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DIFFERENTIAL POLYNOMIALS AND UNIQUENESS OF MEROMORPHIC FUNCTIONS

Dedicated with deepest respect to Mother Teresa

By

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Abstract. We prove a uniqueness theorem for meromorphic functions involving differential polynomials.

1. Introduction and Definitions

Let f, g be two transcendental meromorphic functions defined in the open complex plane C. If for some $a \in C$ the zeros of $f-a$ and $g-a$ coincide in locations and multiplicities, we say that f and g share the value a CM (counting multiplicities).

Recently Yi and Yang [7] proved the following theorem.

Theorem A. *If f, g are such that* (i) $\Theta(\infty;f)=\Theta(\infty;g)=1$, (ii) $f^{(n)}$, $g^{(n)}$ *share* 1 *CM and* (iii) $\delta(0;f)+\delta(0;g)>1$ *then either* $f\equiv g$ *or* $f^{(n)}\cdot g^{(n)}\equiv 1$, *where n is a nonnegative integer.*

Considering $f=\frac{-1}{2^n}\exp(2z)+\exp(z)$, $g=-\frac{(-1)^n}{2^n}\exp(-2z)+(-1)^n\exp(-z)$ they [7] claimed that the condition (iii) of Theorem A is necessary. Their claim is true for $n=0$ but for $n\geq 1$ it is questionable as we find in the following example.

Example 1. Let $f=1+\exp(z)$ and $g=1+(-1)^n\exp(-z)$ where n is a positive integer. Then $f^{(n)}$, $g^{(n)}$ share 1 CM, $\Theta(\infty;f)=\Theta(\infty;g)=1$ and $\delta(0;f)=\delta(0;g)=0$ but $f^{(n)}\cdot g^{(n)}\equiv 1$.

So it is natural to think that for $n\geq 1$ Theorem A deserves some improvement. In this connection we note that for the functions of the example of Yi and Yang $\sum_{a\neq\infty}\delta(a;f)+\sum_{a\neq\infty}\delta(a;g)=1$ where as for those of Example 1 the

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sum is greater than one.

In the paper we prove a uniqueness theorem involving differential polynomials generated by f and g. As a consequence of this theorem we give an improvement of Theorem A for $n \geq 1$. Throughout the paper we use standard notations and definitions of the value distribution theory [3].

Definition 1. We denote by $\Psi(D)$ the linear differential operator $\Psi(D) = \sum_{i=1}^{n} \alpha_i D^i$ where $\alpha_i \varepsilon C (i = 1, 2, \ldots, n)$, $\alpha_n \neq 0$ and $D \equiv \frac{d}{dz}$.

Definition 2. We denote by E the exceptional set of finite linear measure that appears in the second fundamental theorem and by $S(r; f_1, f_2, \ldots, f_n)$ a function of r such that $S(r; f_1, f_2, \ldots, f_n) = o\{\sum_{i=1}^{n} T(r, f_1)\}$ as $r \to \infty$ ($r \notin E$)where f_i's are meromorphic functions defined on C.

2. Lemmas

In this section we present some lemmas which will be required in the sequel.

Lemma 1.[2] *Let $f_j (j = 1, 2, \ldots, p)$ be linearly independent meromorphic functions such that $\sum_{j=1}^{p} f_j \equiv 1$. Then for $j = 1, 2, \ldots, p$*

$$
\begin{aligned}
T(r, f_j) < & \sum_{i=1}^{p} N(r, 0; f_i) + N(r, f_j) + N(r, \Delta) \\
& - \sum_{i=1}^{p} N(r, f_i) - N(r, 0; \Delta) + S(r; f_1, f_2, \ldots, f_p)
\end{aligned}
$$

where Δ is the wronskian determinant of $f_1, f_2, \ldots, f_p$.

Lemma 2.[5] *If $\beta_i (\not\equiv 0; i = 1, 2)$ are meromorphic funcitons such that $T(r, \beta_i) = S(r; f, g)$ and $\beta_1 f + \beta_2 g \equiv 1$ then*

$$T(r, f) \leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, f) + S(r; f, g)$$

and

$$T(r, g) \leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, g) + S(r; f, g).$$

Lemma 3.[6] *Let f_1, f_2, f_3 be three nonconstant meromorphic functions satisfying $\sum_{i=1}^{3} f_i \equiv 1$ and let $g_1 = -(f_1/f_2)$, $g_2 = 1/f_2$, $g_3 = -(f_3/f_2)$. If f_1, f_2, f_3 are linearly independent then g_1, g_2, g_3 are linearly independent.*

Lemma 4. *If f is of finite order then*

(i) $\sum_{a \neq \infty} \delta(a; f) \leq \liminf_{r \to \infty} \dfrac{T(r, \Psi(D) f)}{T(r, f)}$ *and*

(ii) $\sum_{a\neq\infty}\delta(a;f)\leq\{1+n(1-\Theta(\infty;f))\}\cdot\delta(0;\Psi(D)f)$.

We omit the proof because it can be proved by the inequality 2.1 (p. 33, [3]) and Milloux theorem (p. 55, [3]).

Lemma 5. *If f is of finite order then* $\Theta(\infty;\Psi(D)f)\geq 1-\dfrac{1-\Theta(\infty;f)}{\sum_{a\neq\infty}\delta(a;f)}$, *where* $\sum_{a\neq\infty}\delta(a;f)>0$.

Proof. By Lemma 4 we get

$$\begin{aligned}\Theta(\infty;\Psi(D)f)&\geq 1-\limsup_{r\to\infty}\frac{\overline{N}(r,\infty;f)}{T(r,f)}\cdot\limsup_{r\to\infty}\frac{T(r,f)}{T(r,\Psi(D)f)}\\&\geq 1-\frac{1-\Theta(\infty;f)}{\displaystyle\sum_{a\neq\infty}\delta(a;f)}.\end{aligned}$$

This proves the lemma. ■

Note 1. With modified characteristic functions and deficiencies as introduced in [1], [4] Lemma 4 and Lemma 5 can be proved similarly for functions of unrestricted order.

3. Theorems

In this section we give the main results.

Theorem 1. *Let $\Psi(D)f$, $\Psi(D)g$ be nonconstant and*

(i) *f, g share ∞ CM,*

(ii) *$\Psi(D)f$, $\Psi(D)g$ share 1 CM,*

(iii) $$\left\{1-\frac{1-\Theta(\infty;g)}{\sum_{a\neq\infty}\delta(a;g)}\right\}\cdot\left\{\frac{\sum_{a\neq\infty}\delta(a;f)}{1+n(1-\Theta(\infty;f))}-\frac{2(1-\Theta(\infty;f))}{\sum_{a\neq\infty}\delta(a;f)}\right.$$
$$\left.+\frac{\sum_{a\neq\infty}\delta(a;g)}{1+n(1-\Theta(\infty;g))}\right\}>1,\text{ where }\sum_{a\neq\infty}\delta(a;f)>0,\ \sum_{a\neq\infty}\delta(a;g)>0.$$

Then either (a) $[\Psi(D)f]\cdot[\Psi(D)g]\equiv 1$ *or* (b) $f-g\equiv s$, *where $s\equiv s(z)$ is a solution of $\Psi(D)w=0$. If, further,* (iv) *f has at least one pole, the case* (a) *does not arise.*

Proof. First we suppose that f, g are of finite order. Let us put $F = \Psi(D)f$ and $G = \Psi(D)g$. Then by Lemma 4 and Lemma 5 the condition (iii) implies

$$\Theta(\infty; G)\{\delta(0; F) + 2\Theta(\infty; F) - 2\} + \delta(0; G) > 1. \tag{1}$$

This gives $\delta(0; F)+\Theta(\infty; F) > 1$ and $\delta(0; G)+\Theta(\infty; G) > 1$. In view of condition (ii) we get by the second fundamental theorem

$$\begin{aligned} T(r, F) &\leq N(r, 0; F) + N(r, 1; F) + \overline{N}(r, \infty; F) + S(r, F) \\ &= N(r, 0; F) + N(r, 1; G) + \overline{N}(r, \infty; F) + S(r, F) \\ &\leq N(r, 0; F) + T(r, G) + \overline{N}(r, \infty; F) + S(r, F) \end{aligned}$$

i.e., $\delta(0; F) + \Theta(\infty; F) - 1 \leq \liminf_{r\to\infty} \dfrac{T(r, G)}{T(r, F)}$.

Also by the second fundamental theorem and condition (ii) we get $T(r, G) \leq N(r, 0; G) + T(r, F) + \overline{N}(r, \infty; G) + S(r, G)$ and so

$$\limsup_{r\to\infty} \frac{T(r, G)}{T(r, F)} \leq \frac{1}{\delta(0; G) + \Theta(\infty; G) - 1}.$$

Combining these two we get

$$\begin{aligned} \delta(0; F) + \Theta(\infty; F) - 1 &\leq \liminf_{r\to\infty} \frac{T(r, G)}{T(r, F)} \leq \limsup_{r\to\infty} \frac{T(r, G)}{T(r, F)} \\ &\leq \frac{1}{\delta(0; G) + \Theta(\infty; G) - 1}. \end{aligned} \tag{2}$$

Let $H = \frac{F-1}{G-1}$. Then $H \not\equiv 0$ and by conditions (i) and (ii) $\overline{N}(r, H) + \overline{N}(r, 0; H) = S(r; F, G)$. Put $F_1 = F$, $F_2 = H$ and $F_3 = -GH$. Then

$$F_1 + F_2 + F_3 \equiv 1. \tag{3}$$

If possible suppose that F_1, F_2, F_3 are linearly independent.

If Δ is the wronskian determinant of F_1, F_2, F_3, and if z_0 is a zero F_i with multiplicity $p(> 2)$, it is a zero of Δ of multiplicity $p - 2$. So $\sum_{i=1}^{3} N(r, 0; F_i) - N(r, 0; \Delta) \leq \sum_{i=1}^{3} N_2(r, 0; F_i)$, where in $N_2(r, 0; F_i)$ a zero of F_i with multiplicity p is counted p times if $p \leq 2$ and is counted twice if $p > 2$. Hence

$$\begin{aligned} \sum_{i=1}^{3} N(r, 0; F) - N(r, 0; \Delta) &\leq N_2(r, 0; F) + N_2(r, 0; H) + N_2(r, 0; -GH) \\ &\leq N(r, 0; F) + N(r, 0; G) + 2N_2(r, 0; H) \\ &\leq N(r, 0; F) + N(r, 0; G) + 4\overline{N}(r, 0; H). \end{aligned}$$

So by Lemma 1 we get

$$(4)\quad T(r,F) < N(r,0;F) + N(r,0;G) + N(r,\Delta) - \sum_{i=2}^{3} N(r,F_i) + S(r;F,G).$$

Now $N(r,\Delta)-\sum_{i=2}^{3} N(r,F_i) \le 2\overline{N}(r,F_2)+2\overline{N}(r,F_3) \le 2\overline{N}(r,F)+S(r,F,G)$ because $\Delta = F_2^{(1)}\cdot F_3^{(2)} - F_2^{(2)}\cdot F_3^{(1)}$ and by condition (i). So form (4) we get in view of (2)

$$(5)\qquad T(r,F) < N(r,0;F) + N(r,0;G) + 2\overline{N}(r,F) + S(r,F).$$

Since by Lemma 3 $G_1 = -F/H$, $G_2 = 1/H$, $G_3 = G$ are linearly independent and $G_1 + G_2 + G_3 \equiv 1$, proceeding as above we get

$$(6)\qquad T(r,G) < N(r,0;F) + N(r,0;G) + 2\overline{N}(r,F) + S(r,F).$$

In view of (1) we choose an ε, $0 < \varepsilon < \frac{1}{4}\{\delta(0;F) + \delta(0;G) + 2\Theta(\infty;F) - 3\}$. Then from (5) and (6) we get for sufficiently large values of r

$$\begin{aligned}\max\{T(r,F),T(r,G)\} &< \{4 + 3\varepsilon - \delta(0;F) - \delta(0;G) - 2\Theta(\infty;F) + o(1)\}\\ &\quad\times \max\{T(r,F),T(r,G)\}\\ &\le \{1 - \varepsilon + o(1)\}\max\{T(r,F),T(r,G)\}\end{aligned}$$

which is a contradiction. Hence F_1, F_2, F_3 are linearly dependent and so there exist constants c_1, c_2, c_3, not all zero, such that

$$(7)\qquad c_1F_1 + c_2F_2 + c_3F_3 \equiv 0.$$

If $c_1 = 0$, we get $(c_2 - c_3G)H \equiv 0$ and so G is a constant, which is not the case. So $c_1 \ne 0$ and hence from (3) and (7) we get

$$(8)\qquad cF_2 + dF_3 \equiv 1, \text{ where } c = 1 - \frac{c_2}{c_1},\ d = 1 - \frac{c_3}{c_1}.$$

Now we consider the following cases.

Case I. Let $c\cdot d \ne 0$. Then from (8) we get $1/(cH) + (d/c)(G) \equiv 1$. So by Lemma 2 we get

$$\begin{aligned}T(r,G) &\le \overline{N}(r,H) + \overline{N}(r,0;G) + \overline{N}(r,G) + S(r;G,H)\\ &= \overline{N}(r,0;G) + \overline{N}(r,G) + S(r;F,G)\end{aligned}$$

and so by (2), $T(r,G) \le \overline{N}(r,0;G) + \overline{N}(r,G) + S(r,G)$. This gives $\delta(0;G) + \Theta(\infty;G) \le 1$, which is a contradiction. Hence the case $c\cdot d \ne 0$ does not arise.

Case II. Let $c \cdot d = 0$.

Sub case (i). Let $c = 0$. Then from (8) we get $dGH \equiv -1$ and so $(1-d)G + dGF \equiv 1$. If $d \neq 1$, we obtain $\frac{d}{1-d}F - \frac{1}{(1-d)G} \equiv 1$. So by Lemma 2 we get in view of (2) and condition (i) that

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,G) + \overline{N}(r,F) + S(r;F,G) \\ &\leq \overline{N}(r,0;F) + 2\overline{N}(r,F) + S(r,F) \end{aligned}$$

and so $\delta(0;F) \leq 2(1-\Theta(\infty;F))$. This gives by (1) that $\delta(0;G) > 1$, which is a contradiction. Therefore, $d = 1$ and so $GF \equiv 1$ i.e., $[\Psi(D)f] \cdot [\Psi(D)G] \equiv 1$

Sub case (ii). Let $d = 0$. Then from (8) we get $cF - G \equiv c - 1$. If $c \neq 1$, we obtain $\frac{c}{c-1}F - \frac{1}{c-1}G \equiv 1$. So by Lemma 2 and condition (i) we get in view of (2)

$$T(r,F) \leq \overline{N}(r,F) + N(r,0;G) + N(r,0;F) + S(r,F)$$

and

$$T(r,G) \leq \overline{N}(r,F) + N(r,0;G) + N(r,0;F) + S(r,F).$$

In view of (1) we choose an ε, $0 < \varepsilon < \frac{1}{2}\{\Theta(\infty;F) + \delta(0;F) + \delta(0;G) - 2\}$.

Then from above we get for sufficiently large values of r

$$\begin{aligned} \max\{T(r,F), T(r,G)\} &< \{3 + \varepsilon - \Theta(\infty;F) - \delta(0,F) - \delta(0;G) \\ &\quad + o(1)\} \cdot \max\{T(r,F), T(r,G)\} \\ &< \{1 - \varepsilon + o(1)\} \cdot \max\{T(r,F), T(r,G)\}, \end{aligned}$$

a contradiction.

Therefore, $c = 1$ and so $F \equiv G$ i.e., $\Psi(D)(f-g) \equiv 0$ so that $f - g \equiv s$, where $s \equiv s(z)$ is a solution of $\Psi(D)w = 0$.

Further if f has at least one pole, say z_0, then it is a pole of g also. Hence z_0 is a common pole of $\Psi(D)f$ and $\Psi(D)g$ which is impossible if $[\Psi(D)f] \cdot [\Psi(D)g] \equiv 1$. So the case (a) does not arise if f has at least one pole.

Let at least one of f and g be of infinite order. Since $\Theta(\infty;f) \leq \Theta_0(\infty;f)$, $\Theta(\infty;g) \leq \Theta_0(\infty;g)$, $\delta(a;f) \leq \delta_0(a;f)$ and $\delta(a;g) \leq \delta_0(a;g)$ (cf. Proposition 5, [4]), where $\Theta_0(\infty;f)$, $\delta_0(a;f)$ etc. denote the modified deficiencies (cf. [1], [4]), condition (iii) of the theorem implies that

$$\begin{aligned} \left\{1 - \frac{1-\Theta_0(\infty;g)}{\sum_{a\neq\infty}\delta_0(a;g)}\right\} \cdot \Bigg\{ &\frac{\sum_{a\neq\infty}\delta_0(a;f)}{1+n(1-\Theta_0(\infty;f))} \\ &- \frac{2(1-\Theta_0(\infty;f))}{\sum_{a\neq\infty}\delta_0(a;f)} + \frac{\sum_{a\neq\infty}\delta_0(a;g)}{1+n(1-\Theta_0(\infty;g))}\Bigg\} > 1, \end{aligned}$$

and so in view of Note 1 we get

$$\Theta_0(\infty; G)\{\delta_0(0; F) + 2\Theta_0(\infty; F) - 2\} + \delta_0(0; G) > 1$$

which can play the role of inequality (1). Now the theorem can be proved in a like manner by integrating the inequalities appearing in the above proof. This proves the theorem. ■

Remark 1. Condition (iii) of Theorem 1 is necessary. For if $f = \exp(z) + \frac{-1}{2^n}\exp(2z)$, $g = (-1)^n \exp(-z) - \frac{(-1)^n}{2^n}\exp(-2z)$ and $\Psi(D) = D^n$ $(n \geq 1)$, we see that $\sum_{a\neq\infty} \delta(a; f) = \frac{1}{2}$, $\sum_{a\neq\infty} \delta(a; g) = \frac{1}{2}$, f, g share ∞ CM and $\Psi(D)f$, $\Psi(D)g$ share 1 CM. Also $\Theta(\infty; f) = \Theta(\infty; g) = 1$ and neither $f - g$ is a polynomial nor $[\Psi(D)f] \cdot [\Psi(D)g] \equiv 1$.

Remark 2. Let $f = \exp(z)$, $g = (-1)^n \exp(-z)$ and $\Psi(D) = D^n$ $(n \geq 1)$. Then f has no pole. Also we see that f, g share ∞ CM, $\Psi(D)f$, $\Psi(D)g$ share 1 CM and $\sum_{a\neq\infty} \delta(a; f) + \sum_{a\neq\infty} \delta(a; g) = 2 > 1$. So the conditions (i), (ii), (iii) of Theorem 1 are satisfies and we note that $[\Psi(D)f][\Psi(D)g] \equiv 1$ which is case (a) of the theorem. Therefore the condition (iv) of Theorem 1 is necessary for nonoccurrence of case (a).

Now as a consequence of Theorem 1 we prove the next theorem which improves Theorem A for $n \geq 1$.

Theorem 2. *If f, g are such that*

(i) $\Theta(\infty; f) = \Theta(\infty; g) = 1$.

(ii) $f^{(n)}$, $g^{(n)}$ *(n is a positive integer) share* 1 *CM,*

(iii) $\sum_{a\neq\infty} \delta(a; f) + \sum_{a\neq\infty} \delta(a; g) > 1$ *and*

(iv) $\Theta(0; f) + \Theta(0; g) > 1$ *then either* (a) $f \equiv g$ *or* (b) $f^{(n)} \cdot g^{(n)} \equiv 1$.

Proof. Let $\Psi(D) = D^n$ $(n \geq 1)$ and as in Theorem 1 we suppose that $F = \Psi(D)f$, $G = \Psi(D)g$, $H = \frac{F-1}{G-1}$. By conditions (i) and (ii) of this theorem it follows that $\overline{N}(r, H) + \overline{N}(r, 0; H) = S(r; F, G)$ and so in this case the condition (i) of Theorem 1 is not needed. Also since $\Theta(\infty; f) = \Theta(\infty; g) = 1$, the condition (iii) of Theorem 1 reduces to the conditon (iii) of this theorem. Now by Theorem 1 we see that either $f - g \equiv Q$ or $f^{(n)} \cdot g^{(n)} \equiv 1$, where Q is a polynomial of degree at most $n - 1$. Since f, g are transcendental, if $Q \not\equiv 0$ then by Theorem 2.5 [3] we get $\Theta(0; f) + \Theta(0; g) \leq 1$. This contradiction proves the theorem. ■

Remark 3. The condition (iv) of Theorem 2 is necessary for the validity of case (a). For if $f = 1+\exp(z)$ and $g = (-1)^n \exp(-z)$ then $\Theta(0; f)+\Theta(0; g) = 1$, $\sum_{a\neq\infty} \delta(a; f)+\sum_{a\neq\infty} \delta(a; g) > 1$, $\Theta(\infty; f) = \Theta(\infty; g) = 1$ and $f^{(n)}$, $g^{(n)}$ $(n \geq 1)$ share 1 CM, but in this case $f^{(n)} \cdot g^{(n)} \equiv 1$. Also if $f = 1 + \exp(z)$ and $g = \exp(z)$ then $\Theta(0; f) + \Theta(0; g) = 1$, $\sum_{a\neq\infty} \delta(a; f) + \sum_{a\neq\infty} \delta(a; g) > 1$, $\Theta(\infty; f) = \Theta(\infty; g) = 1$ and $f^{(n)}$, $g^{(n)}$ $(n \geq 1)$ share 1 CM but in this case $f - g \equiv 1$.

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