{g} : \bar{M} \to \bar{M}$ such that $qg = \bar{g}q$.

Suppose that there is a fixed point $y \in \bar{M}$ of $\bar{g}$. Then for $x \in M$ such that $q(x) = y$, we have $q(g(x)) = \bar{g}q(x) = \bar{g}(y) = y$. Therefore $h^i g(x) = x$ for some integer i, and x is a fixed point of the action of $D_{2,p}$ on M. Since the fixed point set of the action of $D_{2,p}$ on M, consists of a finite number of points, Fix$(\bar{g}, \bar{M})$ also consists of a finite number of points. Hence by Lemma, there exists a Heegaard splitting $(\bar{M}_1, \bar{M}_2 : \bar{F})$ of $\bar{M}$ such that $\bar{g}(\bar{M}_1) = \bar{M}_2$ and $\bar{g}(\bar{F}) = \bar{F}$. Put $M_i = q^{-1}(\bar{M}_i)$, $i = 1, 2$, and $F = q^{-1}(\bar{F})$. Then M_1 amd M_2 are handlebodies. Hence $(M_1, M_2 : F)$ is a Heegaard splitting of M satisfying the equations $g(M_i) = M_{3-i}$ and $h(M_i) = M_i$ $(i = 1, 2)$ by construction.

This completes the proof of Proposition 1. □

Let g be an involution of a 3-manifold M and $(M_1, M_2 : F)$ a Heegaard splitting of M such that $g(M_i) = M_{3-i}$ $(i = 1, 2)$. Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ be a basis of $H_1(F)$ so that $I(x_1), I(x_2), \cdots, I(x_n)$ is a basis of $H_1(M_1)$ and $I(y_i) = 0$ $(i = 1, 2, \cdots, n)$, where I is the homomorphism from $H_1(F)$ to $H_1(M_1)$ induced by the inclusion map from F to M_1. Then we have a matrix representation $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $(g|_F)_*$ corresponding to the basis $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$, where A, B, C, D are $\dim H_1(M_1) \times \dim H_1(M_1)$ matrices.

Proposition 2. *Let M, M_1, M_2, A, B, C, D be as above, then*

$$\beta_1(M) = \dim H_1(M_1) - \mathrm{rank} B.$$

Proof. There is the following exact sequence of homology groups.

$$\cdots \to H_2(M) \to H_1(F) \overset{I\oplus J}{\to} H_1(M_1) \oplus H_1(M_2) \to H_1(M) \to \cdots$$

where J is the homomorphism from $H_1(F)$ to $H_1(M_2)$ induced by the inclusion map F to M_2. Therefore $H_1(M) \cong H_1(M_1) \oplus H_1(M_2)/\mathrm{Im}(I \oplus J)$. Note that the elements $g_*I(x_1)$, $g_*I(x_2)$, $\cdots$, $g_*I(x_n)$ are a basis for $H_1(M_2)$ and $g_*I(y_i) = 0$ $(i = 1, 2, \cdots, n)$. The elements $g_*(x_1)$, $g_*(x_2)$, $\cdots$, $g_*(x_n)$, $g_*(y_1)$, $g_*(y_2)$, $\cdots$, $g_*(y_n)$ are also a basis for $H_1(F)$. Let a_{ij} (b_{ij}, c_{ij}, d_{ij}, resp.) be the ij-element of the matrix A (B, C, D, resp.). Then $H_1(M)$ has a group presentation as follows;

$$\begin{aligned} H_1(M) = \langle & x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, \\ & g_*(x_1), g_*(x_2), \cdots, g_*(x_n), g_*(y_1), g_*(y_2), \cdots, g_*(y_n) \\ & \mid y_i = 0, \quad g_*(y_i) = 0, \\ & \Sigma_{j=1}^n a_{ji}x_j + \Sigma_{j=1}^n c_{ji}y_j = g_*(x_i), \\ & \Sigma_{j=1}^n b_{ji}x_j + \Sigma_{j=1}^n d_{ji}y_j = g_*(y_i) \\ & (i = 1, 2, \cdots, n)\rangle \\ = \langle & x_1, x_2, \cdots, x_n, g_*(x_1), g_*(x_2), \cdots, g_*(x_n) \\ & \mid \Sigma_{j=1}^n a_{ji}x_j = g_*(x_i), \\ & \Sigma_{j=1}^n b_{ji}x_j = 0 \\ & (i = 1, 2, \cdots, n) > \\ = < & x_1, x_2, \cdots, x_n \mid \Sigma_{j=1}^n b_{ji}x_j = 0 \ (i = 1, 2, \cdots, n)\rangle. \end{aligned}$$

Therefore we have $\beta_1(M) = n - \mathrm{rank}B = \dim H_1(M_1) - \mathrm{rank}B$.
This completes the proof of Proposition 2. □

Proof of Theorem 1. By Proposition 1, there exists a Heegaard splitting $(M_1, M_2 : F)$ of M such that $g(M_i) = M_{3-i}$ $(i = 1, 2)$. Note that $g|_F$ is an orientation preserving involution of F, and $h|_F$ is an orientation preserving fixed point free homeomorphism of F with period p.

In [7] [5] cyclic actions of a 3-dimensional handlebody are studied, and their results imply that the homeomorphism $h|_{M_1}$ is conjugate to a homeomorphism which is a restriction of $2\pi/p$-rotation with respect to z-axis of R^3 to a handlebody in equivariant position as indicated in Figure 2. Therefore we may assume that $h|_{M_1}$ is as indicated in Figure 2.

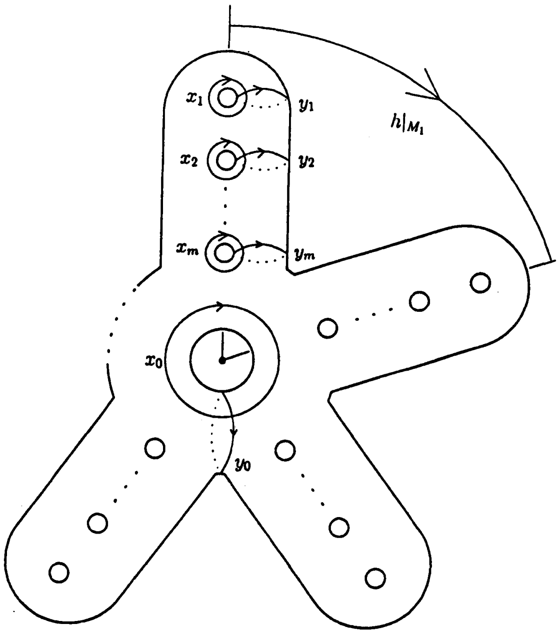

Figure 2

In Figure 2, m is an integer with genus$(M_1) = pm + 1$. Let x_0, x_1, x_2, $\cdots$, x_m, y_0, y_1, y_2, $\cdots$, y_m be cycles represented by essential simple closed curves as indicated in Figure 2.

Define the vectors

$$X_{i,k} = \frac{1}{\sqrt{p}} \Sigma_{j=0}^{p-1} \eta^{-kj} h_*^j(x_i), \quad i = 1, 2, \cdots, m; \ k = 0, 1, \cdots, p-1$$

and

$$Y_{i,k} = \frac{1}{\sqrt{p}} \Sigma_{j=0}^{p-1} \eta^{-kj} h_*^j(y_i), \quad i = 1, 2, \cdots, m; \ k = 0, 1, \cdots, p-1.$$

where $\eta = \exp(2\pi i/p)$, $i^2 = -1$.

Then $h_*(X_{i,k}) = \eta^k X_{i,k}$, $h_*(Y_{i,k}) = \eta^k Y_{i,k}$ $(i = 1, 2, \cdots, m;\ k = 0, 1, \cdots, p-1)$, $h_*(x_0) = x_0$, and $h_*(y_0) = y_0$. By easy calculation we can see that the eigenvalues of $h_* : H_1(F; \mathbf{C}) \to H_1(F; \mathbf{C})$ are η^k, $k = 0, 1, \cdots, p-1$. Let G_0 (H_0, resp.) be the subspace generated by x_0, $X_{1,0}$, $X_{2,0}$, $\cdots$, $X_{m,0}$ (y_0, $Y_{1,0}$, $Y_{2,0}$, $\cdots$, $Y_{m,0}$, resp.) and G_k (H_k, resp.) the subspace generated by $X_{1,k}, X_{2,k}, \cdots, X_{m,k}$ ($Y_{1,k}, Y_{2,k}, \cdots, Y_{m,k}$, resp.) $(k = 1, 2, \cdots, p-1)$. Then $G = \bigoplus_{k=0}^{p-1} G_k$ is a half space of $H_1(F; \mathbf{C})$ such that $G \cong I(G) \cong I(H_1(F; \mathbf{C})) \cong H_1(M_1; \mathbf{C})$, and $H = \bigoplus_{k=0}^{p-1} H_k \cong \mathrm{Ker} I$. Since the generating elements of G_k, H_k $(k = 0, 1, 2, \cdots, p-1)$ are linearly independent, $\dim G_0 = \dim H_0 = m+1$ and $\dim G_k = \dim H_k = m$ $(k = 1, 2, \cdots p-1)$. Note that $G_k \cup H_k$ is the eigenspace of h_* corresponding to the eigenvalue η^k, $k = 0, 1, 2, \cdots p-1$.

We define an intersection form $[,] : H_1(F; \mathbf{C}) \times H_1(F; \mathbf{C}) \to \mathbf{C}$ as follows:

Let ϕ be an isomorphism of $H_1(F;\mathbf{C})$ to $H_1(F;\mathbf{Z})\otimes\mathbf{C}$ and for $x,y\in H_1(F;\mathbf{C})$ with $\phi(x)=x'\otimes\lambda$ and $\phi(y)=y'\otimes\tau$ ($x',y'\in H_1(F;\mathbf{Z})$, $\lambda,\tau\in\mathbf{C}$), put

$$[x,y]=[x'\otimes\lambda,y'\otimes\tau]=\lambda\tau\mathrm{Int}(x',y')$$

where Int(,) is the intersection form on $H_1(F;\mathbf{Z})\otimes H_1(F;\mathbf{Z})$.

Then $[,]$ is a skew-symmetric bilinear form over $\mathbf{C}$.

We can easily see that $[,]|_{G\times G}=[,]|_{H\times H}\equiv 0$, $[x_0,Y_{i,k}]=[X_{i,k},y_0]=0$, $[x_0,y_0]=1$, and

$$[h_*^k(x_i),h_*^s(y_r)]=\begin{cases}1 & \text{if } i=r \text{ and } k\equiv s \bmod p\\ 0 & \text{otherwise.}\end{cases}$$

Therefore for $i\neq r$, $[X_{i,k},Y_{r,s}]=0$ ($i,r=1,2,\cdots,m$; $k,s=0,1,\cdots,p-1$).

For $X_{i,k}$ and $Y_{i,s}$ ($i=1,2,\cdots,m$; $k,s=0,1,\cdots,p-1$),

$$\begin{aligned}[X_{i,k},Y_{i,s}]&=[\frac{1}{\sqrt{p}}\Sigma_{j=0}^{p-1}\eta^{-kj}h_*^j(x_i),\frac{1}{\sqrt{p}}\Sigma_{j=0}^{p-1}\eta^{-sj}h_*^j(y_i)]\\&=\Sigma_{j=0}^{p-1}[\frac{1}{\sqrt{p}}\eta^{-kj}h_*^j(x_i),\frac{1}{\sqrt{p}}\eta^{-sj}h_*^j(y_i)]\\&=\Sigma_{j=0}^{p-1}\frac{1}{p}\eta^{-j(k+s)}\\&=\frac{1}{p}\times\Sigma_{j=0}^{p-1}(\eta^{-(k+s)})^j.\end{aligned}$$

Therefore, if $k+s\not\equiv 0 \bmod p$, then

$$[X_{i,k},Y_{i,s}]=\frac{1}{p}\frac{1-(\eta^{-(k+s)})^p}{1-\eta^{-(k+s)}}=0,$$

and if $k+s\equiv 0 \bmod p$, then

$$[X_{i,k},Y_{i,s}]=\frac{1}{p}\Sigma_{j=0}^{p-1}1=\frac{p}{p}=1.$$

Hence if $k+s\not\equiv 0 \bmod p$, then $[\ ,\ ]|_{G_k\times H_s}=0$, and the intersection form $[,]|_{G_k\times H_{p-k}}$ ($k=1,\cdots,p-1$) corresponding to the above basis is represented by the $m\times m$ identity matrix E_m, and the intersection form $[,]|_{G_0\times H_0}$ on the above basis is represented by the $(m+1)\times(m+1)$ identity matrix E_{m+1}.

Since the homomorphisms g_* and h_* satisfy the relations $(g_*h_*)^2=g_*^2=\mathrm{id.}$, for $x\in G_k\oplus H_k$, the following holds;

$$\begin{aligned}(h|_F)_*((g|_F)_*(x))&=(g|_F)_*(h|_F)_*^{-1}(x)\\&=(g|_F)_*(\eta^{-k}x)\\&=\eta^{-k}(g|_F)_*(x)\end{aligned}$$

and $(g|_F)_*(x) \in G_{p-k} \oplus H_{p-k}$. The representation matrix of $(g|_F)_*$ corresponding to the above basis for $G_0 \oplus H_0 \oplus G_1 \oplus H_1 \oplus \cdots \oplus G_{p-1} \oplus H_{p-1}$ is as follows;

$$
\begin{pmatrix}
N_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & N_1 \\
\vdots & \vdots & & & \cdot & N_2 & 0 \\
\vdots & \vdots & & \cdot & \cdot & \cdot & \vdots \\
\vdots & \vdots & \cdot & \cdot & \cdot & & \vdots \\
\vdots & 0 & \cdot & \cdot & & & \vdots \\
0 & N_{p-1} & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix},
$$

where N_k is $\dim(G_k \oplus H_k) \times \dim(G_k \oplus H_k)$ matrix $(k = 0, 1, 2, \cdots, p-1)$. Since $g_*^2 = \text{id.}$, $N_{p-k} = N_k^{-1}$.

Put $N_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$, where A_k, B_k, C_k and D_k are $\dim G_k \times \dim G_k$ matrices $(k = 0, 1, 2, \cdots, p-1)$. Then by appropriate changes of orders of rows and columns, we can see that the representation matrix of $(g|_F)_*$ corresponding to the above basis of $G \oplus H = G_0 \oplus G_1 \oplus \cdots \oplus G_{p-1} \oplus H_0 \oplus H_1 \oplus \cdots \oplus H_{p-1}$ is as follows;

$$
\begin{pmatrix}
A_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & B_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & A_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & B_1 \\
\vdots & \vdots & & & \cdot & A_2 & 0 & \vdots & \vdots & & & \cdot & B_2 & 0 \\
\vdots & \vdots & & \cdot & \cdot & \cdot & \vdots & \vdots & \vdots & & \cdot & \cdot & \cdot & \vdots \\
\vdots & \vdots & \cdot & \cdot & \cdot & & \vdots & \vdots & \vdots & \cdot & \cdot & \cdot & & \vdots \\
\vdots & 0 & \cdot & \cdot & & & \vdots & \vdots & 0 & \cdot & \cdot & & & \vdots \\
0 & A_{p-1} & 0 & \cdots & \cdots & \cdots & 0 & 0 & B_{p-1} & 0 & \cdots & \cdots & \cdots & 0 \\
C_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & D_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & C_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & D_1 \\
\vdots & \vdots & & & \cdot & C_2 & 0 & \vdots & \vdots & & & \cdot & D_2 & 0 \\
\vdots & \vdots & & \cdot & \cdot & \cdot & \vdots & \vdots & \vdots & & \cdot & \cdot & \cdot & \vdots \\
\vdots & \vdots & \cdot & \cdot & \cdot & & \vdots & \vdots & \vdots & \cdot & \cdot & \cdot & & \vdots \\
\vdots & 0 & \cdot & \cdot & & & \vdots & \vdots & 0 & \cdot & \cdot & & & \vdots \\
0 & C_{p-1} & 0 & \cdots & \cdots & \cdots & 0 & 0 & D_{p-1} & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix},
$$

therefore for the matrix B in Proposition 2, $B = \oplus_{k=0}^{p-1} B_k$ and rank $B = \Sigma_{k=0}^{p-1}$ rank B_k.

Claim 2. *For the matrix* B_k $(k = 0, 1, 2, \cdots, p-1)$, $\mathrm{rank} B_k$ *is even.*

Proof. Since $g|_F$ preserves the orientation of F, for $x \in G_k \oplus H_k$ and $y \in G_{p-k} \oplus H_{p-k}$, $[x, y] = (\deg g)[g_*(x), g_*(y)] = [g_*(x), g_*(y)]$. And $g_*|_{G_k \oplus H_k} : G_k \oplus H_k \to G_{p-k} \oplus H_{p-k}$ is represented by the matrix N_k $(k = 0, 1, \cdots p-1)$. Since the representation matrix of $[\ ,\ ]|_{G_k \times H_{p-k}}$ is E_m $(k = 1, 2, \cdots p-1)$ and the representation matrix of $[\ ,\ ]|_{G_0 \times H_0}$ is E_{m+1}, representation matrices of $[\ ,\ ] : (G_k \oplus H_k) \times (G_{p-k} \oplus H_{p-k}) \to \mathbf{C}$ $(k = 1, 2, \cdots, p-1)$ is $\begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix}$, and representation matrices of $[\ ,\ ] : (G_0 \oplus H_0) \times (G_0 \oplus H_0) \to \mathbf{C}$ is $\begin{pmatrix} 0 & E_{m+1} \\ -E_{m+1} & 0 \end{pmatrix}$.

Hence for $k = 0, 1, 2, \cdots p-1$, we have the following relation;

$$N_k^{\top} \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} N_{p-k} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

where

$$E = \begin{cases} E_{m+1} & \text{if } k = 0 \\ E_m & \text{if } k = 1, 2, \cdots, p-1 \end{cases}$$

and $P^{\top}$ is the transpose of the matrix P. By $N_{p-k} = N_k^{-1}$ and $N_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$, we have

$$\begin{pmatrix} A_k^{\top} & C_k^{\top} \\ B_k^{\top} & D_k^{\top} \end{pmatrix} \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$$

and

$$\begin{pmatrix} -C_k^{\top} & A_k^{\top} \\ -D_k^{\top} & B_k^{\top} \end{pmatrix} = \begin{pmatrix} C_k & D_k \\ -A_k & -B_k \end{pmatrix}.$$

This means the matrix B_k is alternating. Hence $\mathrm{rank} B_k$ is even.

This completes the proof of Claim 2. □

By Proposition 2, $\beta_1(M) = \dim H_1(M_1) - \mathrm{rank} B = pm + 1 - \Sigma_{k=0}^{p-1} \mathrm{rank} B_k$. By Claim 2, $\Sigma_{k=0}^{p-1} \mathrm{rank} B_k$ is even.

In the case that p is even, pm is even and $\beta_1(M)$ is an odd number.

In the case that p is odd and m is even, pm is even and $\beta_1(M)$ is also an odd number.

In the case that p is odd and m is odd, $\beta_1(M)$ is an even number. Since B_k $(k = 1, 2, \cdots, p-1)$ is an $m \times m$-matrix and has even rank, we have $m - \mathrm{rank} B_k \geq 1$ for $k = 1, 2, \cdots, p-1$. Therefore

$$\begin{aligned}\beta_1(M) &= pm + 1 - \Sigma_{k=0}^{p-1}\mathrm{rank}B_k \\ &= m + 1 - \mathrm{rank}B_0 + \Sigma_{k=1}^{p-1}(m - \mathrm{rank}B_k) \\ &\geq \Sigma_{k=1}^{p-1}1 \\ &= p - 1.\end{aligned}$$

This completes the proof of Theorem 1. □

3. Proof of Theorem 2

In this section, we construct a 3-manifold M such that a dihedral group acts on M and the fixed point set of the action consists of a finite number of points.

Proof of Theorem 2. To prove Theorem 2 it is enough to treat the following three cases.

Case 1) b is odd and p is an integer which is greater than one.

Case 2) p is an odd integer and b is an even integer which is greater than or equal to $p+1$.

Case 3) p is an odd integer and $b = p - 1$.

Case 1)

Let $M = F \times S^1$, where F is a closed orientable surface of genus $(b-1)/2$, and S^1 is a 1-dimensional sphere. F admits an orientation preserving involution g'. (We can find such involution, for example, as a restriction of a π-rotation of R^3 to the surface in a standard position in R^3.) We identify S^1 with $\{e^{i\theta} | \theta \in R\}$. Then we construct homeomorphisms g and h as follows;

$$\begin{aligned} g \quad &: F \times S^1 \to F \times S^1 \\ &g(x, e^{i\theta}) = (g'(x), e^{-i\theta}) \\ h \quad &: F \times S^1 \to F \times S^1 \\ &h(x, e^{i\theta}) = (x, e^{i(\theta+2\pi)/p}) \end{aligned}$$

Then we can see that g and h generate a dihedral group $D_{2,p} = \langle g, h \mid g^2 = h^p = (gh)^2 = 1\rangle$ acting on M. If g' is free then $D_{2,p}$ acts freely, and if g' has a fixed point, the fixed point set of the action of this group consists of a finite number of points.

This completes the proof for Case 1).

Case 2)

Consider a 3-manifold $M = F \times S^1$ and a dihedral group $D_{2,p} = \langle g, h \mid g^2 = h^p = (gh)^2 = 1\rangle$ acting on M as in the Case 1) such that M and $D_{2,p}$ satisfy the following conditions;

(1) p is the given odd number,
(2) the genus of F is $(b-p-1)/2$,
(3) g is constructed from g' with $\text{Fix}(g', F) \neq \emptyset$.

We consider p copies $X_0, X_1, \cdots, X_{p-1}$ of $S^2 \times S^1$. Let I_i be an identification map from X_0 to X_i $(i = 1, 2, \cdots, p-1)$. Note that $X_0 = S^2 \times S^1$ admits an orientation reversing involution f_0 with four fixed points. Take a fixed point x_0 of f_0 in X_0. Let f_i be the involution on X_i defined by $f_i = I_i f_0 I_i^{-1}$ and $x_i = I_i(x_0)$ $(i = 1, 2, \cdots p-1)$. Let y be a point of M with $g(y) = y$. Then there is a g-equivariant regular neighborhood of y, $N(y)$ such that $N(y)$, $hN(y)$, $h^2N(y)$, $\cdots$, $h^{p-1}N(y)$ are mutually disjoint. Furthermore there is an f_i-equivariant regular neighborhood $N(x_i)$ of x_i in X_i $(i = 0, 1, \cdots p-1)$. We attach $X_i - \text{int}(N(x_i))$ $(i = 0, 1, \cdots, p-1)$ to $M - \cup_{i=0}^{p-1} \text{int} h^i(N(y))$ so that identifying maps satisfy the following conditions.

(1) $\partial N(y)$ is identified with $\partial N(x_0)$ by an identifying map $J : \partial N(x_0) \to \partial N(y)$ which satisfies the relation $g|_{\partial N(y)} J = J f_0|_{\partial N(x_0)}$.
(2) $\partial h^i N(y)$ is identified with $\partial N(x_i)$ by the identifying map $h^i J I_i^{-1}|_{\partial N(x_i)}$ $: \partial N(x_i) \to \partial h^i N(y)$.

Then we can obtain a manifold M' which is a connected sum of $F \times S^1$ and X_0, X_1, $\cdots$, X_{p-1}. There is an orientation preserving homeomorphism on M' with period p induced by h on M. There is an orientation reversing involution on M' induced by g on M and f_i on X_i $(i = 0, 1, , \cdots, p-1)$. We can see that these homeomorphisms on M' generate a dihedral group. Note that $\beta_1(M)$ $= 2 \times (b-p-1)/2 + 1 = b - p$, therefore $\beta_1(M') = b$.

This completes the proof for Case 2).

Case 3)

In this case, we construct homeomorphisms of a connected sum of $(p-1)$ $S^2 \times S^1$'s such that the homeomorphisms generate a dihedral group.

As the first step, we define an orientation preserving dihedral group action on S^3. Put $S^3 = \{(s,t,r) | s,t \in R,\ 0 \leq r \geq 1\} / \langle (s,t,r) \sim (s+2\pi, t, r), (s,t,r) \sim (s, t+2\pi, r), (s,t,r) \sim (s,t,r+2\pi), (s,t,0) \sim (s',t,0), (s,t,1) \sim (s,t',1) \rangle$ and give an orientation. Define orientation preserving periodic homeomorphisms k and f by

$$k(s,\ t,\ r) = (s + 2\pi/p,\ t + 2\pi/p,\ r),$$
$$f(s,\ t,\ r) = (-s,\ -t,\ r).$$

Then k and f generate a dihedral group $G = \langle f, k \mid f^2 = k^p = (fk)^2 = 1 \rangle$. Note that k is free and f fixes the curve $\{(s,t,r) | s, t, = 0, \pi\}$. Let B be a regular neighborhood of the point $(0,0,0)$ such that $f(B) = B$ and B, $k(B)$,

$k^2(B)$, $\cdots$, $k^{p-1}(B)$ are mutually disjoint. Then $\cup_{i=0}^{p-1}k^i(B)$ is G-equivariant since $f(k^i(B)) = k^{-i}f(B) = k^{-i}(B)$.

Put $V_1 = S^3 - \cup_{i=0}^{p-1}\text{int}k^i(B)$ and $k_1 = k|_{V_1}$, $f_1 = f|_{V_1}$. Let V_2 be a manifold which is obtained from a copy of V_1 by reversing the orientation of it and I the orientation reversing homeomorphism from V_2 to V_1 which is induced from the identification map. Then V_2 also have an orientation preserving homeomorphisms $k_2 = I^{-1}k_1I$ and $f_2 = I^{-1}f_1I$. The homeomorphisms k_2 and f_2 generate a dihedral group.

We construct a closed 3-manifold M from V_1 and V_2 by identifying ∂V_2 with ∂V_1 through the identifying map f_1I. Define homomorphisms g and h of M as follows;

$$g|_{V_1} = I^{-1},\ g|_{V_2} = I,\ h|_{V_1} = k_1,\ h|_{V_2} = k_2^{-1}.$$

Then M is a connected sum of $(p-1)$ $S^2 \times S^1$'s. We can check that g and h are well defined by the following relations. (Note that f_i^2 =id. and $k_if_i = f_ik_i^{-1}$.) For $x \in \partial V_2$,

$$\begin{aligned} g(f_1I(x)) &= I^{-1}f_1I(x) \\ &= (f_1I)^{-1}(g(x)) \\ h(f_1I(x)) &= k_1f_1I(x) = f_1k_1^{-1}I(x) = f_1Ik_2^{-1}(x). \\ &= f_1I(h(x)). \end{aligned}$$

We can check that g is an orientation reversing involution and h is an orientation preserving free periodic homeomorphism of period p. The homeomorphisms g and h generate a dihedral group $D_{2,p}$ since for $x \in V_1$, $ghgh(x) = Ik_2^{-1}I^{-1}k_1(x) = Ik_2^{-1}k_2I^{-1}(x) = x$, and for $x \in V_2$, $ghgh(x) = I^{-1}k_1Ik_2^{-1}(x) = k_2k_2^{-1}(x) = x$. The fixed point set of the action of this group is finite since it consists of the points corresponding to $\text{Fix}(G, S^3) \cap (\cup_{i=0}^{p-1}k^i(\partial B))$. Therefore we have a dihedral group $D_{2,p} = \langle g, h \mid g^2 = h^p = (gh)^2 = 1\rangle$ which acts on a connected sum of $(p-1)$ $S^2 \times S^1$'s with the fixed point set consisting of $2p$ points.

This completes the proof for Case 3).

This completes the proof of Theorem 2. □

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