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STRONG CONSISTENCY OF THE NUMBER OF VERTICES OF GIVEN DEGREES IN NONUNIFORM RANDOM RECURSIVE TREES

By

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Abstract. We consider the strong convergence of the number of vertices with a given number of degrees and the degree of a given vertex in nonuniform random recursive trees.

1. Introduction and Main Results

A tree with n vertices labeled $1,2,\ldots,n$ is a recursive tree if $n=1$ or if $n\geq 2$ and the ith vertex, $2\leq i\leq n$, is joined to one of the $i-1$ vertices already presented in the partial tree. Such a tree can also be defined as a result of successively joining the ith vertex to one of the first $i-1$ vertices for $i=2,3,\ldots,n$. The usual model of recursive tree is the so-called uniform random recursive tree, in which all $(n-1)!$ recursive trees are equally likely (cf. [6]). Another kind of recursive trees is the nonuniform random recursive trees, which was studied by [9]. In the situation of nonuniform rondom recursive trees, we don't assume that all possible choices of a tree are equiprobable.

We define the degree of a given vertex as the total number of vertices incident to this vertex, more actually, a given vertex with k degree means that there are k vertices connected with this vertex directly. Throughout this paper, we let $X_{n,k}$ be the number of vertices with degree k, $D_{n,i}$ be the degree of the ith vertex, and $Z_{n,k}$ be the number of vertices with the degree larger than k after the nth step, respectively.

Recursive tree models, as a class of data structures, are widely used in practice (cf. [2], [3], [6] and [8]). Under whatever probability model, the degrees of the nodes in recursive trees is an object of prime interest. Some results about the quantities $X_{n,k}$ have been derived in the situation of uniform random recursive trees, for more details, see [7] and [4]. In the situation of nonuniform recursive

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trees, [9] obtained the expected value of $X_{n,k}$, $Z_{n,k}$ and $D_{n,i}$ as well as the variance of $D_{n,i}$. For more details of this field, we refer to [5]. In this paper, first, we deal with a kind of nonuniform random recursive trees in which the probability of joining a new vertex to the vertex i is proportional to the degree of the vertex i (see [9]), namely, we denote $p_{n,d} = d/2(n-1)$ as the probability that the $(n+1)$th vertex is joined to a given vertex with degree d in a random recursive tree of n vertices, where $p_{1,0} = 1$, and $q_{n,d} = 1 - p_{n,d}$. We have the following results.

Theorem 1. *For any constant $\alpha > 1/2$ and $k \geq 1$,*

$$\lim_{n\to\infty} \frac{n^{1/2}}{(\log n)^{\alpha}} \left| \frac{X_{n,k}}{\mathbf{E}X_{n,k}} - 1 \right| = 0 \quad a.s. \tag{1}$$

Theorem 2. *There exists a nondegenerate random variable $Y_i \geq -1$ satisfying $\mathbf{E}Y_i < \infty$ for $i = 1, 2, \ldots$, such that*

$$\lim_{n\to\infty} \left(\frac{D_{n,i}}{\mathbf{E}D_{n,i}} - 1 \right) = Y_i \quad a.s \tag{2}$$

and

$$\lim_{n\to\infty} \mathbf{E} \left| \frac{D_{n,i}}{\mathbf{E}D_{n,i}} - 1 - Y_i \right| = 0. \tag{3}$$

Another important class of nonuniform random recursive trees is called the random plane-oriented recursive tree (see [5]). In this class, The vertices with no descendants are the leaves of the tree, the rest of nodes are internal vertices, and the outdegree of a node is the number of its immediate descendants. Thus the leaves are the nodes of outdegree 0 and the degree of a given vertex is equal to the outdegree of this vertex plus 1. In a random plane-oriented recursive tree a node is the parent of a newly added node with probability proportional to its degree (the root is the only exception to this rule as it has no parent in the tree, instead its chance of being selected as a parent for next node is 1 plus its outdegree). By contrast, we can see that these two kinds of nonuniform random recursive trees are nearly identical. The difference between them is that here we use $p_{n,d} = d/(2n-1)$ as the probability for the $(n+1)$st node to be joined to a given vertex (not the root) with degree d and $p_{n,d} = (d+1)/(2n-1)$ as the probability for the $(n+1)$st node to be joined to the root with degree d in a random recursive tree of n vertices, where $p_{1,1} = 1$. On the random plane-oriented recursive tree, [5] studied the exact and limit distributions of the size and number of leaves in the branches of the tree by using generalized Pólya urn models, the martingale central limit theorem for the distribution of the linear combination of the number of leaves and the number of internal nodes as well as some related results. Under the situation of the random plane-oriented recursive tree, let $X^*_{n,j}$ denote the number of vertices of outdegree j and $D^*_{n,i}$ denote the outdegree of the ith vertex after the nth step, respectively. We have the following results which are analogous to Theorem 1 and 2.

Theorem 3.

$$\lim_{n\to\infty} \frac{n^{1/2}}{(\log n)^\alpha} \left| \frac{X^*_{n,j}}{\mathbf{E}X^*_{n,j}} - 1 \right| = 0 \quad a.s.$$

where $\alpha > 1/2$ *is a constant and* $j \geq 0$.

Theorem 4. *There exists a nondegenerate random variable* $Y^*_i \geq -1$ *satisfying* $\mathbf{E}Y^*_i < \infty$ *for* $i = 1, 2, \ldots$, *such that*

$$\lim_{n\to\infty} \left(\frac{D^*_{n,i} - 1}{\mathbf{E}D^*_{n,i} - 1} - 1 \right) = Y^*_i \quad a.s.$$

and

$$\lim_{n\to\infty} \mathbf{E} \left| \frac{D^*_{n,i} - 1}{\mathbf{E}D^*_{n,i} - 1} - 1 - Y^*_i \right| = 0.$$

2. Proof of Theorems

Prior to proving Theorem 1, we use the idea of [5] to get the expressions of $\mathbf{E}X_{n,k}$ for $n \geq k$ and $k = 1, 2, \ldots$ which were proposed by [9].

Lemma 1. *If* k *is fixed and* $n \to \infty$, *then*

$$\mathbf{E}X_{n,k} = \frac{4(n-1)}{k(k+1)(k+2)} + O(n^{-1/2}). \tag{4}$$

Proof. Let

$$u_n = \Pi_{j=1}^n 2j/(2j-1).$$

It is known that if $n \to \infty$, then (compared with [9])

$$u_n = \sqrt{\pi n} \left(1 + \frac{1}{8n} + \frac{1}{128n^2} + O(n^{-3}) \right).$$

By [9],

$$\mathbf{E}X_{n,1} = \frac{2}{3}(n-1) + \frac{4}{3}u_{n-2}^{-1},$$

hence (4) is true for $k=1$. We will show that the difference

$$\beta_{n,k} \stackrel{\text{def}}{=} \mathbf{E}X_{n,k} - \frac{4(n-1)}{k(k+1)(k+2)}$$

is asymptotically negligible with respect to the leading term in the statement of the Lemma, and in fact is $O(n^{-1/2})$. Based on the definitions of $X_{n,k}$ for $k>1$ and $n \geq k$, it is easy to check that

$$\begin{aligned} \mathbf{E}X_{n+1,k} &= \mathbf{E}(\mathbf{E}(X_{n+1,k}|X_{n,k},\ldots,X_{1,1})) \\ &= p_{n,k-1}\mathbf{E}X_{n,k-1} + q_{n,k}\mathbf{E}X_{n,k}. \end{aligned} \tag{5}$$

Substitute $\beta_{n,k}$ in (5) to get

$$\beta_{n+1,k} = \frac{k-1}{2(n-1)}\beta_{n,k-1} + \frac{2n-k-2}{2(n-1)}\beta_{n,k}. \tag{6}$$

By [9], there is a positive constant c_1 such that

$$|\beta_{n,1}| \leq c_1 n^{-1/2}.$$

If there is a positive constant c_{k-1} for $k-1>1$ such that

$$|\beta_{n,k-1}| \leq c_{k-1}n^{-1/2},$$

then from (6),

$$|\beta_{n+1,k}| \leq \frac{k-1}{2(n-1)}c_{k-1}n^{-1/2} + \frac{2n-k-2}{2(n-1)}|\beta_{n,k}|. \tag{7}$$

Since $\beta_{k,k} = -4(k-1)/(k(k+1)(k+2))$, we have $|\beta_{k,k}| \leq 4k^{-2} \leq 4k^{-1/2}$. Choose $c_k \geq \max\{4, c_{k-1}\}$, then for $k \geq 2$,

$$c_k(n+1)^{-1/2} \geq \frac{k-1}{2(n-1)}c_{k-1}n^{-1/2} + \frac{2n-k-2}{2(n-1)}c_k n^{-1/2} \tag{8}$$

by Mean Value Theorem and some simple calculations. Combining (7), (8) with an induction on n and then on k, we complete the proof of Lemma 1. □

Proof of Theorem 1. First, we show that (1) holds for $k=1$. Based on the definitions of $X_{n,k}$ for $k>1$ and $n \geq k$, it is easy to check that

$$\mathbf{E}\left(X_{n+1,1}|X_{n,1},\ldots,X_{1,1}\right) = q_{n,1}X_{n,1} + 1$$

and $Z_{n,1} = \left(\Pi_{i=1}^{n-1} q_{i,1}\right)^{-1} (X_{n,1} - \mathbf{E}X_{n,1})$ is an $\mathcal{L}^2$ martingale. From the construction of the recursive tree and (5), one can deduce that

$$\begin{aligned}
&\mathbf{E}\left[(Z_{n,1} - Z_{n-1,1})^2 | Z_{n-1,1}, \ldots, Z_{1,1}\right] \\
&\quad = \left(\Pi_{i=1}^{n-1} q_{i,1}\right)^{-2} \mathbf{E}\left[(X_{n,1} - \mathbf{E}(X_{n,1}|X_{n-1,1}, \ldots, X_{1,1}))^2 | X_{n-1,1}, \ldots, X_{1,1}\right] \\
&\quad = \left(\Pi_{i=1}^{n-1} q_{i,1}\right)^{-2} \left(X_{n-1,1}p_{n-1,1} - (X_{n-1,1}p_{n-1,1})^2\right) \\
(9) \quad &\quad \le \left(\Pi_{i=1}^{n-1} q_{i,1}\right)^{-2},
\end{aligned}$$

since $0 \le X_{n-1,1}p_{n-1,1} \le 1$. Furthermore,

$$\begin{aligned}
\frac{n^{1/2}}{(\log n)^\alpha} \left| \frac{X_{n,1}}{\mathbf{E}X_{n,1}} - 1 \right| &= \frac{n^{1/2}}{(\log n)^\alpha \mathbf{E}X_{n,1}} |X_{n,1} - \mathbf{E}X_{n,1}| \\
(10) \qquad &= \frac{n^{1/2}\left(\Pi_{i=1}^{n-1} q_{i,1}\right)}{(\log n)^\alpha \mathbf{E}X_{n,1}} |Z_{n,1}|,
\end{aligned}$$

Because of Lemma 1 and (9), with probability one we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{n(\Pi_{i=1}^{n-1} q_{i,1})^2 \mathbf{E}\left[(Z_{n,1} - Z_{n-1,1})^2 | Z_{n-1,1}, \ldots, Z_{1,1}\right]}{(\log n)^{2\alpha} (\mathbf{E}X_{n,1})^2} \\
&\qquad \le \sum_{n=2}^{\infty} \frac{n}{(\log n)^{2\alpha} (\mathbf{E}X_{n,1})^2} \\
&\qquad \le \sum_{n=2}^{\infty} \frac{3}{n(\log n)^{2\alpha}} \\
(11) \qquad &\qquad < \infty.
\end{aligned}$$

Thus (1) holds for $k = 1$ by (10), (11) and the Hájek-Rényi inequality (see [1]). Now we assume that (1) holds for $k = 1, 2, \ldots, j$. Let us define

$$\begin{aligned}
\mathcal{F}_n &= \sigma\left\{X_{i,r} : i = 1, 2, \ldots, n \text{ and } r = 1, 2, \ldots, i\right\}, \\
V_{n+1} &= (X_{n+1,j+1} - \mathbf{E}X_{n+1,j+1}) - (X_{n,j+1} - \mathbf{E}X_{n,j+1})
\end{aligned}$$

and

$$U_n = p_{n,j}\left(X_{n,j} - \mathbf{E}X_{n,j}\right) - p_{n,j+1}\left(X_{n,j+1} - \mathbf{E}X_{n,j+1}\right),$$

then $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $V_{n+1} - U_n$ is a martingale difference sequence on $\mathcal{F}_n$. Noting the fact that

$$(12) \qquad 0 \le p_{n,l}X_{n,l} \le 1 \quad \text{and} \quad |X_{n+1,l} - X_{n,l}| \le 1,$$

for $n = 1, 2, \ldots$, and $l = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\mathbf{E}\left[(V_{n+1} - U_n)^2 | \mathcal{F}_n\right] &= p_{n,j+1}X_{n,j+1}(1 - p_{n,j+1}X_{n,j+1}) \\
&\quad + p_{n,j}X_{n,j}(1 - p_{n,j}X_{n,j}) + 2p_{n,j+1}X_{n,j+1}p_{n,j}X_{n,j} \\
&\le 4,
\end{aligned}$$

for $n = 1, 2, \ldots,$. Thus an application of the Hájek-Rényi inequality leads to

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(V_{i+1} - U_i)}{n^{1/2}(\log n)^\alpha} &= \lim_{n\to\infty} \frac{1}{n^{1/2}(\log n)^\alpha}\Big((X_{n+1,j+1} - \mathbf{E}X_{n+1,j+1}) \\
&\quad - \sum_{i=1}^{n} p_{i,j}(X_{i,j} - \mathbf{E}X_{i,j}) \\
&\quad + \sum_{i=1}^{n} p_{i,j+1}(X_{i,j+1} - \mathbf{E}X_{i,j+1})\Big) \\
&= 0 \quad a.s. \qquad (13)
\end{aligned}$$

Because of the assumption, there is a set denoted as Ω satisfying $\mathbf{P}(\Omega) = 1$, such that for any $\varepsilon > 0$ and $\omega \in \Omega$, we can find a N_ω satisfying

$$\frac{1}{n^{1/2}(\log n)^\alpha} |X_{n,j}(\omega) - \mathbf{E}X_{n,j}| \le \varepsilon, \quad \text{for} \quad n \ge N_\omega.$$

Furthermore, by the definition of $p_{i,j}$,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^{1/2}(\log n)^\alpha} \sum_{i=1}^{n} p_{i,j}(X_{i,j}(\omega) - \mathbf{E}X_{i,j}) \\
&\le \lim_{n\to\infty} \frac{j}{n^{1/2}(\log n)^\alpha} \sum_{i=1}^{n} \frac{(\log i)^\alpha}{i^{1/2}} \left(\frac{X_{i,j}(\omega) - \mathbf{E}X_{i,j}}{(\log i)^\alpha i^{1/2}}\right) \\
&\le \varepsilon + \lim_{n\to\infty} \frac{j\varepsilon}{n^{1/2}(\log n)^\alpha} \sum_{i=N_\omega}^{n} \frac{(\log i)^\alpha}{i^{1/2}} \\
&\le (2j+1)\varepsilon. \qquad (14)
\end{aligned}$$

Now let us define

$$d_n = \sum_{i=1}^{n} p_{i,j+1}(X_{i,j+1} - \mathbf{E}X_{i,j+1}),$$

$$\mathcal{D} = \{\omega : \lim_{n\to\infty} (1/n^{1/2}(\log n)^\alpha)((X_{n+1,j+1}(\omega) - \mathbf{E}X_{n+1,j+1}) + d_n(\omega)) = 0\}$$

and

$$\mathcal{A} = \{\omega : \lim_{n\to\infty} (1/n^{1/2}(\log n)^\alpha)(X_{n+1,j+1}(\omega) - \mathbf{E}X_{n+1,j+1}) \ne 0\}$$

Combining (13) with (14), we have $\mathbf{P}(\mathcal{D}) = 1$. So it suffices to show that $\mathcal{A} \subseteq \mathcal{D}^c$, where $\mathcal{D}^c$ is the complement set of $\mathcal{D}$. If this is not true, then there exists a sample point ω such that $\omega \in \mathcal{A} \cap \mathcal{D}$, namely,

$$\lim_{n\to\infty} \frac{d_n(\omega)}{n^{1/2}(\log n)^\alpha} \ne 0.$$

Without loss of generality, we assume that

$$\limsup_{n\to\infty} \frac{d_n(\omega)}{n^{1/2}(\log n)^\alpha} > 0. \tag{15}$$

Therefore there exists a positive constant C and subseries $\{n_i\} \uparrow \infty$, such that

$$d_{n_i}(\omega) = \max_{1\le n\le n_i} d_n(\omega), \qquad d_{n_{i-1}}(\omega) < d_{n_i}(\omega)/2$$

and

$$\liminf_{i\to\infty} \frac{d_{n_i}(\omega)}{n_i^{1/2}(\log n_i)^\alpha} \ge C.$$

By (12) and (15), we can choose a subsequence of disjoint intervals $[n_{i1}, n_i]$, such that

$$\frac{d_{n_i}(\omega)}{2} + 1 \ge d_{n_{i1}}(\omega) \ge \frac{d_{n_i}(\omega)}{2},$$

and for any $n \in [n_{i1}, n_i]$,

$$d_n(\omega) \ge \frac{d_{n_i}(\omega)}{2}.$$

Therefore there exists at least one $r_i \in [n_{i1}, n_i]$ such that

$$X_{r_i,j+1}(\omega) - \mathbf{E}X_{r_i,j+1} \ge 0,$$

and further

$$\begin{aligned}
\limsup_{i\to\infty} \frac{X_{r_i,j+1}(\omega) - \mathbf{E}X_{r_i,j+1} + d_{r_i}(\omega)}{r_i^{1/2}(\log r_i)^\alpha} &\ge \limsup_{i\to\infty} \frac{d_{r_i}(\omega)}{r_i^{1/2}(\log r_i)^\alpha} \\
&\ge \limsup_{i\to\infty} \frac{d_{n_i}(\omega)}{2n_i^{1/2}(\log n_i)^\alpha} \\
&\ge \frac{C}{2},
\end{aligned}$$

which is a contradiction with $\omega \in \mathcal{D}$. This concludes the proof. □

Based on Theorem 1, it is not difficult to prove the following Corollary.

Corollary 1.

$$\lim_{n\to\infty} \frac{n^{1/2}}{(\log n)^\alpha} \left| \frac{Z_{n,k}}{\mathbf{E}Z_{n,k}} - 1 \right| = 0 \quad a.s.$$

where $\alpha > 1/2$ is a constant and $k \ge 1$.

Proof of Theorem 2. From the construction of the tree,

$$\begin{aligned}\mathbf{E}D_{n+1,i} &= \mathbf{E}(\mathbf{E}(D_{n+1,i}|D_{n,i},\dots,D_{i,i}))\\ &= \mathbf{E}(D_{n,i}(1-D_{n,i}/2(n-1))+(D_{n,i}+1)(D_{n,i}/2(n-1)))\\ &= ((2n-1)/2(n-1))\mathbf{E}D_{n,i}.\end{aligned}$$

Solving this recurrence with the boundary condition $\mathbf{E}D_{i,i}=1$ we get $\mathbf{E}D_{n+1,i}=nu_{i-1}/(i-1)u_n$. Furthermore,

$$\mathbf{E}\left[\left(\frac{D_{n+1,i}}{\mathbf{E}D_{n+1,i}}-1\right)\middle| D_{n,i},\dots,D_{i,i}\right]=\frac{(2n-1)D_{n,i}/2(n-1)}{nu_{i-1}/(i-1)u_n}-1$$

$$=\frac{D_{n,i}}{(n-1)u_{i-1}/(i-1)u_{n-1}}-1=\frac{D_{n,i}}{\mathbf{E}D_{n,i}}-1,$$

namely, $\{D_{n,i}/\mathbf{E}D_{n,i}-1\}$ is a martingale. Also by [9], it is easy to get

$$\sup_n \mathbf{E}\left(\frac{D_{n,i}}{\mathbf{E}D_{n,i}}-1\right)^2<\infty. \tag{16}$$

So (2) is true by martingale convergence theorem. Now since $\{D_{n,i}/\mathbf{E}D_{n,i}-1\}$ is uniformly integrable because of (16), (3) is also true. We claim that Y_i is not a constant. If this is not true, then there exists a constant C such that $\mathbf{P}(Y_i=C)=1$. Further we know that $C=0$ from (3) and $\mathbf{E}(D_{n,i}/\mathbf{E}D_{n,i})=1$. Thus, by the uniform integrability of the martingale, one has in fact

$$0=\mathbf{E}(Y_i|D_{n,i},\dots,D_{i,i})=(D_{n,i}/\mathbf{E}D_{n,i})-1,$$

hence $\mathbf{E}D_{n,i}=D_{n,i}$. This is not possible, because $D_{n,i}$ is not a constant by the construction of $D_{n,i}$. This concludes the proof of Theorem 2. □

Proof of Theorem 3. By [5],

$$\mathbf{E}X^*_{n,j}=\frac{4n-2}{(j+1)(j+2)(j+3)}+O(n^{-1}),$$

for $k=0,1,\dots,n$. Then we finish the proof of Theorem 3 by the similar way used in the proof of Theorem 1. □

Let $Z^*_{n,j}$ be the number of vertices with the outdegree larger than j after the nth step, we, using the result of Therem 3, have the following Corollary which is analogous to Corollary 1.

Corollary 2.

$$\lim_{n\to\infty}\frac{n^{1/2}}{(\log n)^\alpha}\left|\frac{Z^*_{n,j}}{\mathbf{E}Z^*_{n,j}}-1\right|=0 \quad a.s.$$

where $\alpha > 1/2$ *is a constant and* $j \geq 0$.

Proof of Theorem 4. By the definition of $D^*_{n,i}$, it is easy for us to get that

$$\mathbf{E}(D^*_{n+1,i}|D^*_{n,i},\ldots,D^*_{i,i}) = (2n/2n-1)\mathbf{E}D^*_{n,i} + 1/(2n-1),$$
$$\mathbf{E}D^*_{n+1,i} = \mu_n/\mu_{i-1} - 1,$$

and

$$\begin{aligned}\mathbf{E}(D^*_{n+1,i})^2 &= \frac{2n+1}{2i-1} + \frac{2n+1}{\mu_{i-1}}\left(\sum_{m=i}^{n}\frac{\mu_{m-1}}{(2m+1)(2m-1)}\right) - 2\frac{\mu_n}{\mu_{i-1}} + 1 \\ &= O(n).\end{aligned}$$

Now we can finish the proof of Theorem 4 by the similar arguments in the proof of Theorem 2. □

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