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THE DIAGONAL FLIPS OF TRIANGULATIONS ON THE SPHERE

By

HIDEO KOMURO

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Abstract. It will be shown that any two triangulations with n vertices on the sphere can be transformed into each other by at most $8n-54$ diagonal flips if $n\geq 13$ and $8n-48$ if $n\geq 7$.

1. Introduction

A *triangulation* G on a closed surface F^2 is a simple graph embedded on F^2 so that each face is triangular and any two faces meet along at most one edge. Let abc and acd be two triangular faces of G which have an edge ac in common. The *diagonal flip* of ac is to replace the diagonal ac with bd in the quadrilateral $abcd$ (see Figure 1). We don't carry out this diagonal flip, not to make multiple edges, if there is an edge bd in G.

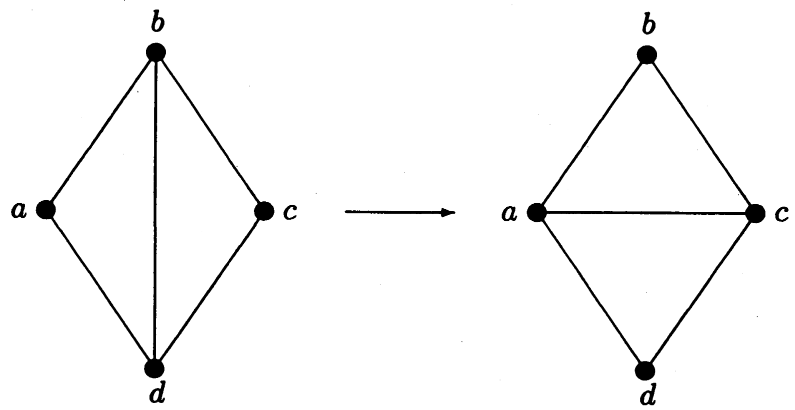

Figure 1 Diagonal flip

Classically, Wagner proved in [8] that any two triangulations on the sphere with the same number of vertices can be transformed into each other by a finite sequence of diagonal flips. Also, Dewdney [2], Negami and Watanabe [5] has

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shown the same results for the torus, the projective plane and the Klein bottle. The same fact does not hold for other surfaces in general, but Negami [3] has shown that there is a natural number $N = N(F^2)$ for each closed surface F^2 such that two triangulations G_1 and G_2 can be transformed into each other by a finite sequence of diagonal flips if $|V(G_1)| = |V(G_2)| \geq N$. Moreover, there are several papers, for example [1] and [4], which include interesting theorems on diagonal flips.

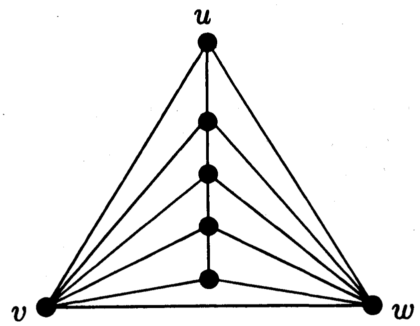

Figure 2 The standard form of triangulations on the sphere

In this paper, we shall focus on how many diagonal flips are needed to transform two triangulations into each other. For example, the proof of Wanger's theorem found in Ore's book [6] on "Four Color Theorem" (also in [4]) gives us an easy algorithm to transform a given triangulation with n vertices on the sphere into the standard form Δ_{n-3}, shown in Figure 2. (The notation Δ_m is used for the consistency with [3], [4] and [5], meaning that the triangle contains m vertices inside.) Roughly speaking, his algorithm decreases the degree of vertices so that they form the vertical path $\Delta_{n-3} - \{v, w\}$ afterwards and suggests a quadratic upper bound for the number of diagonal flips with respect to the number of vertices n. However, we shall give a linear upper bound for it as follows:

Theorem 1. *Any two triangulations with n vertices on the sphere can be transformed into each other, up to ambient isotopy, by at most $8n - 54$ diagonal flips if $n \geq 13$ and by at most $8n - 48$ diagonal flips if $n \geq 7$.*

Unfortunately, we have never known whether or not these bounds are best possible, yet. It seems to be difficult to decide it. For example, Sleator, Tarjan and Thurston [7] have given a very big theory with 3-dimensional hyperbolic geometry and computer experiments to show a precise upper bound for the length of shortest sequences of diagonal flips which transform a given pair of polygons triangulated with only diagonals into each other. In Section 3, we shall show that the order of our bounds cannot not be improved with respect to the number of vertices n.

2. Proof of Theorem

First, we shall show two lemmas on spherical triangulations. In particular, the first one is a core of our proof of Theorem 1. For the proof of the lemma, we define a constant $d_G(v,w)$ by:

$$d_G(v,w) = 3\deg v + \deg w$$

Lemma 2. *Let G be a triangulation with n vertices on the sphere and let v and w be any pair of adjacent vertices of G. Then G can be transformed into Δ_{n-3}, up to ambient isotopy, by $4n-4-(3\deg v+\deg w)$ diagonal flips.*

Proof. Let uvw be a face sharing the edge vw and let $w, w_1, w_2, \ldots, w_l, v$ be the neighbors of u lying around u in this order. First suppose that $\deg u \geq 4$. If w_2 is not adjacent to w, then we replace uw_1 with ww_2. If w_2 is adjacent to w, then we replace uw with vw_1. In these cases, $d_G(v,w)$ increases by 1 or 2, respectively, with one diagonal flip.

Now suppose that $\deg u = 3$ and let u_1 be the unique common neighbour of u, v and w. If $\deg u_1 \geq 5$, then we shall deform G as follows. Let $u, w, h_1, h_2, \ldots, h_k, v$ be the neighbors of u_1 lying around u_1 in this order. If h_2 is not adjacent to w, then we replace u_1h_1 with h_2w and $d_G(v,w)$ increases by 1 with this diagonal flip. If h_2 is adjacent to w, then we replace u_1w with h_1u, and uu_1 with h_1v. (Since h_2 is adjacent to w, v is not adjacent to h_1.) In this case, $d_G(v,w)$ increases by 2 with two diagonal flips.

The remaining case is that $\deg u_1 = 3$ or 4. If $\deg u_1 = 3$, then G consists of only four vertices $\{u, v, w, u_1\}$ and is isomorphic to Δ_1. If $\deg u_1 = 4$, then there is another common neighbor u_2 of u_1, v and w, different from u. Then we can carry out the same deformation inside the triangle u_1vw as we did for the triangle uvw.

Repeating these deformations, we shall have a sequece of vertices $u, u_1, u_2, \ldots,$ u_{n-3} so that they form a path and are adjacent to both v and w. This algorithm stops when $\deg u_{n-3} = 3$ and we shall obtain the final form isomorphic to Δ_{n-3}. In this Δ_{n-3}, both v and w have degree $n-1$, and $d_{\Delta_{n-3}}(v,w) = 4n-4$. Since one diagonal flip corresponds to the increment 1 or 2 of $d_G(v,w)$ through the above deformations, the total number of those diagonal flips in this algorithm does not exceed:

$$d_{\Delta_{n-3}}(v,w) - d_G(v,w) = 4n-4-(3\deg v+\deg w)$$

Thus, the lemma follows. ■

Obviously, the bigger the value of $3\deg v + \deg w$ is, the smaller the number of diagonal flips in Lemma 2 is. So we would like to find two adjacent vertices which have large degree.

Lemma 3. *Let G be a triangulation with n vertices on a closed surafce and $n \geq 6$. Then, any vertex of degree at least* 5 *in G is adjacent to a vertex of degree at least* 5 *unless G is isomorphic to $C_{n-2} + \overline{K_2}$ on the sphere.*

Proof. Let v be a vertex of degree $k \geq 5$ in G and let $u_1, \ldots, u_k$ be its neighbors lying around v in this order. Suppose that $\deg u_i \leq 4$ for all i. If $\deg u_3 = 3$, then u_2 and u_4 are adjacent so that $u_2u_3u_4$ forms a face of G. In this case, $\deg u_2 = \deg u_4 = 4$, and u_1 would coincide with u_5 so that $u_1u_2u_4$ forms a face, which implies that $\deg v \leq 4$, a contradiction. Thus, $\deg u_i = 4$ for all i. It however follows that u_i's are adjacent to a common vertex v' outside the star neighborhood of v. In this case, G consists of the cycle $u_1 \cdots u_k$ with two vertices v and v' and is isomorphic to $C_{n-2} + \overline{K_2}$ on the sphere. ■

Furthermore, it can be shown easily that all the vertices of degree at least 5 induce a connected subgraph in any triangulation. Lemma 3 is however enough for our purpose.

Proof of Theorem 1. First, we shall estimate the number of diagonal flips which transform a given triangulation G into the standard form Δ_{n-3}. We would like to find a pair of adjacent vertices u and v so as to maxmize the value of $3\deg v + \deg w$ in Lemma 2. The unique exception $C_{n-2} + \overline{K_2}$ in Lemma 3 can be transformed into Δ_{n-3} by only one diagonal flip. Thus, we may neglect the case that G is isomorphic to $C_{n-2} + \overline{K_2}$.

If $n \geq 13$, there is a vertex of degree at least 6. This follows from the well-known formula

$$\sum_{i \geq 3} (6 - i) V_i = 12$$

where V_i stands for the number of vertices of degree i. Choose such a vertex as v. By Lemma 3, there is a vertex w of degree at least 5 which is adjacent to v.

Thus, $3\deg v + \deg w \geq 3 \times 6 + 5 = 23$ and the number of diagonal flips does not exceed $4n - 27$.

When $7 \leq n \leq 12$, G may contain no vertex of degree at least 6. If all vertices of G had degree at most 4, then G would consist of at most 6 vertices. Thus, G has a vertex v of degree at least 5 and has another vertex w of degree at least 5 adjacent to v. So at most $4n - 24$ diagonal flips are needed to obtain Δ_{n-3} from

G in this case.

Now consider any two triangulations G_1 and G_2 with n vertices on the sphere. Since each of them can be transformed into Δ_{n-3}, G_1 and G_2 can be transformed into each other via Δ_{n-3} by twice many diagonal flips as we showed above. Thus, the theorem follows. ∎

In our proof, we have evaluated only the length of a sequece from G_1 and G_2 which passes through Δ_{n-3}. The reader will expect a shorter sequence from G_1 and G_2, not via Δ_{n-3}.

3. Lower bounds

In this section, we shall estimate some lower bounds for the number of diagonal flips which transform a given triangulation into another and show that the linear order of the bounds in Lemma 2 and also in Theorem 1 are best possible with respect to the number of vertices n of triangulations.

Let G and G' be two triangulations on a closed surface with $V(G) = \{v_1, \dots, v_n\}$ and $V(G') = \{v'_1, \dots, v'_n\}$ and suppose that

$$\deg v_1 \le \cdots \le \deg v_n; \quad \deg v'_1 \le \cdots \le \deg v'_n.$$

Then we define the *degree difference* $D(G, G')$ by:

$$D(G, G') = \sum_{i=1}^{n} |\deg v_i - \deg v'_i|$$

Theorem 4. *Let G and G' be two triangulations on a closed surface. Any sequence of diagonal flips which transforms G into G' contains at least $\frac{1}{4}D(G, G')$ diagonal flips.*

Proof. Let D_σ denote the number of diagonal flips contained in the sequence and suppose that each vertex v_i of G corresponds to a vertex $v'_{\sigma(i)}$ of G' through the sequence. We need at least $|\deg v_i - \deg v'_{\sigma(i)}|$ diagonal flips to adjust the degree of v_i while each diagonal flip changes the degrees of four vertices at the same time. Thus,

$$D_\sigma \ge \frac{1}{4} \sum_{i=1}^{n} |\deg v_i - \deg v'_{\sigma(i)}|.$$

Since the permutation σ over $\{1, \dots, n\}$ is however unknown, we have just

$$D_\sigma \ge \frac{1}{4} \min_\sigma \sum_{i=1}^{n} |\deg v_i - \deg v'_{\sigma(i)}|.$$

It is not difficult to show that $\frac{1}{4}D(G, G')$ attains the right hand of this inequality. Let $d_i = \deg v_i$ and $d'_i = \deg v'_i$, and suppose that $d_1 \leq \cdots \leq d_n$ and $d'_1 \leq \cdots \leq d'_n$. We can show the following inequality only by a routine.

$$(|d_i - d'_k| + |d_j - d'_h|) - (|d_i - d'_h| + |d_j - d'_k|) \leq 0 \quad (i < j;\ k < h)$$

For example, if $d'_k \leq d_i \leq d'_h \leq d_j$, then

$$(|d_i - d'_k| + |d_j - d'_h|) - (|d_i - d'_h| + |d_j - d'_k|) = 2(d_i - d'_h) \leq 0$$

This implies that

$$\sum_{i=1}^{n} |d_i - d'_i| \leq \sum_{i=1}^{n} |d_i - d'_{\sigma(i)}|$$

Thus, the theorem follows. ■

If we confine a pair of triangulations to the standard form Δ_{n-3} and another on the sphere, we shall obtain a more accurate lower bound, as shown in Theorem 5.

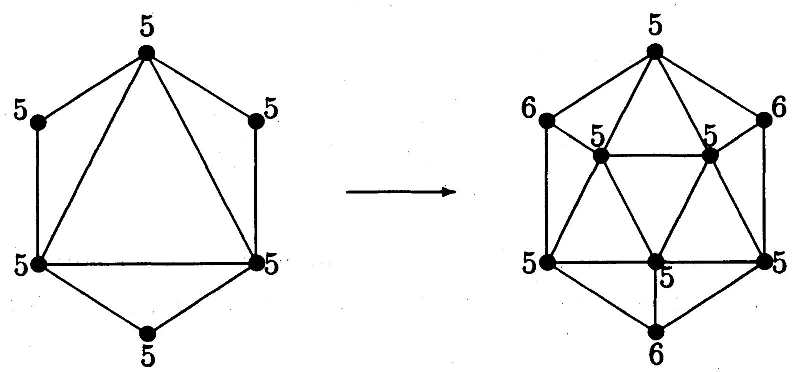

Figure 3 Making three vertices of degree 6

The standard form Δ_{n-3} has the degree sequence $(3, 3, 4, \ldots, 4, n-1, n-1)$ which contains $n-4$ 4's. On the other hand, we can construct a triangulation G_n of sufficiently large size $n = 12 + 3m$ with degree sequence (5,...,5,6,...,6) including twelve 5's and $n - 12$ 6's. For example, we can repeat the operation given in Figure 3 to construct G_n, starting with the icosahedron. (The numbers in the figure indicate the degrees of vertices.) Then we have:

$$\begin{aligned} D(\Delta_{n-3}, G_n) &= 2 \cdot (5-3) + 10 \cdot (5-4) + (n-14) \cdot (6-4) + 2 \cdot (n-1-6) \\ &= 4n - 28 \end{aligned}$$

Thus, we need at least $n - 7$ diagonal flips to transform G_n into Δ_{n-3}, by Theorem 4. But, the following thorem improve this lower bound to be $2n - 15$.

Theorem 5. *Let G be a triangulation with n vertices on the sphere. Then at least $2n-2\Delta(G)-3$ diagonal flips are needed to transform G into the standard form Δ_{n-3}, where $\Delta(G)$ denotes the maximum degree of G.*

Proof. Let v and w be the vertices of G which have degree $n-1$ in the final form Δ_{n-3} through a sequence of diagonal flips. Since $\deg v, \deg w \le n-1$ in G at the initial stage, at least $n-1-\deg v$ diagonal flips in the sequence contributes to increasing the degree of v and $n-1-\deg w$ does for w. If a diagonal flip increases the degrees of v and w at the same time, then the resulting diagonal must be vw, which is unique. Thus, the two groups of diagonal flips mentioned above include at most one diagonal flip in common and the total sequence includes at least

$$(n-1-\deg v) + (n-1-\deg w) - 1 = 2n - (\deg v + \deg w) - 3$$

diagonal flips. Replacing both $\deg v$ and $\deg w$ with $\Delta(G)$, we obtain the lower bound in the theorem. ■

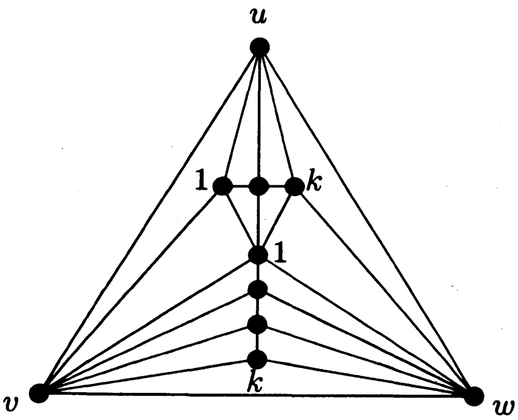

Figure 4 A triangulation T_1 on the sphere

Let T_1 be the triangulation on the sphere given in Figure 4 and let $u_1, \dots, u_h$ and $v_1, \dots, v_k$ denote the vertices lying horizontally with labels 1 to h and vertically with labels 1 to k, respectively. It is clear that precisely $2(h-1)$ diagonal flips transform T_1 to the standard form Δ_{n-3} with a vertical path $uu_1 \cdots u_h v_1 \cdots v_k$ inside uvw, where $n = h + k + 3$. If $h < k$, then we have:

$$\Delta(T_1) = k + 3 > \deg v_1 = h + 3$$

In this case, the number of diagonal flips, $2(h-1)$ is greater than the bound in Theorem 5 by one.

$$2(h-1) = 2h-2 = 2n - 2\Delta(T_1) - 2$$

Now let T_2 be the triangulations obtained from T_1 by replacing the diagonal vw with uv_k. Then $\Delta(T_2) = k+2$. The diagonal flip of uv_k transforms T_2 into T_1 and the same sequnece as used for T_1 above transforms it into Δ_{n-3}. So the number of diagonal flips in this sequence is:

$$2h-1 = 2n - 2\Delta(T_2) - 3$$

This attains the bound for T_2 in Theorem 5, but its value is bigger than that for T_1.

It is easy to construct triangulations with n vertices on the sphere whose maximum degree is 6 for $n \geq 7$, using the previous G_n with $n \geq 13$ for example. By Theorem 5, we need at least $2n-15$ diagonal flips to transform them into Δ_{n-3}. However, we have never known whether or not they attain this bound.

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Department of Computer Science and
Information Mathematics
University of Electorical Communications, 1-5-1 Chofugaoka, Chofu-Shi
Tokyo 182, Japan.
Email: hide@yebisu.im.uec.ac.jp