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ON THE EXISTENCE OF EXTREMAL PERIODIC SOLUTIONS FOR NONLINEAR PARABOLIC PROBLEMS WITH DISCONTINUITIES

By

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Abstract. In this paper we consider a very general second order nonlinear parabolic boundary value problem. Assuming the existence of an upper solution φ and a lower solution ψ satisfying $\psi \leq \varphi$, we show that the problem has extremal periodic solutions in the order interval $K = [\psi, \varphi]$. Our proof is based on a general surjectivity result for the sum of two operators of monotone type and on truncation and penalization techniques. In addition we use a result of independent interest which we prove here and which says that the pseudomonotonicity property of $A(t, \cdot)$ can be lifted to its Nemitsky operator. Finally when we impose stronger conditions on the data, we show that the extremal solutions can be obtained with a monotone iterative process.

1. Introduction

Let $T = [0, b]$ and $Z \subseteq R^N$ a bounded domain in R^N with Lipschitz boundary Γ. In this paper we consider the following nonlinear periodic parabolic problem:

$$(1)\left\{\begin{array}{c} \frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t, z, x, Dx) + a_0(t, z, x) \sum_{k=1}^{N} D_k x = f(x(t, z)) \text{ on } T \times Z \\ x(0, z) = x(b, z) \text{ a.e. on } Z, \quad x|_{T \times \Gamma} = 0 \end{array}\right\}$$

The nonlinearity f is in general discontinuous and is supposed to satisfy a decomposition into the difference of two nondecreasing functions (i.e. $f : R \to R$ is locally of bounded variation). It is well known that under these conditions, problem (1) need not have a solution. To obtain an existence theory, we need to pass to a multivalued version of the problem, which roughly speaking is obtained by filling in the gaps at the discontinuity points of the second nondecreasing

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functions in the decomposition of $f(\cdot)$. In the context of elliptic systems, this problem has been studied by many authors, under different conditions on the nonlinearity and by employing different methods. These methods and result can be traced in the fundamental works of Ambrosetti, Badiale [1], Chang [7], Heikkila [15], Stuart [22] and Stuart, Toland [23] and the references therein. The study of the dynamic version of the problem (parabolic systems) is lagging behind and only recently there have been some papers in this direction. We mention the works of Feireisl [11] and Feireisl, Norbury [12], who treat semilinear problems and the nonlinear work of Carl [6], where the nonlinear differential operator is less general than ours and the method used (based on molification techniques), does not allow the author to obtain the existence of the extremal solutions and forces some unnecessary additional restrictions on the data.

In this paper, we combine techniques from the theory of nonlinear operators of monotone type, with the method of upper and lower solutions. The method of upper and lower solutions, turned out to be a powerful tool for the resolution of nonlinear parabolic problems. The works of Boccardo, Murat, Puel [3], Deuel, Hess [9] and Mokrane [19], were based on this method. However the way this method was implemented in these works, is different from our use of the upper and lower solutions in this paper. It should be mentioned that none of the above works allows for the presence of discontinuous nonlinearities and all three require that the upper and lower solutions are L^∞-functions on $T \times Z$. So it seems that our approach is more suitable to deal with problems involving discontinuities.

2. Preliminaries

In this section, we fix our notation and the hypothesis on the data of the porblem and we also introduce all the relevant notations that we will be using in the sequel.

In what follows as usual $D_k = \frac{\partial}{\partial_{z_k}}$ $k \in \{1, 2, \dots N\}$ and $D = (D_k)_{k=1}^N$ (the gradient). Our hypothesis on the functions a_k, $k \in \{1, 2, \dots N\}$ are the following:

$H(a)$: $a_k : T \times Z \times R \times R^N \to R$, $k \in \{1, 2, \dots N\}$ are functions such that

(i) $(t, z) \to a_k(t, z, x, \eta)$ is measurable;

(ii) $(x, \eta) \to a_k(t, z, x, \eta)$ is continuous;

(iii) $|a_k(t, z, x, \eta)| \le \beta_1(t, z) + c_1 \left(|x|^{p-1} + \|\eta\|^{p-1}\right)$ a.e. on $T \times Z$, for all $(x, \eta) \in R \times R^N$ and with $\beta_1 \in L^q(T \times Z)$, $\beta_1(t, z) \ge 0$, $c_1 > 0$, $2 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$;

(iv) $\sum_{k=1}^N (a_k(t, z, x, \eta) - a_k(t, z, x, \eta'))(\eta_k - \eta'_k) \ge 0$ a.e. on $T \times Z$ for all $x \in R$ and all $\eta, \eta' \in R^N$; and

(v) $\sum_{k=1}^{N} a_k(t,z,x,\eta)\eta_k \geq c\|\eta\|^p$ a.e. on $T \times Z$, for all $x \in R$, all $\eta \in R^N$ and with $c > 0$.

Recall that by an "evolution triple", we understand three space $X \subseteq H \subseteq X^*$ such that:

(a) X is a separable and reflexive Banach space;
(b) H is a separable Hilbert space, identified with its dual (pivot space); and
(c) the embedding of X into H is continuous, (i.e. there exists a constant $\widehat{c} > 0$ such that for all $x \in X|x| \leq \widehat{c}\|x\|$, with $|\cdot|$ (resp. $\|\cdot\|$ dinoting the norm of H (resp. of X)) and dense (see Zeidler [25], definition 23.11, p. 416).

Let $W^{1,p}(Z)$ be the usual Sobolev space and $W^{1,p}(Z)^*$ its dual. Since $p \geq 2$, the spaces $W^{1,p}(Z) \subseteq L^2(Z) \subseteq W^{1,p}(Z)^*$ from an evolution triple with the embedding being in addition compact. Also by $W_0^{1,p}(Z)$ we denote the subspace of $W_0^{1,p}(Z)$, consisting of elements with zero trace. As usual, the dual of $W_0^{1,p}(Z)$ is denoted by $W^{-1,q}(Z)$. Again $W_0^{1,p}(Z) \subseteq L^2(Z) \subseteq W^{-1,q}(Z)$ is an evolution triple with the embeddings being compact.

The following two spaces, will play a prominent role in our subsequent considerations:

$$\widehat{W}_{pq}(T) = \left\{ f \in L^p\big(T, W^{1,p}(Z)\big) : \frac{\partial f}{\partial t} \in L^q\big(T, W^{1,p}(Z)^*\big) \right\} \text{ and}$$

$$W_{pq}(T) = \left\{ f \in L^p\big(T, W_0^{1,p}(Z)\big) : \frac{\partial f}{\partial t} \in L^q\big(T, W^{-1,q}(Z)\big) \right\}.$$

In these definitions, the derivative $\frac{\partial f}{\partial t}$ is understood in the sense of vector-valued distributions. Both spaces equipped with obvious norm $\|f\|_{pq} = \|f\|_p + \left\|\frac{\partial f}{\partial t}\right\|$, are embedded continuously in $C\big(T, L^2(Z)\big)$ and compactly in $L^p(T \times Z)$. For details we refer to Lions [18] (thorem 5.1, p. 58) and Zeidler [25] (proposition 23.23, pp. 422 and 450).

Because of hypothesis $H(a)$, we can define the semilinear form

$$a : L^p\left(T, W_0^{1,p}(Z)\right) \times L^p\left(T, W_0^{1,p}(Z)\right) \to R$$

by setting $a(x,y) = \int_0^b \int_Z \sum_{k=1}^{N} a_k(t,z,x,Dx) D_k y(t,z)\,dz\,dt.$

In what follows, by $\big((\cdot,\cdot)\big)$ we will denote the duality brackets between $L^p\big(T, W^{1,p}(Z)\big)$ and $L^q\big(T, W^{1,p}(Z)^*\big)$ and also between $L^p\big(T, W_0^{1,p}(Z)\big)$ and $L^q\big(T, W^{-1,q}(Z)\big)$. Recall that if X is a reflexive Banach space (or more generally if X^* has the Radon-Nikodym Property (RNP)) and $1 \leq p < \infty$, then $L^p(T,X)^* = L^q(T,X^*)$, with $\frac{1}{p}+\frac{1}{q}=1$ (see Diestel, Uhl [10], theorem 1, p. 98).

Our hypothesis on the discontinuous nonlinearity is the following:

$H(f): f: R \to R$ is a function such that $f = g - h$, with g, h: $R \to R$ being both nondecreasing (so $f(\cdot)$ is locally of bounded variation).

In what follows $g_r(x) = \lim_{\varepsilon \downarrow 0} g(x+\varepsilon)$ and $g_l(x) = \lim_{\varepsilon \downarrow 0} g(x-\varepsilon)$. Similarly we define $h_r(x)$ and $h_l(x)$. Let $\beta(\cdot)$ be the maximal monotone graph in R^2 associated with the nondecreasing function $h(\cdot)$; i.e. $\beta(x) = [h_l(x), h_r(x)]$ for all $x \in R$. Then instead of problem (1), we will study the following multivalued version of it:

$$(2) \qquad \left\{ \begin{array}{l} \frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t,z,x,Dx) + a_0(t,z,x) \sum_{k=1}^{N} D_k x \\ \qquad\qquad + \beta(x(t,z)) \ni g(x(t,z)) \text{ on } T \times Z \\ x(0,z) = x(b,z) \text{ a.e. on } Z, \quad x|_{T\times\Gamma} = 0 \end{array} \right\}$$

Since the functions a_k are not assumed to be smooth, we are forced to interpret problem (2) above in a weak fashion.

Definition. A functions $x \in W_{pq}(T)$ is said to be a "solution" (weak) of (2), if there exists $v \in L^q(T \times Z)$ such that $v(t,z) \in \beta(x(t,z))$ a.e. on $T \times Z$ and for all $w \in L^p\big(T, W_0^{1,p}(Z)\big)$

$$\left(\left(\frac{\partial x}{\partial t}, w\right)\right) + a(x,w) + \int_0^b \int_z a_0(t,z,x(t,z)) \left(\sum_{k=1}^{N} D_k x(t,z)\right) w(t,z)dzdt$$
$$+ \int_0^b \int_z v(t,z)w(t,z)dzdt = \int_0^b \int_z g(x(t,z))w(t,x)dzdt$$

As we mentioned in the introduction our approach will use upper and lower solutions, combined with truncation and penalization techniques. So we need to introduce the concepts and necessary analytical tools associated with this method.

Definition. A function $\varphi \in \widehat{W}_{pq}(T)$ is said to be an "upper solution" of (2), if

$$\left(\left(\frac{\partial \varphi}{\partial t}, w\right)\right) + a(\varphi,w) + \int_0^b \int_z a_0(t,z,\varphi(t,z)) \left(\sum_{k=1}^{N} D_k \varphi(t,z)\right) w(t,z)dzdt$$
$$+ \int_0^b \int_z h_l(\varphi(t,z))w(t,z)dzdt \geq \int_0^b \int_z g(\varphi(t,z))w(t,z)dzdt$$

for all $w \in L^p(T, W_0^{1,p}(Z)) \cap L^p(T \times Z)_+$, $\varphi(0,z) \geq \varphi(b,z)$ a.e. on Z and $\varphi|_{T\times\Gamma} \geq 0$.

Similarly a function $\psi \in \widehat{W}_{pq}(T)$ is said to be a "lower solution" of (2), if in the above definition the inequalities are reversed and h_l is replace by h_r.

We will make the following hypothesis concerning upper and lower solutions.

H_0: there exists an upper solution $\varphi \in \widehat{W}_{pq}(T)$ and a lower solution $\psi \in \widehat{W}_{pq}(T)$ such that $\psi(t,z) \leq \varphi(t,z)$ a.e. on $T \times Z$ and $g_r(\varphi)$, $g_l(\psi)$, $h_r(\varphi)$, $h_l(\psi) \in L^q(T \times Z)$.

Note that in Boccardo, Murat, Puel [3], Deuel, Hess [9] and Mokrane [19] is assumed that $\psi, \varphi \in L^\infty(T \times Z)$.

Remark. Here is a simple situation where hypothesis H_0 is satisfied. Consider the following periodic parabolic problem involving the *p-Laplacian*

$$\left\{ \begin{array}{c} \frac{\partial x}{\partial t} - div\left(\|Dx\|^{p-2}Dx\right) + a_0(z,x)\sum_{k=1}^{N} D_k x = f(x(t,z)) \text{ on } Z \\ x(0,z) = x(b,z) \text{ a.e. on } Z, \quad x|_{T\times\Gamma} = 0 \end{array} \right\}$$

Suppose $f: R \to R$ satisfies hypothesis $H(f)$. Then the above problem fits in the framework of our problem (1). Assume that there exist $s_1 \leq 0 \leq s_2$ such that $h_r(s_1) \leq g(s_1)$ and $g(s_2) \leq h_l(s_2)$. Set $\varphi(t,z) = s_2$ and $\psi(t,z) = s_1$. It is clear that φ, ψ are upper and lower solutions repectively which satisfy hypothesis H_0.

The truncation part of the method, will be based on the following truncation map. Given $x \in L^p(T, W^{1,p}(Z))$, we set

$$\tau(x)(t,z) = \left\{ \begin{array}{ll} \varphi(t,z), & \text{if } \varphi(t,z) \leq x(t,z) \\ x(t,z), & \text{if } \psi(t,z) \leq x(t,z) \quad \leq \varphi(t,z) \\ \psi(t,z), & \text{if } x(t,z) \leq \psi(t,z) \end{array} \right\}$$

Proposition 1. *$\tau: L^p(T, W^{1,p}(Z)) \to L^p(T, W^{1,p}(Z))$ is continuous.*

Proof. From lemma 7.6, P. 146 of Gilbarg, Trudinger [13], we know that for any $x \in L^p(T, W^{1,p}(Z))$, we have that for almost all $t \in T$, $\tau(x)(t,\cdot) \in W^{1,p}(Z)$ and

$$D\tau(x)(t,z) = \left\{ \begin{array}{ll} D\varphi(t,z), & \text{if } \varphi(t,z) \leq x(t,z) \\ Dx(t,z), & \text{if } \psi(t,z) \leq x(t,z) \quad \leq \varphi(t,z) \\ D\psi(t,z) & \text{if } x(t,z) \leq \psi(t,z) \end{array} \right\}$$

Hence it follows that $\tau(x)(\cdot,\cdot) \in L^p(T, W^{1,p}(Z))$.

Now let $x_n \to x$ in $L^p(T, W^{1,p}(Z))$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume that $x_n(t,z) \to x(t,z)$ and $D_k x_n(t,z) \to D_k x(t,z)$ a.e. on $T \times Z$ as $n \to \infty$ for all $k \in \{1, 2. \dots, N\}$. Moreover from theorem 2.8.1, p. 74 of Kufner, John, Fucik [17], we can find functions θ, $\theta_k \in L^p(T \times Z)$ $k \in \{1, 2, \dots, N\}$ such that $|x_n(t,z)| \leq \theta(t,z)$ and $|D_k x_n(t,z)| \leq \theta_k(t,z)$ a.e. on $T \times Z$. Note that for all $n \geq 1$

$|\tau(x_n)(t,z)| \leq \max\{|\varphi(t,z)|, |\psi(t,z)|\}$ a.e. on $T \times Z$ and $|D_k \tau(x_n)(t,z)| \leq \max\{\theta_k(t,z), |D_k\varphi(t,z)|, |D_k\psi(t,z)|\}$ a.e. on $T \times Z$ for all $k \in \{1, 2, \dots, N\}$. Thus the dominated convergence theorem, implies that $\tau(x_n) \to \tau(x)$ in $L^p(T \times Z)$ and $D_k\tau(x_n) \to D_k\tau(x)$ in $L^p(T \times Z, R^N)$ as $n \to \infty$. Therefore we conclude that $\tau(x_n) \to \tau(x)$ in $L^p(T, W^{1,p}(Z))$ as $n \to \infty$, which proves the continuity of $\tau(\cdot)$. ∎

For the penalization aspect of the method, we introduce the penalty function $u : T \times Z \times R \to R$ denined by

$$u(t,z,x) = \left\{ \begin{array}{ll} \big(x - \varphi(t,z)\big)^{p-1}, & \text{if } \varphi(t,z) \leq x \\ 0, & \text{if } \psi(t,z) \leq x \leq \varphi(t,z) \\ -\big(\psi(t,z) - x\big)^{p-1}, & \text{if } x \leq \psi(t,z) \end{array} \right\}$$

From this definition and an elementary calculation, we obtain:

Proposition 2. *The function $u : T \times Z \times R \to R$ is a Caratheodory function, $|u(t,z,x)| \leq \beta_2(t,z) + c_2|x|^{p-1}$ a.e. on $T \times Z$, with $\beta_2 \in L^q(T \times Z)$, $c_2 > 0$, and $\int_0^b \int_z u(t,z,x(t,z))x(t,x)dzdt \geq c_3\|x\|^p_{L^p(T\times Z)} - c_4\|x\|^{p-1}_{L^p(T\times Z)}$, with $c_3, c_4 \geq 0$.*

Our hypothesis on the function $a_0(t,z,x)$, are the following:
$H(a_0) : a_0 : T \times Z \times R \to R$, is a function such that

(i) $(t,z) \to a_0(t,z,x)$ is measurable;
(ii) there exists $k \in L^\infty(T \times Z)$ such that for almost all $(t,z) \in T \times Z$ and $x, x' \in |\psi(t,x), \phi(t,z)|$, $|a_0(t,z,x) - a_0(t,z,x')| \leq k(t,z)|x - x'|$; and
(iii) for all $x \in L^p(T \times Z)$ such that $\psi(t,z) \leq x(t,z) \leq \phi(t,z)$ a.e. on $T \times Z$, the function $(t,z) \to a_0(t,z,x(t,z))$ belongs in $L^\infty(T \times Z)$.

3. Auxiliary abstract results

In this section, we introduce some basic notions and present some abstract results, which will be crucial in the proof of our main theorem in the next section. Our proof of that theorem, will be based on a general surjectivity result for the sum of two operators of monotone type. The application of this theorem, requires an auxiliary result of independent interest, which we prove here and which roughly speaking says that the pseudomonotonicity property of an operator $A(t,x)$, can be lifted to the Nemitsky (superposition) operator.

So let (X, H, X^*) be an evolution triple (see section 2). In what follows by $\langle\cdot,\cdot\rangle$ we will denote the duality brackets for the pair (X, X^*) and by $(\cdot,\cdot)$ the inner product of H. The two are compatible in the sense that $\langle\cdot,\cdot\rangle|_{X\times H} = (\cdot,\cdot)$ Also by $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$), we will denote the norm of X (resp. of H, X^*). We recall the following generalization of a monotone operator (see Zeidler [25], p. 585).

Definition. An operator $A : X \to X^*$ is to said to be "pseudomonotone", if $x_n \xrightarrow{w} x$ in X as $n \to \infty$ and $\overline{\lim}\langle A(x_n), x_n - x\rangle \leq 0$, imply that $\langle A(x), x - y\rangle \leq \underline{\lim}\langle A(x_n), x_n - y\rangle$ for all $y \in X$.

Remark. A monotone hemicontinuous operator or a strongly continuous operator, are pseudomonotone. Pseudomonotonicity is preserved by addition and clearly implies property (M) (i.e. $x_n \xrightarrow{w} x$ in X; $A(x_n) \xrightarrow{w} u^*$ in X^* as $n \to \infty$ and $\overline{\lim}\langle A(x_n), x_n - x\rangle \leq 0$, then $A(x) = u$). For details we refer to Zeidler [25], pp. 583–589.

In what follows, we will be dealing with an operator $A(t,x)$, for which we assume the following:

H(A): $A : T \times X \to X^*$ is an operator such that

(i) $t \to A(t,x)$ is measurable for each $x \in X$;
(ii) $x \to A(t,x)$ is demicontinuous and pseudomonotone (recall that demicontinuity means that if $x_n \to x$ in X, then $A(t,x_n) \xrightarrow{w} A(t,x)$ in X^* as $n \to \infty$);
(iii) $\|A(t,x)\|_* \leq \widehat{\beta}_1(t) + \widehat{c}_1\|x\|^{p-1}$ a.e. on T with $\widehat{\beta}_1 \in L^q(T)_+$, $c_1 > 0$, $2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$; and
(iv) $\langle A(t,x), x\rangle \geq \widehat{c}\|x\|^p - \eta\|x\|^r - \theta(t)$ for almost all $t \in T$, all $x \in X$ and with $\theta(\cdot) \in L^1(T), \widehat{c}, \eta > 0$, $1 \leq r \leq p-1$.

Let $\widehat{A} : L^p(T,X) \to L^q(T,X^*)$ be the Nemitsky (superposition) operator corresponding to $A(t,x)$; i.e. $\widehat{A}(x)(\cdot) = A(\cdot, x(\cdot))$.

We will show that in some sense the pseudomonotonicity property of $A(t,\cdot)$ is passed to $\widehat{A}(\cdot)$. First we need a definition:

Definition. Let Y be a reflexive Banach space, $L : D(L) \subseteq Y \to Y^*$ is a linear densely defined maximal monotone operator and $V : Y \to 2^{Y^*}\backslash\{\emptyset\}$ is a multivalued operator with weakly compact and convex values. We say that $V(\cdot)$ is "pseudomonotone with respect to $D(L)$" (or "L-pseudomonotone"), if for $\{y_n\}_{n\geq 1} \subseteq D(L)$ with $y_n \xrightarrow{w} y$ in Y and $L(y_n) \xrightarrow{w} L(y)$ in Y^* as $n \to \infty$ and for $y_n^* \in V(y_n) n \geq 1$ satisfying $y_n^* \xrightarrow{w} y^*$ as $n \to \infty$ and $\overline{\lim}(y_n^*, y_n) \leq (y^*, y)$, we have $y^* \in V(y)$ and $(y_n^*, y_n) \to (y^*, y)$ as $n \to \infty$.

Remark. Recall that a linear operator $L : D \subseteq Y \to Y^*$ is maximal monotone if and only if it is densely defined in X, L and L^* are both monotone and L is closed (i.e. GrL is closed in $Y \times Y^*$). For a proof of this fact, we refer to Zeidler [25], (theorem 32, p. 897).

Now let $L : D \subseteq L^p(T,X) \to L^q(T,X^*) = L^p(T,X)^*$, be defined by $Lx = \dot{x}$ for all $x \in D = \{x \in L^p(T,X) : \dot{x} \in L^q(T,X^*), x(0) = x(b)\}$. As before, the time derivative of $x(\cdot)$ is defined in the sense of vector-valued distributions. Also since the separable, reflexive Banach space $W_{pq}(T) = \{x \in L^p(T,X) : \dot{x} \in L^q(T,X^*)\}$, is continuously embedded in $C(T,H)$, we see that the equality $x(0) = x(b)$ makes sense. Since $C_0^1(T,X)$ is dense in $L^p(T,X)$ for the norm topology, we deduce that $L(\cdot)$ is densely defined on $L^p(T,X)$. Moreover note that $L^*v = -\dot{v}$ for all $v \in D = \{v \in L^p(T,X) : \dot{v} \in L^q(T,X^*), v(0) = v(b)\}$. So using the integration by parts formula for functions in $W_{pq}(T)$ (see Zeidler [25], proposition 23.23 (iv), pp. 422-423), we see that L and L^* are both monotone. Finally it is easy to see that GrL is closed in $L^p(T,X) \times L^q(T,X^*)$. So according to the previous remark, $L(\cdot)$ is a maximal monotone operator.

The next proposition, is actually a result of independent interest and can be useful in the study of evolution equations and inclusions defined on evoluiton triples.

Proposition 3. *If X is compactly embedded in $H, A : T \times X \to X^*$ is an operator satisfying hypothesis H(A) and $L : D \subseteq L^q(T,X) \to L^q(T,X^*)$ is the linear maximal monotone operator defined by $L(x) = \dot{x}$ for all $x \in D \subseteq L^p(T,X) = \{x \in L^p(T,X) : \dot{x} \in L^q(T,X^*),\ x(0) = x(b)\}$, then the Nemitsky operator $\widehat{A} : L^p(T,X) \to L^q(T,X^*)$ is demicontinuous and pseudomonotone with respect to $D(L) = D$.*

Proof. We will start by showing the demicontinuity of $\widehat{A}(\cdot)$. So let $x_n \to x$ in $L^p(T,X)$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume that $x_n(t) \to x(t)$ a.e. on T in X as $n \to \infty$. Then because of hypothesis H(A)(ii), given $y \in L^p(T,X)$, we have $\langle A(t,x_n(t)), y(t)\rangle \to \langle A(t,x(t)), y(t)\rangle$ a.e. on T as $n \to \infty$. Moreover thanks to hypothesis H(A)(iii), we can apply the generalized dominated convergence theorem (see for example Ash [1], theorem 7.52, p. 295) and obtain that

$$((\widehat{A}(x_n), y)) = \int_0^b \langle A(t,x_n(t)), y(t)\rangle dt \to \int_0^b \langle A(t,x(t)), y(t)\rangle dt = ((\widehat{A}(x_n), y))$$

as $n \to \infty$.

Since $y \in L^p(T,X)$ was arbitrary, we conclude that $\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in

$L^q(T, X^*)$ as $n \to \infty$, which proves the demicontinuity of $\widehat{A}(\cdot)$.

Next we will prove the pseudomonotonicity of $\widehat{A}(\cdot)$ with respect to $D(L)$. So let $x_n \xrightarrow{w} x$ in $L^p(T, X)$ and $x_n \xrightarrow{w} x$ in $L^q(T, X^*)$ as $n \to \infty$ (i.e. $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ as $n \to \infty$) with $x_n \in D$ $n \geq 1$ (i.e. $x_n(0) = x_n(b)$ for all $n \geq 1$). Assume that $\overline{\lim}\big((\widehat{A}(x_n), x_n - x)\big) = \overline{\lim} \int_0^b \langle A(t, x_n(t)), x_n(t) - x(t)\rangle dt \leq 0$. Let $\xi_n(t) = \langle A(t, x_n(t)), x_n(t) - x(t)\rangle$ $n \geq 1$. Since $W_{pq}(T)$ embeds continuously in $C(T, H)$, we have $x_n \to x$ in $C(T, H)$ and so for every $t \in T$ $x_n(t) \xrightarrow{w} x(t)$ in H as $n \to \infty$. On the other hand, let $N \subseteq T$ be the exceptional Lebesgue null set, outside of which hypothesis H(A)(iiii) and (iv) hold. Then for every $t \in T\backslash N$ we have

$$(3) \quad \xi_n(t) \geq \widehat{c}\|x_n(t)\|^p - \eta\|x_n(t)\|^r - \theta(t) - \big(\beta_1(t) + \widehat{c}_1\|x_n(t)\|^{p-1}\big)\|x(t)\|$$

Set $C = \{t \in T : \underline{\lim}\xi_n(t) < 0\}$. This is a Lebesgue measurable subset of T. Suppose that $\lambda(C) > 0$, where $\lambda(\cdot)$ is the Lebesgue measure on T. From (3) above, we see that for fixed $t \in C \cap (T\backslash N) \neq \emptyset$, the sequence $\{x_n(t)\}_{n\geq 1}$ is bounded in X. Since X is reflexive and because we already know that for every $t \in T$ $x_n(t) \xrightarrow{w} x(t)$ is H as $n \to \infty$, we deduce that $x_n(t) \xrightarrow{w} x(t)$ in X as $n \to \infty$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\underline{\lim}\xi_n(t) = \lim \xi_{n_k}(t)$. Then due to the fact that $A(t, \cdot)$ is pseudomonotone, we deduce that $\langle A(t, x_n(t)), x_{n_k}(t) - x(t)\rangle = \xi_{n_k}(t) \to 0$ as $k \to \infty$, a contradiction to the definition of C (recall that $t \in C \cap (T\backslash N)$). So $\lambda(C) = 0$ and so $0 \leq \underline{\lim}\xi_n(t)$ a.e. on T. Then from the generalized Fatou's lemma (see for example Ash [1], theorem 7.5.2, p. 295) we have

$$0 \leq \int_0^b \underline{\lim}\xi_n(t)dt \leq \underline{\lim} \int_0^b \xi_n(t)dt \leq \overline{\lim} \int_0^b \xi_n(t)dt \leq 0$$

hence $\int_0^b \xi_n(t)dt \to 0$ as $n \to \infty$. Also note that since $0 \leq \underline{\lim}\xi_n(t)$ a.e. on T, we have $\xi_n^-(t) \to 0$ a.e. on T. Moreover from (3) above, it is evident that $\gamma_n(t) \leq \xi_n(t)$ a.e. on T with $\{\gamma_n\}_{n\geq 1} \subseteq L^1(T)$ being uniformly integrable. Then $0 \leq \xi_n^-(t) \leq \gamma_n^-(t)$ a.e. on T and of course $\{\gamma_n^-\}_{n\geq 1} \subseteq L^1(T)$ is uniformly integrable. So a new application of the generalized dominated convergence theorem, gives us that $\int_0^b \xi_n^-(t) \to 0$ as $n \to \infty$. Therefore we deduce that

$$\int_0^b |\xi_n(t)|dt = \int_0^b (\xi_n(t) + 2\xi_n^-(t))dt \to 0 \text{ as } n \to \infty;$$

i.e. $\xi_n \to 0$ in $L^1(T)$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume that $\xi_n(t) \to 0$ a.e. on T as $n \to \infty$. Because $A(t, \cdot)$ is pseudomonotone, we have that $A(t, x_n(t)) \xrightarrow{w} A(t, x(t))$ a.e. on T in X^* and $\langle A(t, x_n(t)), x_n(t)\rangle \to \langle A(t, x(t)), x(t)\rangle$ a.e. on T as $n \to \infty$. So a final application fo the generalized

dominated convergence theorem, tells us that
$\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^q(T, X^*)$ and
$((\widehat{A}(x_n), x_n)) = \int_0^b \langle A(t, x_n(t)), x_n(t)\rangle dt \to \int_0^b \langle A(t, x(t)), x(t)\rangle dt = ((\widehat{A}(x), x))$
as $n \to \infty$.
Thus we conclude that $\widehat{A}(\cdot)$ is pseudomonotone with respect to $D(L) = D$. ■

The next surjectivity result is known (see Lions [18], theorem 1.2, p. 319 or B-A. Ton [24]). However for easy reference, we include it here. Recall that if V, W are Hausdorff topological spaces, then a multifunction $G : V \to 2^W \backslash \{\emptyset\}$ is said to be upper semicontinuous (usc for short) if and only if for every $U \subseteq W$ open subset, $G^+(U) = \{v \in V : G(v) \subset U\}$ is open in V. Such a multifunction has a closed graph; i.e. $GrG = \{(v, w)\} \in V \times W : w \in G(v)\}$ is closed in $V \times W$. For details we refer to DeBlasi, Myjak [8].

Theorem 4. *If Y is a reflexive Banach space, $L : D \subseteq Y \to Y^*$ is a linear maximal monotone operator and $G : Y \to 2^{Y^*} \backslash \{\emptyset\}$ is a multivalued operator with weakly compact and convex values, which is bounded (i.e. $G(\cdot)$ maps bounded sets to bounded sets), usc from Y into Y^*_w (here by Y^*_w we denote the reflexive Banach space Y^* furnished with the weak topology), pseudomonotone with respect to $D(L) = D$ and coercive (i.e. $\dfrac{\inf\{(y^*, y) : y^* \in G(y)\}}{\|y\|} \to +\infty$ as $\|y\| \to \infty$, then $R(L + G) = Y^*$; i.e. the operator $(L + G)(\cdot)$ is surjective.*

4. Main theorem

In this section we prove our main theorem. Namely we show that under the hypothesis fixed in the previous section, problem (2) has its extremal solutions in the order interval $K \doteq [\psi, \varphi] = \{y \in L^p(T \times Z) : \psi(t, z) \leq y(t, z) \leq \varphi(t, z)$ a.e. on $T \times Z\}$. Assume $\frac{3N}{N+2} \leq p$, $2 \leq p$.

In order words problem (2) has the greatest solution x_g and the smallest solution x_s within the order interval K, in the sense that if x is any solution of (2) in K, then $x \in [x_s, x_g]$. Moreover we show that under additional hypothesis on the functions a_k, a_0 $k \in \{1, 2, \ldots, N\}$ and on the regularity properties of $g(\cdot)$, these extremal solution can be attained by a monotone iterative process.

Theorem 5. *If hypothesis $H(a)$, $H(f)$, H_0 and $H(a_0)$ hold, then problem (2) has a greatest solution $x_g \in W_{pq}(T)$ and a smallest solution $x_s \in W_{pq}(T)$ in the order interval*
$K = [\psi, \varphi] = \{y \in L^p(T \times Z) : \psi(t, z) \leq y(t, z), \leq \varphi(t, z)$ *a.e. on* $T \times Z\}$
(i.e. problem (2) has extremal solutions in the order interval K).

Proof. Given $y \in K$, we consider the following auxiliary problem

$$(4) \qquad \left\{ \begin{array}{c} \frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t,z,\tau(x),Dx) + a_0(t,z,\tau(x)) \sum_{k=1}^{N} D_k \tau(x) + \\ + \beta(\tau(x)(t,z)) + u(t,z,x(t,z)) \ni g(x(t,z)) \text{ on } T \times Z \\ x(0,z) = x(b,z) \text{ a.e. on } Z, \quad x|_{T\times\gamma} = 0 \end{array} \right\}$$

In what follows for notational simplicity, we set $X = W_0^{1,p}(Z)$ and $X^* = W^{-1,q}(Z)$.

Let $L : D \subseteq L^p(T,X) \to L^q(T,X^*)$ be defined by $Lx = \dot{x}$ for all $x \in D = \{x \in W_{pq}(T) : x(0) = x(b)\}$ (recall that the time-derivative of x is understood in the sense of vector-valued distributions). From our discussion in section 3, we know that $L(\cdot)$ is maximal monotone.

Next let $A_1 : T \times X \to X^*$ be defined by

$$\langle A_1(t,x), y \rangle = \sum_{k=1}^{N} \int_z a_k(t,z,\tau(x),Dx) D_k y(z) dz.$$

Using Fubini's theorem, we see at once that $t \to \langle A_1(t,x), y\rangle$ is measurable. Since $y \in X$ was arbitrary, we deduce that $t \to A_1(t,x)$ is weakly measurable. But $X^* = W^{-1,q}(Z)$ is a separable reflexive Banach space. So from the Pettis measurability theorem (see Diestel, Uhl [10], theorem 2, p. 42), we have that $t \to A_1(t,x)$ is measurable. Also it is clear from hypothesis $H(a)$(iii), that $\|A_1(t,x)\|_* \leq \beta_1'(t) + c_1'\|x\|^{p-1}$ with $\beta_1' \in L^q(T)$ and $c_1' > 0$, while from hypothesis $H(a)$(v), it follows that $\langle A_1(t,x), x\rangle = \sum_{k=1}^{N} \int_z a_k(t,z,\tau(x),Dx) D_k x(z) dz \geq c'\|x\|^p$ for almost all $t \in T$, all $x \in X$ and with $c' > 0$. Note that in both inequalities $\|\cdot\|$ denotes the norm of $X = W_0^{1,p}(Z)$.

Now we will show that $x \to A_1(t,x)$ is demicontinuous. To this end, let $x_n \to x$ in X as $n \to \infty$. By passing to a subsequence if necessary, we may assume that $\tau(x_n)(z) \to \tau(x)(z)$ and $Dx_n(z) \to Dx(z)$ a.e. on Z as $n \to \infty$. Then from hypothesis $H(a)$(iii) and the generalized dominated convergence theorem, we deduce that for all $y \in X$ we have

$$\begin{aligned} &\langle A_1(t,x_n)), y\rangle \\ &= \sum_{k=1}^{N} \int_z a_k(t,z,\tau(x_n),Dx_n) D_k y(z) dz \to \sum_{k=1}^{N} \int_z a_k(t,z,\tau(x),Dx) D_k y(z) dz \\ &= \langle A_1(t,x)), y\rangle \text{ as } n \to \infty, \end{aligned}$$

hence $A_1(t,x) \xrightarrow{w} A_1(t,x)$ in X^* as $n \to \infty$, which proves the demicontinuity of $x \to A_1(t,x)$.

Finally theorem 3, p. 42 of Gossez, Mustonen [14], tells us that $x \to A_1(t,x)$ is pseudomonotone.

Next for every $(t,x) \in T \times X$, define $h(t,x)$ as follows

$$h(t,x)(\cdot) = a_0(t,\cdot,\tau(x)(\cdot)) \sum_{k=1}^{N} D_k\tau(x)(\cdot).$$

Evidently $h(t,x) \in H$. So we can consider the map $h : T \times X \to X^*$. Clearly $t \to h(t,x)$ is measurable. We will also show that $x \to h(t,x)$ is completely continuous (i.e. if $x_n \xrightarrow{w} x$ in X as $n \to \infty$, then $h(t,x_n)$ in X^* as $n \to \infty$). To this end let $x_n \xrightarrow{w} x$ in $X = W_0^{1,p}(Z)$ as $n \to \infty$. Note that since by hypothesis $2 \le p$, in $\frac{3N}{N+2} \le p$, $W_0^{1,p}(Z)$ embeds compactly in $L^{2q}(Z)$ $(1 < q \le 2 \le p < \infty)$. So we have $x_n \to x$ in $L^{2q}(Z)$ as $n \to \infty$.

We need to show that $h(t,x_n) \to h(t,x)$ in X^* as $n \to \infty$. Suppose not. Then we can find $\varepsilon > 0$ and a sequence $\{y_{n_m}\}_{m \ge 1} \subseteq X$ such that $\|y_{n_m}\| \le 1$ for all $m \ge 1$ and $\langle h(t,x_{n_m}) - h(t,x), y_{n_m}\rangle \ge \varepsilon$ for all $m \ge 1$. Passing to a subsequence if necessary, we may assume that $y_{n_m} \xrightarrow{w} y$ in X and so $y_{n_m} \to y$ in $L^{2q}(Z)$ as $m \to \infty$.

Then we have

$$\begin{aligned}
&\Big|\int_z a_0(t,z,\tau(x_{n_m})(z))D_k\tau(x_{n_m})(z)y_{n_m}(z)dz \\
&\quad - \int_z a_0(t,z,\tau(x)(z))D_k\tau(x)(z)y_{n_m}(z)dz\Big| \\
&\le \Big|\int_z a_0(t,z,\tau(x_{n_m})(z)) - a_0(t,z,\tau(x)(z)))D_k\tau(x_{n_m})(z)y_{n_m}(z)dz\Big| \\
&+ \Big|\int_z a_0(t,z,\tau(x)(z))D_k\tau(x_{n_m})(z)(y_{n_m}(z) - y(z))dz\Big| \\
&+ \Big|\int_z a_0(t,z,\tau(x)(z))(D_k\tau(x_{n_m})(z) - D_k\tau(x)(z))y_{n_m}(z)dz\Big| \\
&+ \Big|\int_z a_0(t,z,\tau(x)(z))D_k\tau(x)(z)(y(z) - y_{n_m}(z)dz\Big|.
\end{aligned}$$

From hypothesis $H(a_0)$(ii), we know that

$$|a_0(t,z,\tau(x_{n_m})(z)) - a_0(t,z,\tau(x)(z))| \le k(t,z)|\tau(x_{n_m})(z) - \tau(x)(z)|$$

a.e. on $T \times Z$.

So using Holder's inequality with three factors, we obtain that

$$\begin{aligned}
&\Big|\int_z (a_0(t,z,\tau(x_{n_m})(z)) - a_0(t,z,\tau(x)(z)))D_k\tau(x_{n_m})(z)y_{n_m}(z)dz\Big| \\
&\quad \le \|k\|_\infty \|\tau(x_{n_m}) - \tau(x)\|_{2q}\|D_k\tau(x_{n_m})\|_p\|y_{n_m}\|_{2q} \to 0 \text{ as } n \to \infty,
\end{aligned}$$

since the truncation map $\tau(\cdot)$ is continuous on $L^{2q}(Z)$.

Also for some $\eta > 0$, we have

$$\left|\int_Z a_0(t,z,\tau(x)(z))D_k\tau(x_{n_m})(z)(y_{n_m}(z)-y(z))dz\right|$$
$$\leq \eta\|a(t,\cdot,\tau(x)(\cdot)\|_\infty\|D_k\tau(x_{n_m})\|_p\|y_{n_m}-y\|_{2q}\to 0 \text{ as } m\to\infty.$$

In addition from the continuity of $\tau(\cdot)$ and since $W_0^{1,p}(Z)$ embeds compactly in $L^p(Z)$, we have $\tau(x_n)\to\tau(x)$ in $L^p(Z)$ as $n\to\infty$. Then in $y\in C_0^\infty(Z)$, we have

$$\begin{aligned}\left(D_k\tau(x_{n_m})(z),y\right)_{L^p(Z),L^q(Z)} &= \int_Z D_k\tau(x_{n_m})(z)y(z)dz\\ &= -\int_Z \tau(x_{n_m})(z)D_ky(z)dz \to -\int_Z\tau(x)(z)D_ky(z)dz\\ &= \int_Z D_k\tau(x)(z)y(z)dz\\ &= (D_k\tau(x),y)_{L^p(Z),L^q(Z)} \text{ as } m\to\infty.\end{aligned}$$

Because $C_0^\infty(Z)$ is dense in $L^q(Z)$ (see Kufner, John, Fucik [17] theorem 2.6.1, p. 73), we deduce that $D_k\tau(x_n)\xrightarrow{w} D_k\tau(x)$ in $L^p(Z)$ as $m\to\infty$. So we have

$$\left|\int_Z a_0(t,z,\tau(x)(z))(D_k\tau(x_{n_m})(z)-D_k\tau(x)(z))y(z)dz\right|\to 0 \text{ as } m\to\infty.$$

Finally note that for some $\eta'>0$, we have

$$\left|\int_Z a_0(t,z,\tau(x)(z))D_k\tau(x_{n_m})(z)(y(z)-y_{n_m}(z))dz\right|$$
$$\leq \eta'\left\|a_0(t,\cdot,\tau(x)(\cdot)\right\|_\infty\left\|D_k\tau(x_{n_m})\right\|_p\left\|y-y_{n_m}\right\|_q\to 0 \text{ as } m\to\infty.$$

Combining all these convergences, which are valid for every $k\in\{1,2,\dots,N\}$, we conclude that $\langle h(t,x_{n_m})-h(t,x),y_{n_m}\rangle\to 0$ as $m\to\infty$ where as in section 3, $\langle\cdot,\cdot\rangle$ stands for the duality brackets for the pair (X,X^*).

This last convergence, contradicts the choice of the sequences $\{x_{n_m}\}_{m\geq 1}\{y_{n_m}\}_{m\geq 1}\subseteq X=W_0^{1,p}(Z)$. Therefore we have that $h(t,x_n)\to h(t,x)$ in X^* as $n\to\infty$ and so $x\to h(t,x)$ is completely continuous. If we set $A(t,x)=A_1(t,x)+h(t,x)$, then from proposition 27.6(e), p. 586 of Zeidler [25], we have that $x\to A(t,x)$ is pseudomonotone. Thus proposition 3, tells us that the Nemitsky operator $\widehat{A}:L^p(T,X)\to L^q(T,X^*)$ is pseudomonotone with respect to $D(L)=D$.

Next let $\Phi:L^p(T,X)\to\overline{R}=R\cup\{+\infty\}$ be defined by

$$\Phi(x)=\left\{\begin{array}{cc}\int_0^b\int_Z j(x(t,z))dzdt, & \text{if } j(x(\cdot))\in L^1(T\times Z)\\ +\infty & \text{otherwise}\end{array}\right\}$$

where $j : R \to \overline{R} = R \cup \{+\infty\}$ is the proper, lower semicontinuous and convex function such that $\beta = \partial j$. It is easy to see that $\Phi(\cdot)$ is proper, lower semicontinuous and convex (i.e. $\Phi \in \Gamma_0(L^p(T, X))$). Moreover from corollary 1 of Brezis [4] and theorem 21 of Rockafellar [20], we know that for all $x \in dom\partial\Phi$, $\partial\Phi(x) \subseteq L^1(T \times Z)$ and $v \in \partial\Phi(x)$ if and only if $v(t, z) \in \beta(x(t, z))$ a.e. on $T \times Z$.

Then problem (4), can be equivalently rewritten as the following operator inclusion

$$Lx + \widehat{A}(x) + \partial\Phi(\tau(x)) + U(x) \in \widehat{g}(y)$$

where $U : L^p(T \times Z) \to L^q(T \times Z)$ is defined by

$$U(x)(t, z) = u(t, z, x(t, z))$$

(the Nemitsky operator corresponding to the penalty function u) and

$$\widehat{g}(y)(t, z) = g(y(t, z)) \in L^1(T \times Z) \cap L^q(T, W^{-1,q}(Z)) (\text{see hypothesis } H_0).$$

Let $G : L^p(T, X) \to 2^{L^q(T, X^*)}$ be defined by

$$G(x) = \widehat{A}(x) + \partial\Phi(\tau(x)) + U(x).$$

First note that $G(\cdot)$ has nonempty, weakly compact and convex values. This is an immediate consequence of hypothesis H_0, corollary 1 of Brezis [4] and theorem 21 of Rockafellar [20].

Claim #1. $G(\cdot)$ is pseudomonotone with respect to $D(L)$.

Let $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ as $n \to \infty$, $x_n \in D(L) = D$. Let $g_n \in G(x_n)$, $n \geq 1$ and assume that $g_n \xrightarrow{w} g$ in $L^q(T, X^*)$ as $n \to \infty$ and that $\overline{\lim}((g_n, x_n - x)) \leq 0$ (recall from section 2 that $((\cdot, \cdot))$ denotes the duality brackets for the pair $(L^p(T, X), L^q(T, X^*))$; i.e. $((g, x)) = \int_0^b \langle g(t), x(t)\rangle dt$). By definition $g_n = \widehat{A}(x_n) + v_n + U(x_n)$ $n \geq 1$ with $v_n \in \partial\Phi(\tau(x_n))$. Hence by virtue of hypothesis H_0, $\{v_n\}_{n\geq 1} \subseteq L^q(T, X^*)$ is bounded and so by passing to a subsequence if necessary we may assume that $v_n \xrightarrow{w} v$ in $L^q(T, X^*)$ as $n \to \infty$. Also from the continuity of the penalty function $u(t, z, \cdot)$ (see proposition 2) and because $x_n \to x$ in $L^p(T \times Z)$ as $n \to \infty$ (which is a consequence of the fact that $W_{pq}(T)$ embeds compactly in $L^p(T \times Z)$), we have that $U(x_n) \to U(x)$ in $L^p(T \times Z)$ as $n \to \infty$. Then we have

$$\begin{aligned}
0 &\geq \overline{\lim}((g_n, x_n - x)) = \overline{\lim}\left(\left(\widehat{A}(x_n) + v_n + U(x_n), x_n - x\right)\right) \\
&\geq \overline{\lim}\left(\left(\widehat{A}(x_n), x_n - x\right)\right) + \overline{\lim}((v_n, x_n - x)) + \overline{\lim}((U(x_n), x_n - x)) \\
&\geq \left(\left(\widehat{A}(x_n), x_n - x\right)\right) + \overline{\lim}((v, x_n - x)) + \overline{\lim}((U(x_n), x_n - x)) \\
&= \overline{\lim}\left(\left(\widehat{A}(x_n), x_n - x\right)\right)
\end{aligned}$$

Note the third inequality in the above chain, is a consequence of the monotonicity of the subdifferential operator $\partial\Phi(\cdot)$.

So we have proved that $\overline{\lim}\big((\widehat{A}(x_n), x_n - x)\big) \leq 0$.

But recall that $\widehat{A}(\cdot)$ is pseudomonotone with respect to $D(L)$. So from the above inequality we infer that
$\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^q(T, X^*)$ as $n \to \infty$ and $\big((\widehat{A}(x_n), x_n)\big) \to \big((\widehat{A}(x), x)\big)$ as $n \to \infty$.

Therefore $g_n = \widehat{A}(x_n) + v_n + U(x_n) \xrightarrow{w} g = \widehat{A}(x) + v + U(x)$ in $L^q(T, X^*)$ as $n \to \infty$ and $((g_n, x_n)) \to ((g, x))$ as $n \to \infty$. Finally note that

$$\overline{\lim}((v_n, x_n - x)) = \overline{\lim}\big((g_n - \widehat{A}(x_n) - U(x_n), x_n - x)\big) = 0.$$

Since $\partial\Phi(\cdot)$ is maximal monotone, it is generalized pseudomonotone in the sense of definition 2, p. 253 of Browder, Hess [5] and so $v \in \partial\Phi(x)$. Therefore $g = \widehat{A}(x) + v + U(x) \in \widehat{A}(x) + \partial\Phi(x) + U(x) = G(x)$, which proves claim #1.

Claim #2. $G(\cdot)$ is bounded (i.e. maps bounded sets to bounded sets).

This claim is an immediate consequence of hypothesis H_0 and of the growth properties of the operator $\widehat{A}(\cdot)$ (see the first part of the proof) and of the operator $U(\cdot)$ (see proposition 2).

Claim #3. $G(\cdot)$ is an usc multifunction from $L^p(T, X)$ into $2^{L^q(T,X^*)_w} \setminus \{\emptyset\}$.

In order to prove this claim, we need to show that if $C \subseteq L^q(T, X^*)$ is weakly closed, then $G^-(C) = \{x \in L^p(T, X) : G(x) \cap C \neq \emptyset\}$ is closed in $L^p(T, X)$ (see DeBlasi, Myjak [8]). So let $x_n \in G^-(C) n \geq 1$ and assume that $x_n \to x$ in $L^p(T, X)$ as $n \to \infty$. Let $g_n \in G(x_n) \cap C$ $n \geq 1$. By virtue of claim #2, $\{g_n\}_{n \geq 1}$ is bounded in $L^q(T, X^*)$ and so by passing to a subsequence, we may assume that $g_n \xrightarrow{w} g$ in $L^q(T, X^*)$ as $n \to \infty$. By definition we have

$$g_n = \widehat{A}(x_n) + v_n + U(x_n), \ v_n \in \partial\Phi(x_n) \ n \geq 1.$$

By virtue of hypothesis H_0, we may assume that $v_n \xrightarrow{w} v$ in $L^q(T, X^*)$ as $n \to \infty$. So from the demiclosedness of the subdifferential operator (since it is maximal monotone; see Zeidler [25], p. 915), we have that $v \in \partial\Phi(x)$. Also $\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ and $U(x_n) \to U(x)$ in $L^q(T, X^*)$ as $n \to \infty$. Thus in the limit as $n \to \infty$, we have $g = \widehat{A}(x) + v + U(x)$, $v \in \partial\Phi(x)$.

Claim #4. $G(\cdot)$ is coercive.

Using hypothesis $H(a)$(v), we have

$$\begin{aligned}\big((\widehat{A}(x), x)\big) &\geq c\|x\|^p_{L^p(T,X)} + \int_0^b \int_z a_0(t, z, x(t, x)) \left(\sum_{k=1}^N D_k x(t, z)\right) x(t, z) dz dt \\ &\geq c\|x\|^p_{L^p(T,X)} - \widehat{c}\|a_0(\cdot, \cdot, x(\cdot, \cdot))\|_\infty \|x\|_{L^p(T,X)} \|x\|_{L^p(T \times Z)}\end{aligned}$$

for some $\widehat{c} > 0$.

Using Young's inequality with $\varepsilon > 0$ and setting $\widehat{c}_1 = \widehat{c}\|a_0(\cdot,\cdot,x(\cdot,\cdot))\|_\infty$, we have $\widehat{c}_1\|x\|_{L^p(T,X)}\|x\|_{L^p(T,Z)} \leq \widehat{c}_1\frac{\varepsilon^p}{p}\|x\|^p_{L^p(T,X)} + \widehat{c}\frac{1}{\varepsilon^q q}\|x\|^q_{L^p(T\times Z)}$

So we have

$$((\widehat{A}(x),x)) \geq \left(c - \widehat{c}_1\frac{\varepsilon^p}{p}\right)\|x\|^p_{L^p(T,X)} - \widehat{c}_1\frac{1}{\varepsilon^q q}\|x\|^q_{L^p(T\times Z)}. \tag{5}$$

Also from proposition 2, we know that

$$((U(x),x)) \geq c_3\|x\|^p_{L^p(T\times Z)} - c_4\|x\|^{p-1}_{L^p(T\times Z)}. \tag{6}$$

Finally from hypothesis H_0 and Young's inequality with $\varepsilon > 0$, we have

$$|((\partial\Phi(\tau(x)),x))| \leq \xi\|x\|_{L^p(T,X)} \leq \left(\frac{\xi}{\varepsilon}\right)^q\frac{1}{q} + \frac{\varepsilon^p}{p}\|x\|^p_{L^p(T,X)}$$

hence

$$((\partial\Phi(\tau(x)),x)) \geq -\frac{\varepsilon^p}{p}\|x\|^p_{L^p(T,X)} - \widehat{\xi}(\varepsilon) \text{ with } \widehat{\xi}(\varepsilon) = \left(\frac{\xi}{\varepsilon}\right)^q\frac{1}{q},\ \xi > 0. \tag{7}$$

Putting together (5), (6) and (7), we obtain

$$((G(x),x)) \geq \left(c - \widehat{c}_1\frac{\varepsilon^p}{p} - \frac{\varepsilon^p}{p}\right)\|x\|^p_{L^p(T,X)} + \|x\|^{p-1}_{L^p(T\times Z)}\left(c_3\|x\|_{L^p(T\times Z)} - c_4\right) \tag{8}$$

Choose $\varepsilon > 0$ so that $c - (\widehat{c}_1 + 1)\frac{\varepsilon^p}{p} > 0$.

Then from (8) we infer that $G(\cdot)$ is coercive.

Now we are in a position to apply theorem 4 and obtain $x \in D(L) = D$ such that

$$Lx + \widehat{A}(x) + \partial\Phi(\tau(x)) + U(x) \ni \widehat{g}(y). \tag{9}$$

Let $S(y) \subseteq X = W_0^{1,p}(Z)$ be the solution set of (9). We have just seen that for every $y \in K$, $S(y)$ is nonempty.

Claim #5. $S(K) \subseteq K$

Let $y \in K$ and let $x \in S(y)$. Since $\psi \in \widehat{W}_{pq}(T)$ is a lower solution, we have

$$((\psi,w)) + ((\widehat{A}(\psi),w)) + ((h_\tau(\psi),w)) + ((U(\psi),w)) \leq ((g(\psi),w)) \tag{10}$$

for all $w \in \widehat{W}_{pq}(T) \cap L^p(T\times Z)_+$ and $\psi(0,z) \leq \psi(b,z)$ a.e. on Z, $\psi|_{T\times Z} \leq 0$.

Also for some $v \in \partial\Phi(x)$, we have

$$((x,w)) + ((\widehat{A}(x),w)) + ((v,w)) + ((U(x),w)) = ((g(y),w)). \tag{11}$$

Multiplying (10) with -1 and then adding it to (11) and using as test function $w = (\psi - x)^+ \in \widehat{W}_{pq}(T)$, we obtain

$$\begin{aligned}
&((x-\psi,(\psi-x)^+)) + ((\widehat{A}(x)-\widehat{A}(\psi),(\psi-x)^+)) \\
&\qquad + ((v-h_r(\psi),(\psi-x)^+)) + ((U(x)-U(\psi),(\psi-x)^+)) \\
&\qquad \geq ((\widehat{g}(y)-\widehat{g}(\psi),(\psi-x)^+)).
\end{aligned} \tag{12}$$

From the integration by parts formula for functions in $\widehat{W}_{pq}(T)$ (see Zeidler [25], proposition 23.23(iv), pp. 422–423), we have

$$((x-\psi,(\psi-x)^+)) = -\frac{1}{2}\|(\psi-x)^+(b)\|^2_{L^2(Z)} + \frac{1}{2}\|(\psi-x)^+(0)\|^2_{L^2(Z)} \leq 0. \tag{13}$$

Also recall that $v \in \partial\Phi(\tau(x))$ implies that $v(t,z) \in \beta(\tau(x)(t,z))$ a.e. on $T \times Z$.

Hence from the definition of the truncation map $\tau(x)$, we have

$$((v-h_r(\psi),(\psi-x)^+)) \leq 0. \tag{14}$$

Moreover from Gilbarg, Trudinger [13], lemma 7.6, p. 145, we know that

$$D_k(\psi-x)^+(t,z) = \left\{ \begin{array}{cl} D_k(\psi-x)(t,z), & \text{if } x(t,z) \leq \psi(t,z) \\ 0 & \text{if } \psi(t,z) \leq x(t,z) \end{array} \right\}$$

So using hypothesis $H(a)$(iv), we have

$$\begin{aligned}
&((\widehat{A}(x)-\widehat{A}(\psi),(\psi-x)^+)) \\
&\quad = \int_0^b\int_Z \sum_{k=1}^N (a_k(t,z,\tau(x),Dx) - a_k(t,z,\psi,D\psi))D_k(\psi-x)^+ dzdt \\
&\quad + \int_0^b\int_Z \left(a_0(t,z,\tau(x))\left(\sum_{k=1}^N D_k\tau(x)\right)\right. \\
&\quad \left. - a_0(t,x,\psi)\left(\sum_{k=1}^N D_k\psi\right)\right)(\psi-x)^+ dzdt \\
&\quad = \int\int_{\{\psi\leq x\}} \sum_{k=1}^N (a_k(t,z,\psi,Dx) - a_k(t,z,\psi,D\psi))D_k(\psi-x)dzdt \leq 0.
\end{aligned} \tag{15}$$

Finally because $y \in K$ and $g(\cdot)$ is nondecreasing, we have

$$((\widehat{g}(y)-\widehat{g}(\psi),(\psi-x)^+)) \geq 0. \tag{16}$$

Using inequalities (13) $\rightarrow$ (16) in (12), we obtain $((U(x) - U(\psi), (\psi - x)^+)) \geq 0$ hence $\int_0^b \int_z -(\psi - x)^{p-1}(\psi - x)^+ dzdt \geq 0$ and so $0 \leq \int \int_{\{\psi \geq x\}} -(\psi - x)^p dzdt \leq 0$, from which it follows that $\int \int_{\{\psi \geq x\}} (\psi - x)^p = 0$. Therefore we deduce that

$$\widehat{\lambda}\{(t,z) \in T \times Z : x(t,z) \leq \psi(t,z)\} = 0$$

with $\widehat{\lambda}(\cdot)$ being the Lebesgue product measure on $T \times Z$. In a similar way, working with the upper solution φ, we obtain

$$\widehat{\lambda}\{(t,z) \in T \times Z : x(t,z) \geq \varphi(t,z)\} = 0.$$

Thus we have shown that $\psi(t,z) \leq x(t,z) \leq \varphi(t,z)$ a.e. on $T \times Z$; i.e. $x \in K$. So $S(K) \subseteq K$. In particular if $y = \psi$, then $S(\psi) \subseteq K$ and so for every $x \in S(\psi)$, $\psi \leq x$.

Next let $y_1 \leq y_2$ and $x_1 \in S(y_1)$ with $y_1 \leq x_1$.

Claim #6. Therefore exists $x_2 \in S(y_2)$ such that $x_1 \leq x_2$.

To this end consider the following auxiliary problem:

$$(17) \qquad \left\{ \begin{array}{l} \frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t,z,\tau_1(x), Dx) + \beta(\tau_1(x)) + u_1(t,z,x) \\ \qquad\qquad\qquad\qquad \ni g(y_2(t,z)) \text{ on } T \times Z \\ x(0,z) = x(b,z) \text{ a.e. on } Z, \quad x|_{T \times \Gamma} = 0 \end{array} \right\}$$

Here τ_1 is the truncation map at $\{x_1, \varphi\}$ and u_1 the penalty function corresponding to the same pair $\{x_1, \varphi\}$. Working exactly as for problem (4), we can show that problem (13) above has at least one solution $x_2 \in [x_1, \varphi]$. Evidently $x_2 \in S(y_2)$ and $x_2 \geq x_1$.

Now consider the multifunction $S : K \rightarrow 2^K \setminus \{\emptyset\}$. Note that the values of $S(\cdot)$ are bounded in $W_{pq}(T)$, hence relatively compact in $L^p(T \times Z)$ and since it is easy to see from our previous considerations that $S(y)$ is closed in $L^p(T \times Z)$, is compact there. Thus we can apply proposition 2.2, p. 121 of Heikkila, Hu [16] and deduce the existence of $x \in K$ such that $x \in S(x)$. Evidently $x \in W_{pq}(T)$ is a solution of problem (2). Thus we have proved that problem (2) has at least one solution in the order interval K. In what follows by $\widehat{S}$ we will denote the set of solutions of (2) in K.

Next we will show that $\widehat{S}$ has a greatest and smallest element (extremal soluitons of (2) in K). Indeed by Zorn's lemma $\widehat{S}$ has a maximal element $x_u \in K$ for the pointwise ordering inherited from $L^p(T \times Z)$. We claim that x_u is the greatest element of $\widehat{S}$ in K. This will follow immediately if we can show that $\widehat{S} \subseteq K$ is directed. Indeed let $x_1, x_2 \in \widehat{S}$ and set $x = \max\{x_1, x_2\} \in W_{pq}(T)$,

$x \in K$. Also we see that $x = x_1 + (x_2 - x_1)^+$ and so

$$\frac{dx}{dt} = \frac{d}{dt}(x_1 + (x_2 - x_1)^+)$$
$$= \frac{dx_1}{dt} + \frac{d(x_2 - x_1)^+}{dt} = \left\{ \begin{array}{ll} x_2(t), & \text{if } x_2(t,\cdot) \geq x_1(t,\cdot) \\ x_1(t), & \text{if } x_2(t,\cdot) \leq x_1(t,\cdot) \end{array} \right\}$$

Solving as above the equation

$$x + \widehat{A}_1(t, x(t)) + h(t, \tau_0(x)) - \sum_{k=1}^{2} |h(t, \tau_k(x)) - h(t, \tau_0(x))| + \widehat{\beta}(\tau(x)) \ni \widehat{g}(x),$$

we obtain $\widehat{x}$ solution of the problem such that $\widehat{x} \geq x$. So S is directed and threfore x_u is the greatest element of $\widehat{S}$ in K. Similarly we can produce the smallest element x_1 of $\widehat{S}$ in K. ■

Now will consider the following special case of problem (1)

$$(18) \quad \left\{ \begin{array}{c} \frac{\partial x}{\partial t} = \sum_{k=1}^{N} D_k a_k(t, z, Dx) + a_0(t, z, x) = f(x(t, z)) \text{ on } T \times Z \\ x(0, z) = x(b, z) \text{ a.e. on } Z, \ x|_{T\times\Gamma} = 0 \end{array} \right\}$$

The hypothesis on the data, are now the following:
$H(a)_1 : a_k : T \times Z \times R^N \to R$ $k \in \{1, 2, \ldots, N\}$ are functions such that

(i) $(t, z) \to a_k(t, z, \eta)$ is measurable;
(ii) $\eta \to a_k(t, z, \eta)$ is continuous;
(iii) $|a_k(t, z, \eta)| \leq \beta_1(t, z) + c_1 \|\eta\|^{p-1}$ a.e. on $T \times Z$ for all $\eta \in R^N$ and with $\beta_1 \in L^q(T \times Z)$, $c_1 > 0$, $2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1.$;
(iv) $\sum_{k=1}^{N} (a_k(t, z, \eta) - a_k(t, z, \eta'))(\eta_k - \eta'_k) > 0$ a.e. on $T \times Z$ for all $\eta, \eta' \in R^N$, $\eta \neq \eta'$; and
(v) $\sum_{k=1}^{N} a_k(t, z, \eta)\eta_k \geq c\|\eta\|^p$ a.e. on $T \times Z$, for all $\eta \in R^N$ and with $c > 0$.

Remark. In hypothesis $H(a)_1$, we recognize the well-known Leray, Lions conditions (see Lions [18]).

$H(a_0)_1 : a_0 : T \times Z \times R \to R$ is a function such that

(i) $(t, z) \to a_0(t, z, x)$ is measurable;
(ii) there exists $k \in L^\infty(T \times Z)$ such that for almost all $T \times Z$ and all $x, x' \in [\psi(t, z), \varphi(t, x)]$, $|a_0(t, z, x) - a_0(t, z, x')| \leq k(t, z)|x' - x|$;

(iii) for almost all $(t,z) \in T \times Z$, $x \to a_0(t,z,x)$ is nondecreasing on $[\psi(t,z), \varphi(t,z)]$; and

(iv) for all $x \in L^p(T \times Z)$ such that $\psi(t,z) \leq x(t,z) \leq \varphi(t,z)$ a.e. on $T \times Z$, the function $(t,z) \to a_0(t,z,x(t,z))$ belongs in $L^\infty(T \times Z)$.

$H(f)_1$: $f : R \to R$ is a function such that $f = g - h$ with $g, h : R \to R$ are nondecreasing functions and g is right (resp. left) continuous.

As before, to guarantee existence of solutions, we pass to the following multivalued version of (18):

$$\left\{ \begin{array}{c} \frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t,z,Dx) + a_0(t,z,Dx) + \beta(x(t,z)) \\ \qquad\qquad\qquad\qquad \ni g(x(t,x)) \text{ on } T \times Z \\ x(0,z) = x(b,z) \text{ a.e. on } Z, \quad x|_{T\times\Gamma} = 0 \end{array} \right\} \tag{19}$$

In the next proposition we show that the gratest (resp. smallest) solution of (19) in K can be obtained by a monotone iterative process (see Sattinger [21] forsemilinear systems and classical solutions).

Proposition 6. *If hypothesis $H(a)_1$, $H(a_0)_1$, $H(f)_1$ and H_0 hold, then the greatest (resp. smallest) solution is obtained as the limit of a decreasing (resp. increasing) sequence in K.*

Proof. In this case the map $S : K \to K$ considred in the proof of theorem 5 is single-valued. Moreover from the proof of that theorem, we know $S(\cdot)$ is nondecreasing and $S(K)$ is compact in $L^p(T \times Z)$. Then $y_0 = \varphi$ and $y_n = S(y_{n-1})$ for $n \geq 1$. Evidently $\{y_n\}_{n\geq 1} \subseteq S(K) \subseteq K$ and is nonincreasing. So we have that $y_n \to x_u$ in $L^p(T \times Z)$ and also $y_n \xrightarrow{w} x_u$ in $W_{pq}(T)$ as $n \to \infty$ (recall that $S(K)$ is bounded in $W_{pq}(T)$). We have

$$y_n + \widehat{A}(y_n) + v_n = \widehat{g}(y_{n-1}),\ u_n \in \partial\Phi(x_n)\ n \geq 1.$$

Then by virtue of hypothesis H_0, we may assume that $v_n \xrightarrow{w} v$ in $L^q(T \times Z)$ and as in the proof of theorem 5 we have that $v \in \partial\Phi(x)$. Also

$$\big((y_n, y_n - x_u)\big) + \big((\widehat{A}(x_n), y_n - x_u)\big) + \big((v_n, y_n - x_u)\big) = \big((\widehat{g}(y_{n-1}), y_n - x_u)\big).$$

Note that from the integration by parts formula for functions in $W_{pq}(T)$ (see Zeidler [25], proposition 23.23(iv), p. 423), we have

$$\big((y_n, y_n - x_u)\big) = \big((x_u, y_n - x_u)\big) \to 0 \text{ as } n \to \infty.$$

Also $\big((v_n, y_n - x_u)\big) = \big(v_n, y_n - x_u\big)_{L^p, L^q} \to 0$ as $n \to \infty$. Finally exploiting the right continuity of $g(\cdot)$ and hypothesis H_0, we have that $\widehat{g}(y_{n-1}) \to \widehat{y}(x_u)$

in $L^q(T \times Z)$ and $((\widehat{g}(y_{n-1}), y_{n-1} - x_u)) = (\widehat{g}(y_{n-1}), y_{n-1} - x_u)_{L^p, L^q} \to 0$ as $n \to \infty$.

So finally we have $\overline{\lim}\big((\widehat{A}(y_n), y_n - x_u)\big) = 0$.

Recalling that $\widehat{A}(\cdot)$ is pseudomonotone, we deduce that $\widehat{A}(y_n) \xrightarrow{w} \widehat{A}(x_u)$ in $L^q(T, X^*)$ as $n \to \infty$. So in the limit as $n \to \infty$, we have

$x_u + \widehat{A}(x_u) + v = \widehat{g}(x_u)$, $v \in \partial\Phi(x_u)$, $x_u(0, z) = x(b, z)$ a.e. on Z.

Therefore $x_u \in W_{pq}(T)$ solves (19). In fact we claim that x_u is the greatest solution of (19) in K. Indeed let $\widehat{x} \in K$ be any solution of (19). In particular then $\widehat{x}$ is a lower solution (19) satisfying $\widehat{x} \le \varphi$. Starting again the iteration $y_0 = \varphi$, $y_n = S(y_{n-1})n \ge 1$ we obtain $y_n \to x_u \ge \widehat{x}$ in $L^p(T \times Z)$ as $n \to \infty$, of a nondecreasing sequence in K. ■

Remark. In particular if $g(\cdot)$ is continuous, then both extremal solutions $x_u, x_s \in W_{pq}(T)$ can be obtained through monotone iterative processes.

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