YOKOHAMA MATHEMATICAL
JOURNAL VOL. 44, 1997

CENTRAL LIMIT THEOREMS FOR WEIGHTED $D[0,1]$-VALUED MIXING SEQUENCES II. FUNCTIONAL CENTRAL LIMIT THEOREMS FOR INTEGRATED VARIABLES

By

KEN-ICHI YOSHIHARA

(Received Semtember 4, 1996)

Abstract. As a continuation of [8], in this paper, we establish functional central limit theorems for weighted and integrated random functions of the type

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}h_{n,i}(s)M_{n,i}(t) \quad \text{and} \quad \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{0}^{t}h_{n,i}(s)dM_{n,i}(s)$$

satisfying some strong mixing condition where $\{\{M_{n,i}(s):0\leq s\leq 1\};n\geq 1\}$ is a triangular array of mean-zero martingales and $\{h_{n,i}\}$ is a triangular array of nonrandom functions.

1. Main results

(I) *Weak convergence of weighted sums.*

Let $(\vee, \mathbf{F}, P)$ be a probability space. For each n (≥ 1) and i $(1\leq i\leq n)$ let $\{M_{n,i}(s):0\leq s\leq 1\}$ be a martingale with filtration $\mathbf{F}_{n,s}$ $(\subset \mathbf{F})$. Firstly, we consider weak convergence of weighted sums of martingales of the type

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}M_{n,i}(s)h_{n,i}(t) \quad s,t\in[0,1].$$

We consider the following conditions.

Condition A. $\{\{M_{n,i}(t);0\leq t\leq 1, i=1,\cdots,n\}:n\geq 1\}$ is a strong mixing triangular array of martingales with respect to the filtration $\mathbf{F}_{n,s}$ which satisfies the following requirements:

(i) For each n $M_{n,i}(0)=0$ $(i=1,\cdots,n)$, $EM_{n,i}(t)=0$ $(i=1,\cdots,n)$ for all t in $[0,1]$ and for all i and j $(i\neq j)$ $M_{n,i}(s)$ and $M_{n,j}(s)$ have no common jumps;

1991 Mathematics Subject Classification: 60F05

Key words and phrases: weighted sum, integrated variable, martingale, strong mixing, weak convergence

(ii) $\sum_{i=1}^{n} EM_{n,i}^2(1) = O(n)$ and there exist some positive constants C_0 and κ $(\in (0,1])$

$$\sup_{n\geq 1}\max_{1\leq i\leq n} E\mid M_{n,i}(t)-M_{n,i}(s)\mid^2 \leq C_0 \mid t-s\mid^{\kappa} \quad s,t\in[0,1];$$

(iii) There exist positive constants p and δ such that $\kappa p \geq 7/2$,

$$\sup_{n\geq 1}\max_{1\leq i\leq n} E\mid M_{n,i}(1)\mid^{p+\delta} < \infty$$

and

$$b(p,\delta) = \sum_{n=1}^{\infty}(n+1)^{j(p)-2}\frac{\delta}{j(p)+\delta}(n) < \infty$$

where $j(p) = 2\min\{k\in \mathsf{Z}_+ : 2k\geq p\}$.

Condition B. $\{h_{n,i}(t) : 1\leq i\leq n : n\geq 1\}$ is a triangular array of real-valued functions of bounded variation on $[0,1]$ and there exist β $(> 1/\sqrt{2})$ and C_0 (>0) such that

$$\sup_{n\geq 1}\max_{1\leq i\leq n}\mid h_{n,i}(s)-h_{n,i}(t)\mid < C_0\mid s-t\mid^{\beta}. \tag{1}$$

Condition C. There exists a centered Gaussian process $\{W(s,t) : [0,1]\times[0,1]\}$ with

$$EW(s,t_1)W(s,t_2) = G(s;t_1,t_2) \quad (s,t_1,t_2\in[0,1]).$$

and for all ℓ, j $(=1,2,\cdots,k)$ and s,t_ℓ,t_j $(\in[0,1])$

$$E\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}M_{n,i}(s)h_{n,i}(t_\ell)\frac{1}{\sqrt{n}}\sum_{i'=1}^{n}M_{n,i'}(s)h_{n,i'}(t_j)\right\} \rightarrow G(s,t_\ell,t_j) \tag{2}$$

We prove the following theorem.

Theorem 1. *Suppose Conditions A, B and C hold. Then*

$$\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}M_{n,i}(s)h_{n,i}(t);\ (s,t)\in[0,1]^2\right\} \xrightarrow{D} W = \{W(s,t):\ (s,t)\in[0,1]^2\} \quad \text{in } \mathcal{D}[0,1]^2 \tag{3}$$

where W is a centered Gaussian process with covariance

$$EW(s,t)W(s',t') = G(s\wedge s';\ t,t'). \tag{4}$$

(II) *Central limit theorems for integrated random variables.*

We note here that

$$\text{(5)}\qquad \frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_{n,i}(t)h_{n,i}(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^t M_{n,i}(s)dh_{n,i}(s) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^t h_{n,i}(s)dM_{n,i}(s).$$

When $M_{n,i}(s)$ is of bounded variation, the last integral in (5) can also be interpreted as a pathwise Lebesgue-Stieltjes integration.

If we can show the joint normality of the left side and the second term in the right side in (5), then we obtain the weak convergence of random elements of the type

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^t M_{n,i}(s)dh_{n,i}(s).$$

Now, we consider another condition.

Condition D. (i) There exists a bounded function $h(t)$ such that

$$\sup_{n\geq 1}\max_{1\leq i\leq n} \mid h_{n,i}(t) \mid \leq h(t) < \infty \quad {}^{\forall}t\ (\in [0,1]).$$

(ii) The sequence

$$\frac{1}{n}\sum_{i=1}^{n} h_{n,i}(u)\int_0^t h_{n,i}(s)d\langle M_{n,i}(s)\rangle$$

converges in probability for all t, u in $[0,1]$.

Concerning the above problem, we prove the following theorem.

Theorem 2. *Suppose Conditions A-D hold. Then, the two processes*

$$\left\{\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_{n,i}(t)h_{n,i}(t);\ 0\leq t\leq 1\right\}\ :\ n\geq 1\right\}$$

and

$$\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^t h_{n,i}(s)dM_{n,i}(s);\ 0\leq t\leq 1\right\}$$

are jointly weakly convergent in $\mathcal{D}^2[0,1]$ *to a Gaussian limiting process.*

2. Auxiliary results

To prove Theorems 1 and 2, we need the following general limit theorem.

Theorem 3. *Let* $\{Z_n(s,t) : (s,t) \in [0,1]^2\}$ *be a sequence of* $\mathcal{D}[0,1]^2$*-valued random processes, defined on a probability space* $(\vee, \mathbf{F}, P)$. *Suppose the following conditions hold:*

(i) *For every fixed* s $(\in [0,1])$, $\{Z_n(s,t)\}$ *is a stationary strong mixing sequence of random functions of* t.

(ii) *For arbitrarily fixed* $s_1, \cdots, s_k$, $Z_n = (Z_n(s_1,\cdot), \cdots, Z_n(s_k,\cdot))$, *as a random element taking values in* $\mathcal{D}^k[0,1]$, *converges weakly to* $(W(s_1,\cdot), \cdots, W(s_k,\cdot))$, *where* $W(\cdot,\cdot)$ *is a mean-zero Gaussian process defined on* $[0,1]^2$.

(iii) *There exist constants* r (> 1), q (≥ 2), C (> 0) *and a function* $B_n(s)$ *such that for any* s, s' $(\in [0,1],\ s > s')$

$$(6) \qquad \sup_{0 \leq t \leq 1} E \mid Z_n(s,t) - Z_n(s',t) \mid^r \leq C\{B_n(s) - B_n(s')\}^{\frac{q}{2}} + Cn^{-\frac{q}{4}}.$$

Moreover, $B_n(t)$ *is a non-decreasing step-function with maximum jump size equal to* $O(n^{-\frac{1}{2}})$ *such that* $B_n(t) \to B(t)$ $(t \in [0,1])$ *where* $B(t)$ *is a non-decreasing continuous function on* $[0,1])$.

(iv) *For every* ϵ (> 0) *there exist* $t_0, t_1, \cdots, t_m$ $(0 = t_0 < t_1 < \cdots < t_m = 1)$ *with* $m = O(n^\beta)$ $(\frac{1}{2} \leq \beta < q/4)$ *such that*

$$(7) \qquad P\left(\sup_{0 \leq s \leq 1} \max_{1 \leq j \leq m} \sup_{t_{j-1} \leq t < t_j} \mid Z_n(s,t) - Z_n(s,t_{j-1}) \mid > \epsilon \right) \to 0$$

as $n \to \infty$.

Then

$$(8) \qquad \{Z_n(s,t) : (s,t) \in [0,1]^2\} \xrightarrow{D} \{W(s,t) : (s,t) \in [0,1]^2\} \quad \text{in } \mathcal{D}[0,1]^2$$

as $n \to \infty$.

Proof. By Condition (ii), to prove Theorem 3, it suffices to show that for every ϵ (> 0) there exist $s_0, s_1, \cdots, s_\ell$ $(0 = s_0 < s_1 < \cdots < s_\ell = 1)$ with $\ell = O(n^{\beta_1})$ $(\frac{1}{2} \leq \beta_1 < q/4)$ and $t_0, t_1, \cdots, t_m$ $(0 = t_0 < t_1 < \cdots < t_m = 1)$ with $m = O(n^{\beta_2})$ $(\frac{1}{2} \leq \beta_2 < q/4)$ such that

$$(9) \qquad P\left(\max_{1 \leq i \leq \ell} \sup_{s_{i-1} \leq s < s_i} \max_{1 \leq j \leq m} \sup_{t_{j-1} \leq t < t_j} \mid Z_n(s,t) - Z_n(s_i,t_j) \mid > \epsilon \right) \to 0$$

as $n \to \infty$. Since

(10) *L. H. S. of* (9)

$$\leq P\left(\sup_{0\leq s\leq 1}\max_{1\leq j\leq m}\sup_{t_{j-1}\leq t<t_j}\mid Z_n(s,t)-Z_n(s,t_{j-1})\mid>\frac{\epsilon}{2}\right)$$
$$+P\left(\max_{1\leq j\leq m}\max_{1\leq i\leq \ell}\sup_{s_{i-1}\leq s<s_j}\mid Z_n(s,t_{j-1})-Z_n(s_{i-1},t_{j-1})\mid>\frac{\epsilon}{2}\right)$$
$$\leq P\left(\sup_{0\leq s\leq 1}\max_{1\leq j\leq m}\sup_{t_{j-1}\leq t<t_j}\mid Z_n(s,t)-Z_n(s,t_{j-1})\mid>\frac{\epsilon}{2}\right)$$
$$+m\sup_{0\leq t\leq 1}P\left(\max_{1\leq i\leq \ell}\sup_{s_{i-1}\leq s<s_j}\mid Z_n(s,t)-Z_n(s_{i-1},t)\mid>\frac{\epsilon}{2}\right)$$
$$=I_{1,n}+I_{2,n}\quad \text{(say)},$$

to prove (9) it suffices to show that

$$\max(I_{1,n},I_{2,n})\ \to\ 0\quad (n\to\infty). \tag{11}$$

But, using the method of proof Theorem 15.6 in [2] and (6) we can prove the fact that $I_{2,n}\to 0$. On the other hand, the fact that $I_{1,n}\to 0$ is Condition (iv). Hence, we have the desired conclusion. □

Let $g_n(t)$ be a step-function from $[0,1]\to[0,1]$ such that

$$g_n(1)\equiv 1;\quad g_n(t)=\frac{\ell}{N}\ \left(\frac{\ell}{N}\leq t\leq\frac{\ell+1}{N}\right)\ (\ell=1,\cdots,N-1)$$

where $N=[n^\gamma]$ and γ is chosen so that $1/2\leq\gamma\leq 1/\sqrt{2}$, $\gamma\beta>1/2$ and β is the one defined in (1) and put

$$X_n(s,t)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}M_{n,i}(s)h_{n,i}(g_n(t)).$$

It is easy to see that

$$\sup_{0\leq t\leq 1}\mid t-g_n(t)\mid\leq\ \frac{1}{N}$$

and

$$g_n(t)\ \to\ t\quad \text{uniformly for } t\in[0,1]$$

with maximum jump size $1/N$ of $g_n(t)$.

The following lemma, due to [5], shows that the two processes $X_n(s,t)$ and

$$(1/\sqrt{n})\sum_{i=1}^{n}M_{n,i}(s)h_{n,i}(t)$$

are equivalent in the limit and it is enough to prove the weak convergence for $X_n(s,t)$.

Lemma A. *Suppose Conditions A, B and C hold. Then*

$$(12)\quad P\left(\sup_{0\le s\le 1}\sup_{0\le t\le 1}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_{n,i}(s)h_{n,i}(t)-X_n(s,t)\right|>\epsilon\right)\to 0\quad (n\to\infty).$$

To prove theorems (below) we need the following theorem, due to [5].

Theorem A. *Suppose $\{\xi_i\}$ is a strong mixing sequence of zero-mean random variables such that for some p (≥ 2) and δ (>0)*

$$\max_{1\le i\le n} E\mid \xi_i \mid^{p+\delta}<\ \infty.$$

Put

$$L_t(n,\delta)=\sum_{i=1}^{n}\|\ \xi_i\ \|_{t+\delta}^{t}\ \ (t\ge 1),\quad D_n^2(\delta)=L_2(n,\delta),$$
$$Q_p(n,\delta)=\max\{L_p(n,\delta),D_n^2(\delta)\}.$$

Then

$$E\mid\sum_{i=1}^{n}\xi_i\mid^p\le\ cb(p,\delta)Q_p(n,\delta).$$

where $b(p,\delta)$ is the one defined in Condition A (iii).

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. We use Theorem 3. In view of Lemma 1 and Conditions A and B, it suffices to verify that the auxiliary process $\{X_n(s,t)\}$ satisfies the following conditions:

(i) for all $s_1,\cdots,s_k$ $(\in[0,1])$

$$(X_n(s_1,\cdot),\cdots,X_n(s_k,\cdot))\xrightarrow{D}(W(s_1,\cdot),\cdots,W(s_k,\cdot))\quad \text{in } \mathcal{D}([0,1]^k);$$

(ii) the processes $\{\{X_n(s,t)\};n\ge 1\}$ satisfy (6) and (7).

(i) is clear from Conditions A-C.

To show that (ii) holds, take the equally spaced points $0=t_0<t_1<\cdots<t_{b_m}$ to be the same as the jump points of $g_n(t)$. Then, $b_m=[n^\gamma]$. We note that for all s $(\in[0,1])$

$$\max_{1\le j\le b_m}\ \sup_{t_{j-1}\le t<t_j}\mid X_n(s,t)-X_n(s,t_{j-1})\mid=0$$

since

$$g_n(t) = g_n(t_j) \quad (t \in [t_{j-1}, t_j), j = 1, \cdots, b_n).$$

Thus, (7) is satisfied.

Further, for arbitrary points $0 = s_0 < s_1 < \cdots < s_{a_\ell} = 1$ we have

$$\max_{1\le i\le a_\ell} \sup_{s_{i-1}\le s< s_i} \max_{1\le j\le b_m} \sup_{t_{j-1}\le t<t_j} | X_n(s,t) - X_n(s_{i-1}, t_{j-1}) |$$
$$\le \max_{1\le i\le a_\ell} \sup_{s_{i-1}\le s< s_i} \max_{1\le j\le b_m} | X_n(s,t) - X_n(s_{i-1}, t_{j-1}) |,$$

and hence

(13) $$P\left(\max_{1\le i\le a_\ell} \sup_{s_{i-1}\le s< s_i} \max_{1\le j\le b_m} \sup_{t_{j-1}\le t<t_j} | X_n(s,t) - X_n(s_{i-1}, t_{j-1}) | > \epsilon \right)$$
$$\le P\left(\max_{1\le i\le a_\ell} \sup_{s_{i-1}\le s< s_i} \max_{1\le j\le b_m} | X_n(s,t_{j-1}) - X_n(s_{i-1}, t_{j-1}) | > \epsilon \right)$$
$$\le b_m \max_{1\le j\le b_m} P\left(\max_{1\le i\le a_\ell} \sup_{s_{i-1}\le s< s_i} | X_n(s,t_{j-1}) - X_n(s_{i-1}, t_{j-1}) | > \epsilon \right)$$
$$\le a_\ell b_m \sup_{0\le t\le 1} \max_{1\le i\le a_\ell} P\left(\sup_{s_{i-1}\le s< s_i} | X_n(s,t) - X_n(s_{i-1}, t_{j-1}) | > \epsilon \right)$$
$$\le \frac{a_\ell b_m}{\epsilon^p} \sup_{0\le t\le 1} \max_{1\le i\le a_\ell} \max_{0\le j\le b_m} E\left[\sup_{s_{i-1}\le s< s_i} | X_n(s,t) - X_n(s_{i-1}, t) | \right]^p$$
$$\le \frac{a_\ell b_m}{\epsilon^p} \sup_{0\le t\le 1} \max_{1\le i\le a_\ell} \max_{0\le j\le b_m} E |X_n(s_i,t) - X_n(s_{i-1}, t)|^p,$$

since for any fixed t $\{X_n(s,t) - X_n(s_{i-1},t); s_{i-1} \le s \le s_i\}$ is a martingale.

Let τ be a positive number such that $(4\gamma/3) < \tau < 1$ and put $a_\ell = [n^\tau]$. Let $s_i - s_{i-1} = [n^\tau]^{-1}$ $(i = 1, \cdots, a_\ell)$. By Condition A (ii) we have that for any r (> 0)

$$\| h_{n,j}(t)M_{n,j}(s_i) - h_{n,j}(t)M_{n,j}(s_{i-1}) \|_r$$
$$= | h_{n,j}(t) | \| M_{n,j}(s_i) - M_{n,j}(s_{i-1}) \|_r \le c \, | s_i - s_{i-1} |^{\frac{\kappa}{2}} \le cn^{\frac{-\tau\kappa}{2}}.$$

Hence, by Theorem A we have

$$E \mid X_n(s_i,t) - X_n(s_{i-1},t) \mid^p \le c \max\left(n^{-\frac{p}{2}+1} n^{-\frac{\tau\kappa p}{2}}, n^{-\frac{\tau\kappa p}{2}} \right),$$

which implies (6). Noting that $p > 2$ and $\tau + \gamma < (\tau\kappa p/2)$, we have

$$L.\ H.\ S. \text{ of } (21) = o(1).$$

Now, the desired conclusion follows from Theorem 3. □

Proof of Theorem 2. Let

$$U_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_0^t h_{n,i}(s) dM_{n,i}(s), \quad 0 \le t \le 1.$$

By Condition B the function $n_{n,i}^{(N)}(t)$ defined by

$$h_{n,i}^{(N)}(t) = h_{n,i}\left(\frac{[Nt]}{N}\right)$$

satisfies

$$\sup_{0 \le t \le 1} | h_{n,i}(t) - h_{n,i}^{(N)}(t) | \le C_0 N^{-\beta} \tag{14}$$

for each $N \ge 1$, $n \ge 1$ and $1 \le i \le n$. Let $U_n^{(N)}(t)$ be a process defined by

$$U_n^{(N)}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_0^t h_{n,i}^{(N)}(s) dM_{n,t}(s).$$

By Burkholder inequality with Condition A (ii), we have

$$E\left|\int_0^t (h_{n,i}(s) - h_{n,i}^{(N)}(s)\right|^{2+\delta} \le C_\delta M \left\{ \sup_{0 \le t \le 1} | h_{n,i}(t) - h_{n,i}^{(N)}(t) | \right\}^{2+\delta}, \tag{15}$$

where

$$M = \sup_{n,i} E(| M_{n,i}(1) |^{2+\delta})$$

and C_δ is a constant depending only on δ. For each n and N define

$$Y_{n,i}^{(N)}(t) = \frac{1}{\sqrt{n}} \int_0^t (h_{n,i}(t) - h_{n,i}^{(N)}(s)) dM_{n,i}(s).$$

Then, for each N $\{Y_{n,i}^{(N)}(t) : 0 \le t \le 1, 1 \le i \le n\}$ is a strong mixing sequence satisfying $EY_{n,i}^{(N)}(t) = 0$. It follows from (14) and (15) that

$$E(| Y_{n,i}^{(N)}(t) |^{2+\delta}) \le C_\delta C_0^{2+\delta} M n^{-\frac{2+\delta}{2}} N^{-(2+\delta)\beta}$$

and therefore

$$\sum_{i=1}^{n} \| Y_{n,i}^{(N)}(t) \|_{2+\delta}^2 \le (C_\delta M)^{\frac{2}{2+\delta}} C_0^2 N^{-2\beta}.$$

Thus, we can apply Theorem A to show the existence of a constant C satisfying

$$E \mid U_n(t) - U_n^{(N)}(t) \mid^2 \leq E\left|\sum_{i=1}^{n} Y_{n,i}^{(N)}(t)\right|^2 \leq CN^{-2\beta} \tag{16}$$

for each N, n and $0 \leq t \leq 1$.

Note that the process $U_n^{(N)}(t)$ is written as follows:

$$U_n^{(N)}(t) = \frac{1}{\sqrt{n}}\left[\sum_{j=1}^{[Nt]} h_{n,i}\left(\frac{j-1}{N}\right)\left\{M_{n,i}\left(\frac{j}{N}\right) - M_{n,i}\left(\frac{j-1}{N}\right)\right\}\right.$$
$$\left. + h_{n,i}\left(\frac{[Nt]}{N}\right)\left\{M_{n,i}(t) - M_{n,i}\left(\frac{[Nt]}{N}\right)\right\}\right].$$

Define a centered Gaussian process $U^{(N)}(t)$ by

$$U^{(N)}(t) = \sum_{j=1}^{[Nt]}\left\{W\left(\frac{j}{N}, \frac{j-1}{N}\right) - W\left(\frac{j-1}{N}, \frac{j-1}{N}\right)\right\}$$
$$+W\left(t, \frac{[Nt]}{N}\right) - W\left(\frac{[Nt]}{N}, \frac{[Nt]}{N}\right).$$

Then, it is follows from Theorem A that the finite dimensional distributions of $U_n^{(N)}(t)$ converge to those of $U^{(N)}(t)$. Together with (16) this implies that the finite dimensional distributions of $U_n(t)$ converge to those of a centered Gaussian process $U(t)$. Furthermore, the finite dimensional distributions of vector valued process $(W_n(t,t), U_n(t))$ converge to those of the Gaussian process $(W(t,t), U(t))$ where

$$W_n(s,t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_{n,i}(s)h_{n,i}(t).$$

Now, we will prove that for arbitrary numbers t, s $(0 \leq s < t \leq 1,\ t-s \geq C_1 n^{-1})$

$$E\left|U_n(t) - U_n(s)\right|^p \leq C_2(t-s)^{1+\vartheta} \tag{17}$$

where ϑ, C_3 and C_4 are some positive constants independent of n, t and s.

Let

$$V_i(s,t) = V_{n,i}(s,t) = \int_s^t h_{n,i}(u)dM_{n,i}(u).$$

From Condition A it is obvious that

$$EV_i(s,t) = 0 \qquad {}^\forall i\ (\geq 1)\ {}^\forall s,\ t\ (\in [0,1]))$$

and for any r (> 2)

$$E \mid V_i(s,t) \mid^r \leq c \left[E \int_s^t h_{n,i}^2(u) d\langle M_{n,i} \rangle_u \right]^{\frac{r}{2}}$$
$$\leq c \sup_{s \leq u \leq t} h^r(u) \mid t-s \mid^{\frac{r\kappa}{2}} \leq c \mid t-s \mid^{\frac{r\kappa}{2}} .$$

Thus, we have

$$D_n^2(\delta) = L_2(n,\delta) = \sum_{i=1}^n \| V_i(s,t) \|_{2+\delta}^2 \leq cn \mid t-s \mid^{\kappa}, \tag{18}$$

$$L_p(n,\delta) = \sum_{i=1}^n \| V_i(s,t) \|_{p+\delta}^p \leq cn \mid t-s \mid^{\frac{\kappa p}{2}} . \tag{19}$$

Noting that for fixed s and t $\{V_i(s,t)\}$ is a triangular array of random variables satisfying the same mixing condition as that of $\{M_{n,i}\}$ and that $p > 2$, from Theorem A, Condition A (iii), (18) and (19) we obtain

$$E \mid U_n(t) - U_n(s) \mid^p \leq c \max \left\{ n^{-\frac{p}{2}+1} \mid t-s \mid^{\frac{\kappa p}{2}}, \mid t-s \mid^{\frac{\kappa p}{2}} \right\} \leq c \mid t-s \mid^{\frac{\kappa p}{2}},$$

which implies (17).

Hence the processes $\{U_n(t)\}$ are tight. The tightness of $\{W_n(t,t)\}$ follows from Theorem 1 and therefore we obtain the desired result. □

Acknowledgement. I would like to thank Prof. Tosio Mori of Yokohama City University for a particularly careful reading of the paper and for pointing out some simplifications in my proof.

References

[1] P. Billingsley, *Convergence of probability measures*, Wiley, New York, 1968.

[2] A. Dvoretzkey, Asymptotic normality of sums of dependent random vectors, *Multivariate Analysis-IV* (Ed. by Krishnaiah) 24-34, North Holland Publishing Comp. 1977.

[3] R. Rebolledo, Central limit theorems for local martingales, *Z. Wahr. verw. Geb.* **51** (1980) 269-286.

[4] C. Srinivasan, and M. Zhou, A central limit theorem for weighted and integrated martingales, *Scand. J. Statist.* **22** (1995), 493-504.

[5] S. A. Utev, Inequalities for sums of weakly dependent random variables and estimates of the convergence rate in the invariance principle. Adv. Probab. Th. Limit theorems for sums of random variables (Ed. by Borovkov), 73-114, Optimization Software, Inc., New York 1985.

[6] K. Yoshihara, Moment inequalities for strong mixing sequences, *Kodai Math. J.* **1** (1978), 316-328.

[7] K. Yoshihara, *Weakly dependent stochastic sequences and their applications, Vol. I, Summation theory for weakly dependent sequences.* Sanseido, Tokyo, 1992.

[8] K. Yoshihara, Central limit theorems for weighted $D[0,1]$-valued mixing sequences I. Functional central limit theorems for weighted sums, (1996) (to appear).

Department of Mathematics
Faculty of Engineering
Yokohama National University
Tokiwadai, Hodogaya-ku
Yokohama, 240 Japan