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# $L^2$ TRANSVERSAL CONFORMAL AND PROJECTIVE FIELDS OF RIEMANNIAN FOLIATIONS ON COMPLETE RIEMANNIAN MANIFOLDS

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**Abstract.** In [12], the notion of $\lambda$-automorphisms of harmonic Riemannian foliations on closed Riemannian manifolds was extended to general Riemannian foliations. Besides, certain characterizations of $\lambda$-automorphisms to be transversal Killing were obtained. These results were generalized to the complete case ([13]). As applications, we study the problem when $L^2$ transversal conformal or projective fields are to be transversal Killing. Our main results extend those in [10], [11], [15], [16].

## 1. Introduction

The study on geometric transversal infinitesimal automorphisms such as transversal Killing, affine, conformal, projective fields of Riemannian foliations has been attacked by many differential geometers. One way of doing this study was to extend well-known results concerning those infinitesimal automorphisms on Riemanninan manifolds to foliated versions.

There have been obtained lots of results in the case that the foliation is harmonic (all the leaves are minimal submanifolds). However, in this situation we do not see what the given foliated structures have influences on the properties of geometric transversal infinitesimal automorphisms. Indeed, the mean curvature plays an important role on the study of the transversal geometry for Riemannian foliations. Recently, several authors studied them in a more general situation that the foliation is not harmonic.

In the space of transversal infinitesimal automorphisms, there is a crucial subspace consisting of $\lambda$-automorphisms. In [12], this notion was treated with in a general Riemannian foliation. Furthermore, certain characterizations of $\lambda$-

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automorphisms to be transversal Killing were obtained in the case where the ambient space is close ([12]) and then complete ([13]). In this paper, we are mainly interested in the problem when $L^2$ transversal conformal or projective fields are to be transversal Killing. For the harmonic foliation, our results are found in [10], [11], [15], [16].

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## 2. $L^2$ $\lambda$-automorphisms

Let $(M, g, \mathcal{F})$ be an $m$-dimensional oriented, connected, complete Riemannian manifold with transversally oriented Riemannian foliation $\mathcal{F}$ of codimension $q := m - p$ and a bundle-like metric $g$. It is given by an exact sequence of vector bundles

$$0 \to \mathcal{V} \to TM \xrightarrow{\pi} Q \to 0, \tag{2.1}$$

where $\mathcal{V}$ is the tangent bundle and $Q$ the normal bundle of $\mathcal{F}$. The metric $g$ determines an orthogonal decomposition $TM = \mathcal{V} \oplus \mathcal{H}$. We may identify $\mathcal{H}$ with $Q$ by an isometric splitting

$$\sigma : (Q, g_Q := \sigma^* g_{\mathcal{H}}) \to (\mathcal{H}, g_{\mathcal{H}}). \tag{2.2}$$

We have an associated exact sequence of Lie algebras

$$0 \to \Gamma(\mathcal{V}) \to V(\mathcal{F}) \xrightarrow{\pi} \overline{V}(\mathcal{F}) \to 0, \tag{2.3}$$

where $V(\mathcal{F}) := \{Y \in \Gamma(TM) | [V, Y] \in \Gamma(\mathcal{V}) \text{ for all } V \in \Gamma(\mathcal{V})\}$ and $\overline{V}(\mathcal{F}) := \{s \in \Gamma(Q) | s = \pi(Y),\ Y \in V(\mathcal{F})\}$, called the space of transversal infinitesimal automorphisms of $\mathcal{F}$. Here and hereafter, we denote by $\Gamma(\cdot)$ the space of all smooth sections of a vector bundle $(\cdot)$. The transversal Levi-Civita connection $D$ on $Q$ is a unique torsion free and metric connection with respect to $g_Q$ ([17]).

Throughout this paper, we use the following notations:

$\tau$: the tension field on $\mathcal{F}$,
$\mathrm{div}_D s$: the transversal divergence of $s \in \Gamma(Q)$,
$\mathrm{grad}_D f$: the transversal gradient of a function $f \in C^\infty(M)$,
$R_D$: the transversal curvature tensor of $D$,
$\rho_D$: the transversal Ricci operator,
$c_D$: the transversal scalar curvature,
$\Delta := d_D^* d_D$: the Laplacian acting on $\Gamma(Q)$,
$\theta_Y$: the transversal Lie derivative operator for $Y \in V(\mathcal{F})$,

$A_D(Y) := \theta_Y - D_Y$ for $Y \in V(\mathcal{F})$.

The basic complex $(\Omega_B, d_B := d|_{\Omega_B})$ is a subcomplex of the de Rham complex $(\Omega(M), d)$, where

$$\Omega_B := \{\omega \in \Omega(M) | i_V\omega = \theta_V\omega = 0 \quad \text{for all} \quad V \in \Gamma(\mathcal{V})\}.$$

There is a codifferential operator $\delta_T : \Omega_B^r \to \Omega_B^{r-1}$ defined by

$$\delta_T := (-1)^{r(q+1)+1}\bar{*}d_B\bar{*}, \tag{2.4}$$

where $\bar{*}$ is the star operator associated to the holonomy-invariant metric $g_Q$ on $Q$. It should be noted that in general $d_B$ and $\delta_T$ are not formal adjoint on $\Omega_B$ unless $\mathcal{F}$ is harmonic.

Recall the following operators $\delta$, $\delta^*$ appeared in [6]. Throughout this paper, we denote by $\{E_A\} = \{E_i, E_a\}$, $E_i \in \Gamma(\mathcal{V})$, $E_a \in V(\mathcal{F})$ the special local orthonormal frame field about $x \in M$ with $(E_A)_x = e_A$ introduced in [6]. $\delta : \Gamma(S^2Q^*) \to \Gamma(Q^*)$, $S^2Q^*$ being the symmetric tensor product of $Q^*$ of order 2, is given by the local formula

$$\delta\beta := -\sum_a (D_{E_a}\beta)(E_a, \cdot), \quad \text{for} \quad \beta \in \Gamma(S^2Q^*)$$

and $\delta^* : \Gamma(Q^*) \to \Gamma(S^2Q^*)$ by

$$(\delta^*\omega)(s,t) := \frac{1}{2}\{(D_{\sigma(s)}\omega)(t) + (D_{\sigma(t)}\omega)(s)\}, \quad \text{for} \quad \omega \in \Gamma(Q^*),\ s,t \in \Gamma(Q).$$

The space of basic 1-forms (resp. basic symmetric 2-forms) may be identified with a subspace of $\Gamma(Q^*)$ (resp. $\Gamma(S^2Q^*)$). We denote by $(\ ,\ )_x$ the local scalar product on $\Gamma(Q)$ or $\Gamma(Q^*)$ at a point $x \in M$ and $|\cdot|_x^2 := (\cdot,\cdot)_x$. The local scalar product may be extended on $\Gamma(\otimes^{r_1}Q \otimes^{r_2} Q^*)$. Let $\Gamma_c(Q)$ (resp. $\Gamma_c(Q^*)$) be the space of all sections of $Q$ (resp. $Q^*$) with compact supports. Let $\langle\ ,\ \rangle$ be the global scalar product on $\Gamma_c(Q)$ or $\Gamma_c(Q^*)$ and $\|\cdot\|^2 := \langle\cdot,\cdot\rangle$. The global scalar product may be also extended on $\Gamma_c(\otimes^{r_1}Q \otimes^{r_2} Q^*)$. Let $L^2(Q)$ (resp. $L^2(Q^*)$) be the completion of $\Gamma_c(Q)$ (resp. $\Gamma_c(Q^*)$) with respect to $\langle\ ,\ \rangle$.

**Definition.** We say that an element $s \in L^2(Q) \cap \overline{V}(\mathcal{F})$ is a $L^2$ transversal infinitesimal automorphism of $\mathcal{F}$.

The following fundamental identities were derived in [12].

**Proposition 2.1.** *For $s \in \overline{V}(\mathcal{F})$ and $\omega$ the $g_Q$-dual of $s$ we have*

$$2\delta\delta^*\omega = -trD^2\omega - \rho_D(\omega) + d_B\delta_T\omega,$$

$$(\mathrm{div}_D s)_x = -(\delta_T\omega)_x = (\delta^*\omega, g_Q)_x,$$

$$|\delta^*\omega + \frac{1}{q}(\delta_T\omega)g_Q|_x^2 = |\delta^*\omega|_x^2 - \frac{1}{q}(\delta_T\omega)_x^2.$$

**Definition.** Given $Y \in V(\mathcal{F})$, $s = \pi(Y)$ is called a $\lambda$-automorphism for $\lambda \in \mathbf{R}$ if it satisfies

$$\Delta s - D_{\sigma(\tau)}s - \rho_D(s) - \lambda \mathrm{grad}_D \mathrm{div}_D s = 0, \tag{2.5}$$

or equivalently the $g_Q$-dual $\omega$ satisfies

$$-\, trD^2\omega - \rho_D(\omega) + \lambda d_B \delta_T \omega = 0. \tag{2.6}$$

**Proposition 2.2.** **([14])** *Let $s = \pi(Y) \in \overline{V}(\mathcal{F})$.*

(*i*) *If $s$ is a transversal Killing field, i.e., $\theta_Y g_Q = 0$ then*

$$\mathrm{div}_D s = 0, \quad \Delta s = D_{\sigma(\tau)}s + \rho_D(s).$$

(*ii*) *If $s$ is a transversal conformal field, i.e., $\theta_Y g_Q = 2f_Y g_Q$ where $f_Y$ is a function on $M$, then*

$$\mathrm{div}_D s = qf_Y, \quad \Delta s = D_{\sigma(\tau)}s + \rho_D(s) + \left(1 - \frac{2}{q}\right) \mathrm{grad}_D \mathrm{div}_D s.$$

(*iii*) *If $s$ is a transversal projective field, i.e., $(\theta_Y D)_Z t = \phi_Y(Z)t + \phi_Y(\sigma(t))\pi(Z)$ for any $Z \in \Gamma(TM)$ and $t \in \Gamma(Q)$ where $\phi_Y$ is a 1-form on $M$, then*

$$d(\mathrm{div}_D s) = (q+1)\phi_Y, \quad \Delta s = D_{\sigma(\tau)}s + \rho_D(s) - \frac{2}{q+1} \mathrm{grad}_D \mathrm{div}_D s.$$

We remark that every transversal Killing field is a $\lambda$-automorphism for all $\lambda$, every transversal conformal field a $(1 - \frac{2}{q})$-automorphism and every transversal projective field a $(-\frac{2}{q+1})$-automorphism.

## 3. $L^2$ transversal conformal and projective fields

Let $(M, g, \mathcal{F})$ be an $m$-dimensional oriented, connected, complete Riemannian manifold with transversally oriented Riemannian foliation $\mathcal{F}$ of codimension $q := m - p \geq 2$ and a bundle-like metric $g$. From now on we suppose that $\mathcal{F}$ admits at least one compact leaf $\mathcal{L}_0$ passing through a point $x_0 \in M$. Since a geodesic orthogonal to a leaf is orthogonal to leaves, we can define the distance function $\mathrm{dist}(x)$ between $\mathcal{L}_0$ and the leaf through any point $x$ in $M$. Let

$$w_k(x) := w(\mathrm{dist}(x)/k), \quad k = 1, 2, 3, \cdots,$$

where $w$ is a smooth function defined on $\mathbf{R}$ such that $0 \leq w(t) \leq 1$, $w(t) = 1$ for $t \leq 1$, $w(t) = 0$ for $t \geq 2$. It is well-known that each $w_k$ is basic Lipschitz continuous on $M$ and satisfies

$$
\begin{aligned}
&0 \leq w_k(x) \leq 1, \\
&\operatorname{supp} w_k \subset B(2k) \text{ (geodesic ball of radius } 2k \text{ centered at } x_0), \\
&w_k(x) = 1 \quad \text{on } B(k), \\
&\lim_{k\to\infty} w_k = 1, \\
&|dw_k| \leq C/k \quad \text{almost everywhere on } M,
\end{aligned}
\tag{3.1}
$$

where $C$ is a positive constant independent of $k$ ([7]).

We recall a useful symmetric operator $B^\lambda_D(s) : \Gamma(Q) \to \Gamma(Q)$ appeared in [12]

$$
B^\lambda_D(s) := A_D(s) + {}^tA_D(s) + \lambda(\operatorname{div}_D s)\mathrm{Id}, \tag{3.2}
$$

for a given $s = \pi(Y) \in \overline{V}(\mathcal{F})$, where Id denotes the identity map of $\Gamma(Q)$. An important feature of $B^\lambda_D(s)$ is the fact that $s$ is transversal Killing (resp. transversal conformal) if and only if $B^0_D(s) = 0$ (resp. $B^{2/q}_D(s) = 0$).

Our main results are based on the following result obtained in [13].

**Theorem 3.1.** *Let $(M, g, \mathcal{F})$ be an $m$-dimensional oriented, connected, complete Riemannian manifold with transversally oriented Riemannian foliation $\mathcal{F}$ of codimension $q := m - p \geq 2$ and a bundle-like metric $g$. If $s$ is an $L^2\lambda$-automorphism satisfying*

$$
d(\operatorname{div}_D s) = 0 \quad \text{and} \quad \langle B^{1-\lambda}_D(s)s, \tau\rangle \geq 0,
$$

*then $s$ is an $L^2$ transversal Killing field.*

We first observe from Proposition 2.1 that if $\omega$ is the $g_Q$-dual of a $\lambda$-automorphism $s$ then

$$
2\delta\delta^*\omega + (\lambda - 1)d_B\delta_T\omega = 0.
$$

This yield the following formula ([13])

$$
\begin{aligned}
(2l_1 + (\lambda-1)l_2)\frac{C^2}{k^2}\|\omega\|^2_{B(2k)} &\geq (2 - \frac{2}{l_1})\|w_k\delta^*\omega\|^2_{B(2k)} \\
&+ (\lambda-1)(1 - \frac{1}{l_2})\|w_k\delta_T\omega\|^2_{B(2k)} + \langle w_k^2 B^{1-\lambda}_D(s)s, \tau\rangle_{B(2k)},
\end{aligned}
\tag{3.3}
$$

for $l_1, l_2 > 0$. The proof of Theorem 3.1 is relied on (3.3).

**Remark 3.2.** In particular, it is obvious that if $s$ is an $L^2\lambda$-automorphisms satisfying

$$\mathrm{div}_D s = 0 \quad \text{and} \quad \langle B_D^0(s)s, \tau\rangle \geq 0,$$

then $s$ is an $L^2$ transversal Killing field. When $\mathcal{F}$ is harmonic, this is found in [1], [9].

We prepare a lemma, which is an immediate consequence of (3.3).

**Lemma 3.3.** *If $s$ is an $L^2\lambda$-automorphism with $\lambda \neq 1$ satisfying $\langle B_D^{1-\lambda}(s)s, \tau\rangle \geq 0$, then $\mathrm{div}_D s$ is an $L^2$ function.*

We also need the following divergence formula at a point $x \in M$

$$\begin{aligned}(\mathrm{div}_{\nabla^M} Z)_x &:= \sum_A g\left(\nabla^M_{e_A} Z, e_A\right) \\ &= (\mathrm{div}_D \pi(Z))_x - (\pi(Z), \tau)_x.\end{aligned} \tag{3.4}$$

Now we are in a position to consider our main problem when $L^2$ transversal conformal or projective fields are to be transversal Killing. For our purpose we impose on the mean curvature $\kappa$ (:= the $g_Q$-dual to $\tau$) the following conditions:

(3.5)
$\mathcal{F}$ is isoparametric, i.e., $\kappa$ is a basic 1-form,
$\delta_B \kappa = 0$, where $\delta_B$ is the formal adjoint of $d_B$ which is given by
$\delta_B = \delta_T + \iota_\kappa$,
$\|\kappa\|_\infty := \text{ess sup}_{x\in M}|\kappa|_x < \infty$.

We first consider the problem which is related to the classical result of Lichnerowicz ([8]) for the point foliations on closed Riemannian manifolds. Assume that the transversal scalar curvature $c_D$ for $\mathcal{F}$ is a nonpositive constant. Let $s$ be an $L^2\lambda$-automorphism satisfying $-1 < \lambda < 1$, $\langle B_D^{1-\lambda}(s)s, \tau\rangle \geq 0$ and $\omega$ its $g_Q$-dual. By observing the Weitzenböck formula

$$d_B\delta_T\omega + \delta_T d_B\omega = -trD^2\omega + \rho_D(\omega), \tag{3.6}$$

it is easily seen that (2.6) becomes

$$(1+\lambda)d_B\delta_T\omega + \delta_T d_B\omega = 2\rho_D(\omega). \tag{3.7}$$

In our situation we notice that $\kappa$ being basic implies $d\kappa = 0$ ([5], [7]). It follows that the formal adjoint $d_T := d_B - \kappa\wedge$ of $\delta_T$ satisfies $d_T^2 = 0$. Thus (3.7) gives rise to

$$(1+\lambda)\delta_T d_B\delta_T\omega = 2\delta_T\rho_D(\omega) \tag{3.8}$$

In particular, if $s$ is an $L^2$ transversal conformal field, we have

$$\delta_T \rho_D(\omega) = \frac{c_D}{q} \delta_T \omega. \tag{3.9}$$

Indeed, a local computation yields

$$\begin{aligned} \delta_T \rho_D(\omega) &= -\sum_a (D_{e_a} \rho_D(\omega))(\pi(E_a)) \\ &= -\sum_a (D_{e_a} S_D)(\pi(E_a), s) - \sum_a S_D(\pi(E_a), D_{e_a} s). \end{aligned}$$

Here and hereaftr, $S_D$ is the transversal Ricci curvature defined as $S_D(t,u) := g_Q(\rho_D(t), u)$ for $t, u \in \Gamma(Q)$. The first term $\sum_a (D_{e_a} S_D)(\pi(E_a), s)$ vanishes because of the identity $\delta S_D = \frac{1}{2} d_B c_D$ and $c_D =$ constant. It follows by Propositions 2.1 and 2.2 that

$$\begin{aligned} \delta_T \rho_D(\omega) &= -\sum_a S_D(\pi(E_a), D_{e_a} s) \\ &= -\sum_{a,b} g_Q(D_{e_a} s, \pi(E_b)) S_D(\pi(E_a), \pi(E_b)) \\ &= -\frac{1}{2} \sum_{a,b} 2 g_Q(D_{e_a} s, \pi(E_b)) S_D(\pi(E_a), \pi(E_b)) \\ &= -\frac{1}{2} \sum_{a,b} (\theta_s g_Q)(\pi(E_a), \pi(E_b)) S_D(\pi(E_a), \pi(E_b)) \\ &= \frac{c_D}{q} \delta_T \omega. \end{aligned}$$

It should be noted that the notion $\theta_s$ for $s \in \overline{V}(\mathcal{F})$ makes sense. Therefore, every $\omega$ dual to transversal conformal field $s$ satisfies

$$\delta_T d_B \delta_T \omega = \frac{c_D}{q-1} \delta_T \omega. \tag{3.10}$$

Now we deduce by a direct computation that

$$\begin{aligned} &(\delta_T d_B \delta_T \omega, w_k^2 \delta_T \omega)_x \\ &\quad = -\sum_a (D_{e_a} d_B \delta_T \omega)(e_a)(w_k^2 \delta_T \omega) \\ &\quad = -(\mathrm{div}_D u)_x + 2(w_k d_B \delta_T \omega, \delta_T \omega d w_k)_x + |w_k d_B \delta_T \omega|_x^2, \end{aligned}$$

where $u$ is the $g_Q$-dual to the basic 1-form $w_k^2 \delta_T \omega d_B \delta_T \omega$. The divergence formula (3.4) yields

$$\begin{aligned} \langle \delta_T d_B \delta_T \omega, w_k^2 \delta_T \omega \rangle_{B(2k)} &= -\langle u, \tau \rangle_{B(2k)} \\ &\quad + 2\langle w_k d_B \delta_T \omega, \delta_T \omega d w_k \rangle_{B(2k)} + \| w_k d_B \delta_T \omega \|_{B(2k)}^2. \end{aligned} \tag{3.11}$$

Furthermore, by using the assumption (3.5) imposed on the mean curvature, we evaluate

$$\begin{aligned} -\langle u, \tau\rangle_{B(2k)} &= \langle (w_k\delta_T\omega)\kappa, \delta_T\omega dw_k\rangle_{B(2k)} \\ &\geq -\frac{C}{k}\|\kappa\|_\infty \|\delta_T\omega\|^2_{B(2k)}. \end{aligned} \tag{3.12}$$

Hence from (3.11), (3.12) and the inequality

$$2\langle w_k d_B\delta_T\omega, \delta_T\omega dw_k\rangle_{B(2k)} \geq -\frac{1}{2}\|w_k d_B\delta_T\omega\|^2_{B(2k)} - \frac{2C^2}{k^2}\|\delta_T\omega\|^2_{B(2k)},$$

we see that (3.10) implies

$$\begin{aligned} \frac{c_D}{q-1}\|w_k\delta_T\omega\|^2_{B(2k)} &= \langle \delta_T d_B\delta_T\omega, w_k^2\delta_T\omega\rangle_{B(2k)} \\ &\geq \frac{1}{2}\|w_k d_B\delta_T\omega\|^2_{B(2k)} - \left\{\frac{C}{k}\|\kappa\|_\infty + \frac{2C^2}{k^2}\right\}\|\delta_T\omega\|^2_{B(2k)}. \end{aligned}$$

By letting $k \to \infty$ and using Lemma 3.3, we conclude that $d_B\delta_T\omega = 0$. Moreover, every transversal conformal field $s$ satisfies $B_D^{1-\lambda}(s) = 0$. Therefore Theorem 3.1 leads to that $s$ is an $L^2$ transversal Killing field. Summing up,

**Theorem 3.4.** *Let $(M, g, \mathcal{F})$ be an $m$-dimensional oriented, connected, complete Riemannian manifold with transversally oriented Riemannian foliation $\mathcal{F}$ of codimension $q := m - p \geq 2$ and a bundle-like metric $g$. Let the transversal scalar curvature $c_D$ be a non-positive constant. Assume that $\mathcal{F}$ admits at least one compact leaf and the mean curvature satisfies the condition* (3.5). *If $s$ is an $L^2$ transversal conformal field, then $s$ is an $L^2$ transversal Killing field.*

**Remark 3.5.** If we impose a more stronger condition on the transversal structure for $\mathcal{F}$ than $c_D$ being non-positive constant, that is, if $\mathcal{F}$ is transversal Einstein with

$$S_D = \frac{c_D}{q}g_Q \quad \text{and} \quad c_D \leq 0,$$

we have a more stronger conclusion that every $L^2\lambda$-automorphisms $s$ with $-1 < \lambda < 1$ satisfying $\langle B_D^\lambda(s)s, \tau\rangle \geq 0$ is a transversal Killing field. Indeed, it suffices to notice that in such situation every $\lambda$-automorphism satisfies the formula (3.10).

**Corollary 3.6.** **([10], [11])** *Let $(M, g, \mathcal{F})$ and $c_D$ be as in Theorem* 3.4. *If $\mathcal{F}$ is harmonic, then every $L^2$ transversal conformal field is a transversal Killing field.*

Next, we consider $\lambda$-automorphisms preserving the transversal Ricci curvature $S_D$. For the point foliations on closed Riemannian manifolds this is related to the well-known result of Ishihara ([4]).

Let $s$ be an $L^2$ transversal conformal field such that $\theta_s g_Q = 2f_s g_Q$. Then we have the following identity (cf. [16])

$$\sum_a (\theta_s S_D)(\pi(E_a), \pi(E_a)) = 2(q-1)\delta_T d_B f_s. \tag{3.13}$$

The preserving the transversal Ricci curvature implies that $\delta_T d_B f_s = 0$. A similar way as in Theorem 3.4 gives rise to

$$\begin{aligned} 0 &= 2(q-1)\langle \delta_T d_B f_s, w_k^2 f_s\rangle_{B(2k)} \\ &\geq (q-1)\left\{ \|w_k d_B f_s\|^2_{B(2k)} - 2\|f_s\|^2_{B(2k)} \left( \frac{C}{k}\|\kappa\|_\infty + \frac{2C^2}{k^2}\right)\right\}, \end{aligned}$$

so that letting $k \to \infty$ implies $d_B f_s = 0$. This means that $d(\mathrm{div}_D s) = 0$ by Proposition 2.2. From Theorem 3.1 we conclude that $s$ is an $L^2$ transversal Killing field.

If $s$ is an $L^2$transversal projective field such that $(\theta_s D)_Z t = \phi_s(Z)t + (\sigma(t))\pi(Z)$, it holds the corresponding identity (cf. [16])

$$\sum_a (\theta_s S_D)(\pi(E_a), \pi(E_a)) = (q-1)\delta_T \phi_s - \phi_s(\tau). \tag{3.14}$$

It follows from Proposition 2.2 that the preserving the transversal Ricci curvature implies that

$$\begin{aligned} 0 &= \langle \delta_T d_B \delta_T \omega - d_B \delta_T \omega(\tau), w_k^2 \delta_T \omega\rangle_{B(2k)} \\ &\geq \frac{1}{2}\|w_k d_B \delta_T \omega\|^2_{B(2k)} - \left(\frac{2C}{k}\|\kappa\|_\infty + \frac{2C^2}{k^2}\right)\|\delta_T \omega\|^2_{B(2k)}. \end{aligned} \tag{3.15}$$

We have proved that

**Theorem 3.7.** *Let $(M, g, \mathcal{F})$ be an $m$-dimensional oriented, connected, complete Riemannian manifold with transversally oriented Riemannian foliation $\mathcal{F}$ of codimension $q := m - p \geq 2$ and a bundle-like metric $g$. Assume that $\mathcal{F}$ admits at least one compact leaf and the mean curvature satisfies the condition* (3.5). *Either if $s$ is an $L^2$ transversal conformal field or if $s$ is an $L^2$ transversal projective field satisfying $\langle B_D^{(q+3)/(q+1)}(s)s, \tau\rangle \geq 0$ and satisfies $\theta_s S_D = 0$, then $s$ is an $L^2$ transversal Killing field.*

**Corollary 3.8. ([15], [16])** *Let $(M, g, \mathcal{F})$ be as in Theorem* 3.7. *If $\mathcal{F}$ is harmonic, then every $L^2$ transversal conformal or projective field $s$ satisfying $\theta_s S_D = 0$ is a transversal Killing field.*

**Remark 3.9.** (*i*) As is obviously seen in the proof, Theorem 3.7 and so Corollary 3.8 are still true even if we omit the assumption that $\mathcal{F}$ admits at least one compact leaf. Indeed, it suffices to notice that the above assumption serves to obtain that $\kappa$ being basic implies $d\kappa = 0$, so that $d_T^2 = 0$.

(*ii*) It is known in [3] that given a Riemannian foliation on a closed manifold there always exists a bundle-like metric with respect to which the foliation is isoparametric. In this case, we can derive our results without the assumption that the foliation is isoparametric. However, for the complete case it is not assured the existence of such a bundle-like metric.

(*iii*) It is worthwhile to note that in the presence of the assumption $\delta_B \kappa_B = 0$, some transversal vanishing theorems were obtained in [2], [7].

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