Yokohama Mathematical
Journal Vol. 44, 1997

ON SPIRAL-LINEAR SYSTEMS

By

Shigeo Ichiraku*

(Received October 23, 1996)

Abstract. The Poincaré-Bendixon Theorem implies that two dimensional flows have no chaotic behavior. L.O. Chua and Brown showed an example of chaotic 2-dimensional flow with "swicthing" ([1], [2]). M. Misiurewicz showed that the unimodal map derived from the Chua-Brown system has negative Schwarzian derivative for certain parameter values ([3]). In this note we will show an analogous result for the system called Spiral-Linear system, proposed by H. Kawakami and Lozi, which is 2-dimensional chaotic system with "switching" apparently simpler than Chua-Brown system ([4]). The main result of this note is that the 1-dimensional map, which determines behaviors of the system, has negative Schwarzian derivatives.

1. Spiral-Linear Systems

The Spiral-Linear system was introduced by H. Kawakami and Lozi as a simple chaotic system ([4]). The system is completely described by 1-dimensional map with finite discontinuity as we will see. The system is described as follows.

On the (x, y)-plane, let H, B denote the half planes defined by the following inequalities:

$$H=\{(x,y)\,|\,y\geq -1\},\ B=\{(x,y)\,|\,y\leq -1+\alpha\},\ (0\leq\alpha\leq 1)$$

The flow with "switching" is defined by the following differential equations on each half plane. On the boundary lines of the planes the solution switches onto another plane. On the half plane B the dynamics is determined by the following equations, which is "spiral":

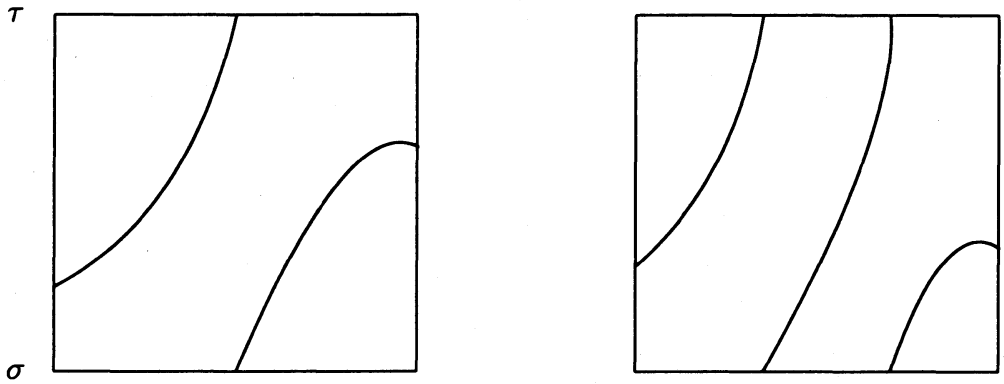
$$(1)\qquad \begin{cases} \dfrac{dx}{dt} = \sigma x - y \\ \dfrac{dy}{dt} = x + \sigma y, \ (\sigma > 0) \end{cases}$$

On the half plane H, the dynamics is determined by the following equations,

*This research was partially supported by Grant-in-Aid for General Scientific Research, Grant Number 06640259.

1991 Mathematics Subject Classification: 58F03, 58F13

Key words and phrases: Schwarzian derivative, chaotic systems



$\sigma = 0.1, \alpha = 0.6$ $\sigma = 0.1, \alpha = 0.4$

Figure 1 Graphs of $f_{\alpha,\sigma}$

which is "linear":

$$
(2) \qquad \begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = 1. \end{cases}
$$

The behavior of this system is descibed by the 1-dimensional smooth map $f_{\sigma,a} = f : L \to L$ with finite discontinuity which is obtained by considering a boundary line $L = \{(x,y) \mid y = -1 + \alpha\}$ of B as a kind of Poincaré section. Typical graphs of maps f's are shown in Fig. 1.

2. An Invariant Interval

First we will show the existence of an invariant interval on which the system may show chaotic behavior. The discontinuous point of the map f corresponds to the tangency of the solution curve of (1) and the line $\partial H = \{(x,y) \mid y = -1\}$. The solution curve of (1) through (x_0, y_0) is given by the spiral described by the following form;

$$
(3) \qquad \begin{cases} x = e^{\sigma t}(x_0 \cos t - y_0 \sin t), \\ y = e^{\sigma t}(x_0 \sin t + y_0 \cos t). \end{cases}
$$

Hence we can easily see the tangential point T has the coordinate $(\sigma, -1)$. Going back to the point on L along the solution curve through $(\sigma, -1)$, we get the point $P : (a, -1 + \alpha)$ and

$$
\sigma = \lim_{x \to a+0} f(x).
$$

Put

$$\tau = \lim_{x \to a-0} f(x).$$

Then the following theorem holds.

Theorem 1. *For a given* $\alpha\,(0 < \alpha < 1)$, *there exists* $\sigma_0 > 0$ *such that if* $0 < \sigma \le \sigma_0$ *then the closed interval* $[\tau, \sigma]$ *is invariant under* $f_{\sigma,a}$.

Proof. It is sufficient to show $\tau \le f(\tau)$ and $f(\sigma) \le \sigma$. We only need to prove $\tau \le f(\tau)$, the other is clear because σ is the maximum value of f. Let P_0 be the point having the coordinate $(\tau, -1+\alpha)$ and $P : (\tau', -1)$ the point where the solution curve of (1) through P_0 meets firstly the half line $x \le 0,\ y = -1+\alpha$. Then we have $f(\tau) = \tau'$.

Putting $\theta = \angle P_0OP$, we have the following;

$$|OP| = e^{\sigma\theta} \cdot |OP_0|,$$
$$\tau^2 = |OP|^2 - (-1)^2,$$
$$\tau'^2 = |OP_0|^2 - (-1+\alpha)^2.$$

Hence,

$$\tau^2 - \tau'^2 = |OP_0|^2 \cdot (e^{2\sigma\theta} - 1) - (2-\alpha)\alpha.$$

Now Theorem 1 follows from the following lemma. ■

Lemma 2.1. *When we make* σ *tend to* 0 *the following hold;*

$$\lim_{\sigma \to +0} |OP_0|^2 \cdot (e^{2\sigma\theta} - 1) = 0, \quad \lim_{\sigma \to +0} |OP_0| = 1$$

and the function of σ, $\rho(\sigma) = |OP_0|^2 \cdot (e^{2\sigma\theta} - 1)$ *is monotone decreasing.*

Proof. Since when $\sigma = 0$ the solution curve are simply circles, the proof of the former half is obvious. To prove the last statement, note that $|OP_0|^2$ is monotone decreasing and $\dfrac{d(e^{2\sigma\theta} - 1)}{d\sigma} = 2\theta\dfrac{d\theta}{d\sigma} \cdot e^{2\sigma\theta}$ is negative for clearly $\dfrac{d\theta}{d\sigma}$ is negative. ■

3. Schwarzian derivatives

Recall that for a C^3-map φ the Schwarzian derivative $S(\varphi)$ of φ is defined as follows;

$$S(\varphi) = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2.$$

For the meaning of Schwarzian drivative, refer, for example, to [4] or [5].

Now we will show the following theorem.

Theorem 2. *For any positive $\sigma < \sigma_0$, Schwarzian derivative of f is negative on the interval $[\tau, \sigma]$ except where f is discontinuous.*

To prove Theorem 2, we need the following lemma proved implicitly in [3].

Lemma 3.1. *Let φ be a C^3-map defined around a point $a \in \boldsymbol{R}$ and ψ^{-1} a C^3-diffeomorphism defined around $\varphi(a)$. Then Schwarzian derivative of $\psi^{-1} \circ \varphi$ at a is negative if and only if the following inequality holds;*

$$\frac{S(\varphi)(a)}{\varphi'(a)^2} < \frac{S(\psi)(\psi^{-1} \circ \varphi(a))}{\psi'(\psi^{-1} \circ \varphi(a))^2}.$$

Proof. Recall the following fundamental result about Schwarzian derivative:

Formula for Schwarzian derivative *Let φ, ψ be C^3-maps and the composition map $\varphi \circ \psi$ be defined. Then the following holds;*

$$S(\varphi \circ \psi) = \psi'^2 \cdot (S(\varphi) \circ \psi) + S(\psi).$$

Hence the compsoition map of maps having negative Schwarzian derivatives has also negative Schwarzian derivative.

Put $F = \psi^{-1} \circ \varphi$ then by the above formula the following holds;

$$S(\varphi) = S(\psi \circ F) = F'^2 \cdot (S(\psi) \circ F) + S(F).$$

Since the chain rule of derivative implies $F' = \dfrac{\varphi'}{\psi'}$, we have;

$$S(F) = S(\varphi) - \left(\frac{\varphi'}{\psi'}\right)^2 \cdot (S(\psi) \circ F).$$

This proves Lemma3.1. ■

Now we prove Theorem 2.

Proof of Theorem 2. On the interval $[\tau, a]$ the map f is given as follows. Let the solution curve through the point $(x, -1+\alpha)$ meet the line at the point $(y, -1)$ and the x-axis at the point at $(x_0, 0)$ then the following holds;

$$\begin{aligned} y &= f(x), \\ \sqrt{x^2 + A^2} &= e^{\sigma \tan^{-1}(A/x_0)} \cdot x_0, \quad \text{where} \quad A = 1 - \alpha. \\ \sqrt{y^2 + 1} &= e^{\sigma \tan^{-1}(1/x_0)} \cdot x_0. \end{aligned}$$

Put $\varphi(x) = X_0$, $\psi(y) = X_0$, then

$$\varphi(x) = e^{-\sigma \tan^{-1}(A/\varphi(x))} \cdot \sqrt{x^2 + A^2},$$
$$\psi(y) = e^{-\sigma \tan^{-1}(1/\psi(x))} \cdot \sqrt{y^2 + 1}$$

and

$$f = \psi^{-1} \circ \varphi.$$

By straightforward calculation we obtain the followings;

$$\varphi'(x) = \frac{x - A\sigma}{x^2 + A^2} \cdot \varphi(x)$$
$$\varphi''(x) = \frac{A^2(1+\sigma^2)}{(x^2 + A^2)^2} \cdot \varphi(x),$$
$$\varphi'''(x) = \frac{-A^2(1+\sigma^2)3x + A\sigma}{(x^2 + A^2)^3} \cdot \varphi(x),$$
$$S(\varphi)(x) = \frac{1}{2}\frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2$$
$$= \frac{-A^2(1+\sigma^2)(6x^2 - 4A\sigma x + A^2(3+\sigma^2)}{2(x^2 + A^2)^2(x - A\sigma)^2)}$$

and

$$\frac{S(\varphi)}{\varphi'^2} = \frac{-A^2(1+\sigma^2)(6x^2 - 4A\sigma x + A^2(3+\sigma^2))}{(x - A\sigma)^4\varphi^2(x)}.$$

Similarly we have;

$$\frac{S(\psi)}{\psi'^2} = \frac{-(1+\sigma^2)(6y^2 - 4\sigma y + (3+\sigma^2))}{(y-\sigma)^4\psi^2(y)}.$$

By Lemma 3.1 we obtain that the map f has negative Schwarzian derivative if and only if;

$$\frac{A^2(6x^2 - 4A\sigma x + A^2(3+\sigma^2))}{(x - A\sigma)^4} > \frac{(6y^2 - 4\sigma y + (3+\sigma^2))}{(y-\sigma)^4}.$$

(Note that $\psi(x) = \varphi(y) = x_0$.)

Put $P(\xi) = \dfrac{6\xi^2 - 4\sigma\xi + (3+\sigma^2)}{(\xi - \sigma)^4}$ and we see f has negative Schwarzian derivative if and only if;

$$P(\xi/A) > P(y).$$

Since the rational function $P(\xi)$ is monotone increasing on $(-\infty, \sigma)$ and monotone decreasing on (σ, ∞), it is sufficient to show that ;

$$x < A\sigma \quad \text{and} \quad x > Ay.$$

The former condition is clearly satisfied (see Fig. 1) and the latter condition is followed from the hypothesis $\sigma < \sigma_0$, because it means $x > y$ by Theorem 1. This proves Theorem 2. ■

References

[1] R. Brown and L. Chua, Horseshoes in the twist-and-flip maps, *International Journal of Bifurcation and Chaos*, **1** (1) (1991), 235–252.

[2] ___ and ___, Generalizing the twist-and-flip paradigm, *International Journal of Bifurcation and Chaos* **1** (2) (1991), 385–416.

[3] M. Misiurewicz, Unimodal interval maps obtained from the modified Chua equations, *International Journal of Bifurcation and Chaos*, **3** (2) (1993), 323–332.

[4] H. Kawakami and Réne Lozi, *Switched Dynamical Systems, Advanced Series in Dynamical Systems Vol. 11, Proceedings of RIMS Conference*, Edited by S. Ushiki, World Scientific.

[5] R.L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison Wesley, Menlo Park, 1986.

[6] Welington de Melo and Sebastian van Strien, *One-dimensional Dynamics*, Springer-Verlag, 1993.

Department of Mathematical Sciences
Yokohama City University
22-2 Seto, Kanazawa-ku
Yokohama 236, Japan