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APPROXIMATE MATRIX ORDER UNIT SPACES

By

ANIL KUAR KARN*AND R. VASUDEVAN

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Introduction. Kadison has shown in [4] that the $C^{\star}$-norm of a self-adjoint element in a unital $C^{\star}$-algebra is the order unit norm. Generalizing Kadison's result to non-unital $C^{\star}$-algebras Ng [6] introduced approximate order unit spaces and has shown that the $C^{\star}$-norm of a self-adjoint element in any $C^{\star}$-algebra is the approximate order unit norm. In [2] Choi and Effros introduced matrix order unit spaces and extended Kadison's result to an arbitrary element of a unital $C^{\star}$-algebra. We introduce approximate matrix order unit spaces. It is shown that any $C^{\star}$-algebra is an approximate matrix order unit space and hence the $C^{\star}$-norm of any element in an arbitrary $C^{\star}$-algebra is the approximate matrix order unit norm.

In Section I, we define matricially normed spaces (mn spaces) and matrix order spaces and prove some properties of theses spaces. In Section II, we introduce matricially Riesz normed spaces (mRn spaces) and approximate matrix order unit spaces (amou spaces) and characterized them. We prove that every $C^{\star}$-algebra is amou space. We also generalized a theorem due to Klee [5] in the matrix order context. For (real) ordered vector spaces we have followed [3] and [11].

Section I

1.1. Throughout this paper, $M_{n,m}$ denotes the space of $n\times m$ matirices of complex numbers. We write M_n for $M_{n,n}$ for all $n\in \mathsf{N}$, and identify it with $B(\mathsf{C}^n)$, whenever norm or order is considered.

A complex vector space V, gives rise to a complex vector space $M_n(V)$, whose elements are $n\times n$ matrices with entries from V, for all $n\in \mathsf{N}$.]Furthermore, for $[\alpha_{i,j}]\in M_n$ and $[v_{i,j}]\in M_n(V)$ we define

$$[\alpha_{i,j}][v_{i,j}]=\left[\sum_{k=1}^{n}\alpha_{i,k}v_{k,j}\right]\quad\text{and}\quad[v_{i,j}][\alpha_{i,j}]=\left[\sum_{k=1}^{n}\alpha_{k,j}v_{i,k}\right]$$

so that $M_n(V)$ becomes a two-sided M_n-bimodule for all $n\in \mathsf{N}$. Also, for

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$v \in V_n(V)$, $w \in M_m(V)$ we define

$$v \oplus w = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \in M_{n+m}(V), \quad n, m \in \mathsf{N}.$$

If $\| \cdot \|_n$ is a (semi) norm on $M_n(V)$ for all $n \in \mathsf{N}$, we say $\{\| \cdot \|_n\}$ is a matrix (semi) norm on V.

A *metrically (semi) normed space* ($m(s)n$ *space*) is a complex vector space V together with a matrix (semi) norm $\{\| \cdot \|_n\}$ satisfying the following properties

(i) $\| v \oplus 0 \|_{n+m} = \| v \|_n$ and
(ii) $\| \alpha v \|_n \| v\alpha \|_n \leq \| \alpha \| \| v \|_n$.

If $v \in M_n(V)$, $0 \in M_m(V)$, $\alpha \in M_n$ and $n, m \in \mathsf{N}$. For $1 \leq p < \infty$ we say that $m(s)n$ space $(V, \{\| \cdot \|_n\})$ is an L^p matricially (semi) norm space (L^p $m(s)n$ space) if $L^p :\| v \oplus w \|_{m+n}^p = \| v \|_n^p + \| w \|_m^p$ where $v \in M_n(V)$, $w \in M_m(V)$ and $m, n \in \mathsf{N}$.

We say that $m(s)n$ space $(V, \{\| \cdot \|\}$ is an L^∞ matricially (semi) normed space (L^∞ $m(s)n$ space) if $L^\infty :\| v \oplus w \|_{n+m} = \max(\| v \|_n, \| w \|_m)$ where $v \in M_n(V)$, $w \in M_m(V)$ and $m, n \in \mathsf{N}$, [9, 10].

We note that in an $m(s)n$ space $(V, \{\| \cdot \|_n\})$, $\| \alpha v \|_n = \| v \|_n$ if $v \in M_n(V)$ and $\alpha \in M_n$ is unitary. Furthermore, if $v = [v_{i,j}] \in M_n(V)$ then

$$\| v_{i,j} \|_1 \leq \| v \|_n \leq \sum_{\ell,k=1}^{n} \| v_{\ell,k} \|_1,$$

[9], so that

(a) if $\| \cdot \|_1$ is a norm on V then $\| \cdot \|_n$ is a norm on $M_n(V)$ for all $n \in \mathsf{N}$,
(b) if $(V, \| \cdot \|_1)$ is complete, then $(M_n(V), \| \cdot \|_n)$ is complete for all $n \in \mathsf{N}$.

1.2. Given complex vector spaces V and W and linear map $\phi : V \to W$, we define $\phi_n : M_n(V) \to M_n(W)$ given by $\phi_n([v_{i,j}]) = [\phi(v_{i,j})]$ for every $[v_{i,j}] \in M_n(V)$, $n \in \mathsf{N}$. Assuming $(V, \{\| \cdot \|_n\})$ and $(W, \{\| \cdot \|_n\})$ to be $m(s)n$ spaces for a linear map $\phi : V \to W$ we define

$$\| \phi \|_{cb} = \sup\{\| \phi_n \| : n \in \mathsf{N}$$

and say, ϕ is *completely bounded* if $\| \phi \|_{cb} < \infty$, ϕ is *completely contractive* if $\| \phi \|_{cb} \leq 1$, and ϕ is *completely isometry* if ϕ_n is an isometry for all $n \in \mathsf{N}$.

1.3. Given a complex vector space V and a dual pair $\langle V, V^d \rangle$, we define, for each $n \in \mathsf{N}$, and $[v_{i,j}] \in M_n(V)$, $[f_{i,j}] \in M_n(V^d)$

$$\langle [v_{i,j}], [f_{i,j}] \rangle = \sum_{i,j=1}^{n} \langle v_{i,j}, f_{i,j} \rangle,$$

[2]. Then, $\langle M_n(V), M_n(V^d)\rangle$ is a dual pair for every $n \in \mathsf{N}$. We shall call this duality to be the *matrix duality* of $\langle V, V^d\rangle$.

In particular, if $(V, \{\|\cdot\|_n\})$ is an $m(s)n$ space and if $(V, \|\cdot\|_1')$ is the Banach dual of $(V, \|\cdot\|_1)$, then giving $M_n(V')$ the dual norm $\|\cdot\|_n'$ of $(M_n(V), \|\cdot\|_n)$ for all $n \in \mathsf{N}$ we get that $(V', \{\|\cdot\|'\})$ is also $m(s)n$ space [9]. If, in addition, $(V, \{\|\cdot\|_n\})$ satisfies L^p-condition $(1 \leq p \leq \infty)$ then $(V', \{\|\cdot\|_n'\})$ satisfies L^q-condition $((1/p) + (1/q) = 1)$ [9]'. $(V', \{\|\cdot\|_n'\})$ will be called the *matrix Banach dual* of $(V, \{\|\cdot\|_n\})$.

1.4. A complex vector space with an involution will be called a $\star$-vector space and the involution will be denoted by "$\star$".

Given a $\star$-vector space V, we define $[v_{i,j}]^\star = [v_{i,j}^\star]$ for every $[v_{i,j}] \in M_n(V)$ so that $M_n(V)$ is also a $\star$-vector space for all n. The real vector space of all self-adjoint elements of $M_n(V)$ will be denoted by $M_n(V)_{sa}$ for all n.

A *cone* in a (real) vector space V is a convex subset C for which $lC \subset C$ for every $l \geq 0$. It is a well known fact that there is a one to one correspondence between the family of cones and the set of vector orderings in a given real vector space [11].

A *complex ordered vector space* is a $\star$-vector space V together with a cone V^+ in V_{sa}.

A *matrix ordered space* is a $\star$-vector space V together with a cone $M_n(V^+)$ in $M_n(V)_{sa}$ for all $n \in \mathsf{N}$ and with the following property: if $v \in M_n(V)^+$ and $\gamma \in M_{n,m'}$ then $\gamma^\star \vee \gamma = M_m(V)^+$ for any $n, m \in \mathsf{N}$.

1.5. Given $\star$-vector spaces V and W, and a linear map $\phi : V \to W$, we define $\phi^\star(v) = \phi(v^\star)^\star$ for every $v \in V$. Then $\phi^\star$ is also a linear map of V into W. We say ϕ is self-adjoint if $\phi^\star = \phi$ or equivalently, $\phi(V_{sa}) \subset W_{sa}$.

In general, $(\phi_n)^\star = (\phi^\star)_n$, so that if ϕ is self-adjoint, then ϕ_n is self-adjoint for every $n \in \mathsf{N}$.

For complex ordered vector spaces (V, V^+) and (W, W^+) a linear map $\phi : V \to W$ is called *positive* if it is self-adjoint and $\phi(V^+) \subset W^+$.

For matrix ordered spaces $(V, \{M_n(V)^+\})$ and $(W, \{M_n(W)^+\})$ a linear map $\phi : V \to W$ is called *completely positive* if ϕ_n is positive for all $n \in \mathsf{N}$.

1.6. With the natural involution $[\alpha_{i,j}]^\star = [\overline{\alpha_{j,i}}]$ in M_n and the natural cone M_n^+, in $(M_n)_{sa}$, M_n is a complex ordered vector space for all $n \in \mathsf{N}$ and $(\mathsf{C}, \{M_n^+\})$ is a matrix ordered space. Thus, replacing $(W, \{M_n(W)^\star\})$ by $(\mathsf{C}, \{M_n^\star\})$ it can be easily seen [2] that

(i) algebraic dual of a $\star$-vector V is a $\star$-vector space $V^\star$,

(ii) algebraic dual of a complex ordered vector space (V, V^+) is a complex ordered vector space $(V^\star, (V^\star)^+)$ where $(V^\star)^+ = \{f \in (V^\star)_{sa} : f(V^+)$

$\geq 0\}$,

(iii) algebraic matrix dual of a matrix ordered space $(V, \{M_n(V)^+\})$ is a matrix ordered space $(V^\star, \{M_n(V^\star)^+\})$.

Let us say, for a $\star$-vector space V and a dual pair $\langle V, V^d\rangle$, that V^d is self-adjoint if $f^\star \in V^d$ whenever $f \in V^d$. In this notation, by considering V^d as a self-adjoint subspace of $V^\star$, $V^\star$ may be replaced by V^d in (i), (ii) and (iii) [2]. $(V^d, \{M_n(V^d)^+\})$ will be called a *matrix ordered dual* of $(V, \{m_n(V)^+\})$.

For example, let $(V, \{\|\cdot\|_n\})$ be an $m(s)n$ space with a matrix order $\{M_n(V)^+\}$ and suppose that $\star$ is an isometry in $(M_n(V), \|\cdot\|_n)$ for all $n \in \mathbb{N}$. Then the matrix Banach dual $(V', \{\|\cdot\|'_n\})$ is also a matrix ordered space in the *dual matrix order* $\{M_n(V')^+\}$, and $\star$ is an isometry in $(M_n(V'), \|\cdot\|'_n)$ for all $n \in \mathbb{N}$. The triple $(V', \{\|\cdot\|'_n\}, \{M_n(V'))^+\})$ is called the *matrix ordered Banach dual* (or simple *matrix Banach dual*, if there is no confusion) of $(V, \{\|\cdot\|_n\}, \{M_n(V)^+\})$.

1.7. In a matrix ordered space $(V, \{\|\cdot\|_n \; (V)^+\})$ we define the following notations:

(a) V^+ is called *proper* if $V^+ \cap (-V^+) = \{0\}$,

(b) V^+ is called *generating* if for any $v \in V$ there is $u \in V^+$ such that

$$\begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+,$$

(c) V^+ is called *Archimedean* if for any $v \in V_{sa}$ and some $u \in V^+$ we have $v \in V^+$ whenever $ku + v \in V^+$ for all $k > 0$.

(d) V^+ is called *almost-Archimedean* if for any $v \in V$ and for some $u \in V^+$ we have $v = 0$ whenever $\begin{pmatrix} ku & v \\ v^\star & ku \end{pmatrix} \in M_2(V)^+$ for all $k > 0$.

1.8. Proposition. *In a matrix ordered space $(V, \{M_n(V)^+\})$ the following properties hold.*

(i) *Let $v \in V$, $u \in V^+$ and suppose* $\begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+$ *, then*

$$\begin{pmatrix} u & v^\star \\ v & u \end{pmatrix} \in M_2(V)^+ \quad \text{and} \quad \begin{pmatrix} u & e^{i\theta}v \\ e^{-i\theta}v^\star & u \end{pmatrix} \in M_2(V)^+$$

for every $\theta \in [0, 2\pi]$. In particular, $u \pm \Re v \in V^+$, $u \pm \Im v \in V^+$. Moreover if $v = v^*$, then

$$\begin{pmatrix} u & v \\ v & u \end{pmatrix} \in M_2(V)^+$$

if and only if $u \pm v \in V^+$. *Thus if* $v \in V^+$ *then*

$$\begin{pmatrix} u & v \\ v & u \end{pmatrix} \in M_2(V)^+$$

if and only if $u - v \in V^+$.

(ii) V^+ *is proper if and only if* $v = 0$ *whenever* $v \in V$ *and*

$$\begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} \in M_2(V)^+.$$

(iii) V^+ *is generating if and only if for every* $v \in V$ *there are* $v_0, v_1, v_2, v_3 \in V^+$ *such that*

$$v = \sum_{k=0}^{3} i^k v_k.$$

(iv) *If* V^+ *is proper, then* $M_n(V)^+$ *is proper for all* n.

(v) *If* V^+ *is generating then* $M_n(V)^+$ *is generating for all* n. *In this case, we say* $(V, \{M_n(V)^+\})$ *is positively generated.*

(vi) *If* V^+ *is almost-Archimedean, it is proper. If* V^+ *is proper and Archimedean it is almost-Archimedean.*

Next, assume that $(V^d, \{M_n(V^d)^+\})$ *is a matrix dual of* $(V, \{M_n(V)^+\})$, *then we have*

(vii) V^+ *is weakly closed if and only if* $V^+ = \{v \in V_{sa} : f(v) \geq 0 \text{ for every} f \in (V^d)^+\}$. *In this case,* V^+ *is Archimedean. Thus,* $M_n(V^d)^+$ *is Archimedean for all* $n \in \mathbb{N}$.

(viii) *If* V^+ *is proper, then* $(V^d)^+$ *is proper.*

(ix) *If* V^+ *is proper then* $(V^d)^+$ *is generating and* V^+ *is weakly closed then* V^+ *is proper.*

Proof. (i) Let $v \in V$ and $u \in V^+$ and suppose that $\begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+$. Then

$$\begin{pmatrix} u & v^\star \\ v & u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(V)^+.$$

If $\theta \in [0, 2\pi]$, then

$$\begin{pmatrix} u & e^{i\theta} v \\ e^{-i\theta} v^\star & u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}^\star \begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in M_2(V)^+.$$

Next,

$$\begin{pmatrix} u & \Re v \\ \Re v & u \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u & v \\ v^\star & u \end{pmatrix}^\star + \frac{1}{2} \begin{pmatrix} u & v^\star \\ v & u \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(V)^+.$$

Thus,

$$u + \Re v = \frac{1}{2}(1\ 1)\begin{pmatrix} u & \Re v \\ \Re v & u \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V^+,$$

and

$$u - \Re v = \frac{1}{2}(1\ -1)\begin{pmatrix} u & \Re v \\ \Re v & u \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \in V^+.$$

Letting $\theta = \pi/2$, we have

$$\begin{pmatrix} u & iv^\star \\ -iv & u \end{pmatrix} \in M_2(V)^+, \quad \text{as} \begin{pmatrix} u & v^\star \\ v & u \end{pmatrix} \in M_2(V)^+,$$

so that

$$\begin{pmatrix} u & iv \\ iv^\star & u \end{pmatrix} \in M_2(V)^+$$

and hence

$$\begin{pmatrix} u & \Im v \\ \Im v & u \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u & iv \\ iv^\star & u \end{pmatrix} + \frac{1}{2}\begin{pmatrix} u & iv^\star \\ -iv & u \end{pmatrix} \in M_2(V)^+.$$

Thus, as above $u \pm \Im v \in V^+$.

If $v = v^\star$, then $\begin{pmatrix} u & v \\ v & u \end{pmatrix} \in M_2(V)^+$ implies $u \pm v \in V^+$. Conversely, let $u \pm v \in V^+$ with $u \in V^+$, $v \in V_{sa}$. Then,

$$\begin{pmatrix} u & v \\ v & u \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(u+v)(1\ 1) + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(u-v)(1\ -1) \in M_2(V)^+.$$

If $v \in V^+$ then $u + v \in V^+$. Hence

$$\begin{pmatrix} u & v \\ v & u \end{pmatrix} \in M_2(V)^+ \quad \text{if and only if } u - v \in V^+.$$

(ii) Let V^+ be proper and suppose $\begin{pmatrix} 0 & v \\ v^\star & 0 \end{pmatrix} \in M_2(V)^+$. Then by (i) $0 \pm \Re v \in V^+$, $0 \pm \Im v \in V^+$. Hence $\Re v = 0 = \Im v$ or equivalently $v = 0$. Next, let $v = 0$, whenever $v \in V$ and $\begin{pmatrix} 0 & v \\ v^\star & 0 \end{pmatrix} \in M_2(V)^+$ and suppose $\pm v \in V^+$ for some $v \in V_{sa}$, then by (i) $\begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} \in M_2(V)^+$ and whence $v = 0$ and V^+ is proper.

(iii) Let V^+ be generating and suppose $v \in V$. Then there is $u \in V^+$ such that $\begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+$, so by (i), $u \pm \Re v \in V^+$, $u \pm \Im v \in V^+$. Putting $v_0 = \frac{1}{2}(u + \Re v)$, $v_1 = \frac{1}{2}(u + \Im v)$, $v_2 = \frac{1}{2}(u - \Re v)$, $v_3 = \frac{1}{2}(u - \Im v)$ we get

$$v = \sum_{k=0}^{3} i^k v_k,$$

where $v_k \in V^+, k = 0, 1, 2, 3$.

Next, to prove the converse of (iii), let $v \in V$ and suppose that there are $v_k \in V^+$ $k = 0, 1, 2, 3$, such that

$$v = \sum_{k=0}^{3} i^k v_k,$$

Then, $\Re v = v_0 - v_2$, $\Im v = v_1 - v_3$, and $v_0 + v_2 \pm \Re v$, $v_1 + v_3 \pm \Im v \in V^+$ so that

$$\begin{aligned}\begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(v_0 + v_2 + \Re v)(1\ 1) + \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}(v_0 + v_2 - \Re v)(1\ \ -1) \\ &+ \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(v_1 + v_3 + \Im v)(1\ 1) + \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}(v_1 + v_3 - \Im v)(1\ \ -i) \\ &\in M_2(V)^+\end{aligned}$$

where $u = v_1 + v_2 + v_3 \in V^+$.

(iv) Proved in [2] as a part of Theorem 4.4.

(v) Assume that V^+ is generating. Fix $n \in \mathsf{N}$ and let $v = [v_{i,j}] \in M_n(V)$. Then, for each pair (i, j), $1 \leq i, j \leq n$, there is $[u_{i,j}] \in V^+$ such that

$$\begin{pmatrix} u_{i,j} & v_{i,j} \\ v_{i,j}^\star & u_{i,j} \end{pmatrix} \in M_2(V)^+.$$

Consider $\gamma_{i,j} \in M_{2,2n}$ with entries 1 at $(1, i)$th and $(2, n + j)$th places and 0 elsewhere for all $i, j = 1, \cdots, n$. Then

$$\sum_{i,j=1}^{n} \gamma_{i,j}^\star \begin{pmatrix} u_{i,j} & v_{i,j} \\ v_{i,j}^\star & u_{i,j} \end{pmatrix} \gamma_{i,j} \in M_2(V)^+.$$

Since

$$u_i = \sum_{j=1}^{n} u_{i,j}, \quad u^j = \sum_{i=1}^{n} u_{i,j} \in V^+$$

for $i, j = 1, \cdots, n$, we have

$$u = \sum_{i=1}^{n} \delta_i^\star u_i \delta_i, \qquad u' = \sum_{j=1}^{n} \delta_j^\star u^j \delta_j \in M_n(V)^+$$

where $\delta_i \in M_{1,n}$ with entries 1 at $(1,i)$th place and 0 elsewhere. In these notations, we have

$$\begin{pmatrix} u & v \\ v^\star & u' \end{pmatrix} = \sum_{i,j=1}^{n} \gamma_{i,j}^\star \begin{pmatrix} u_{i,j} & v_{i,j} \\ v_{i,j}^\star & u_{i,j} \end{pmatrix} \gamma_{i,j} \in M_{2n}(V)^+.$$

Therefore, $\begin{pmatrix} u+u' & v \\ v^\star & u+u' \end{pmatrix} \in M_{2n}(V)^+$, for

$$\begin{pmatrix} u' & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} I_n \\ 0_n \end{pmatrix} (u')(I_n \quad 0_n) + \begin{pmatrix} 0_n \\ I_n \end{pmatrix} (u)(0_n \quad I_n) \in M_{2n}(V)^+.$$

Since $v \in M_n(V)$ and $n \in \mathsf{N}$ are arbitrary, $M_n(V)^+$ is generating for every $n \in \mathsf{N}$.

(vi) Let V^+ be almost-Archimedean and suppose $\begin{pmatrix} 0 & v \\ v^\star & 0 \end{pmatrix} \in M_{2n}(V)^+$, for some $v \in V$. Since $0 \in V^+$ and $k0 = 0$ for all $k > 0$, by the hypothesis $v = 0$. Hence by (ii) V^+ is proper. Next, let V^+ be proper and Archimedean. Suppose that for some $v \in V$, $\begin{pmatrix} ku & v \\ v^\star & ku \end{pmatrix} \in M_2(V)^+$ for all $k > 0$ and a fixed $u \in V^+$. Then by (i) $ku \pm \Re v$, $ku \pm \Im v \in V^+$ for all $k > 0$. Hence, by Archmedean property, $\pm\Re v$, $\pm\Im v \in V^+$. Since V^+ is proper, $\Re v = 0 = \Im v$ whence $v = 0$ so V^+ is almost-Archimedean.

(vii) Let V^+ be weakly closed. If $v \in V^+$ then $f(v) \geq 0$ for every $f \in (V^d)^+$. Suppose that $v \notin V^+$ with $v \in V_{sa}$. Since V^d is self-adjoint $(V_{sa})^d$ may be identified with $(V^d)_{sa}$. Thus, by the Hahn-Banach separation theorem there are $f \in (V^d)_{sa}$ and $\alpha \in \mathsf{R}$ such that $f(v) < \alpha \leq f(V^+)$. Since V^+ is a cone we have $\alpha = 0$ so that $f \in (V^d)^+$ and $f(v) < 0$. Hence

$$V^+ = \{v \in V_{sa} : f(v) \geq 0 \text{ for all } f \in (V^d)^+\}.$$

Next, assume that $V^+ = \{v \in V_{sa} : f(v) \geq 0 \text{ for all } f \in (V^d)^+\}$. Suppose that v lies in the weak closure of V^+. Since V_{sa} is weakly closed, $v \in V_{sa}$. Also we get a net $\{v_l\}$ in V^+ such that $v_l \to v$ weakly. But then for any $f \in (V^d)^+$ we have

$$f(v) = \lim_l f(v_l) \geq 0$$

so that $v \in V^+$, by the hypothesis. Thus, V^+ is weakly closed.

Next, suppose that V^+ is weakly closed. Let $v \in V_{sa}$ be such that $u \in V^+$, $ku + v \in V^+$ for all $k > 0$. Let $f \in (V^d)^+$. Then $kf(u) + f(v) \geq 0$ for all $k > 0$. Since $f(u) > 0$, $f(v) \in \mathsf{R} = \mathsf{C}_{sa}$ and $\mathsf{C}^+ = \mathsf{R}^+$ is Archimedean, we get $f(v) \geq 0$. Since $f \in (V^d)^+$ is arbitrary and V^+ is weakly closed, by the first part of (vii), $v \in V^+$ and hence V^+ is Archimedean. The last part of (vii) follows from the definition of $M_n(V^d)^+$ and the second part of (vii), for all $n \in \mathsf{N}$.

(viii) First, let V^+ be generating and suppose that $\pm f \in (V^d)^+$. Then for every $v \in V^+$, $f(v) = 0$. Since V^+ is generating so by (iii) $f(v) = 0$ for every $v \in V$ whence $f = 0$. Therefore $(V^d)^+$ is proper.

To prove the converse, let V^+ not be generating. Since $V = V_{sa} + iV_{sa}$ we have that

$$\begin{aligned}(V_1)_{sa} &= \left\{ v \in V_{sa} : \begin{pmatrix} u & v \\ v^* & u \end{pmatrix} \in M_2(V)^+ \text{ for some } u \in V^+ \right\} \\ &= \left\{ v \in V_{sa} : u \pm v \in V^+ \text{ for some } u \in V^+ \right\}\end{aligned}$$

is a proper subspace of V_{sa}. Let $v \in V_{sa}$ be such that $v \notin (V_1)_{sa}$. Then, we can find $f \in V^d$ such that $f(v) \neq 0 = f((V_1)_{sa})$. Since $V^d = (V^d)_{sa} + i(V^d)_{sa}$ we may assume that $f \in (V^d)_{sa}$. Then $\pm f \in (V^d)^+$, for $V^+ \subset (V_1)_{sa}$.

Since $f(v) \neq 0$, $f \neq 0$ and consequently, $(V^d)^+$ is not proper.

Finally, using arguments given in the proof (viii), we can prove (ix). □

Section II

2.1. Let $(V, \{M_n(V)^+\})$ be a positively generated matrix ordered space. A (semi) norm $\|\cdot\|$ on V is called a *Riesz (semi) norm* if for each $v \in V$ we have

$$\| v \| = \inf \left\{ \| u \| : u \in V^+ \text{ and } \begin{pmatrix} u & v \\ v^* & u \end{pmatrix} \in M_2(V)^+ \right\}.$$

In this case, $\| v^* \| = \| v \|$ for all $v \in V$ and $\| v \| \leq \| u \|$ if $0 \leq v \leq u$.

A matrix (semi) norm $\{\| \cdot \|_n\}$ on V will be called a *matrix Riesz (semi) norm* if $\| \cdot \|_n$ is a Riesz (semi) norm on $M_n(V)$ for every $n \in \mathsf{N}$. An (L^p-) matricially Riesz (semi) normed space ((L^p-) $mR(s)n$ space) ($1 \leq p \leq \infty$) is an (L^p-)$m(s)n$ space $(V, \{\| \cdot \|_n\})$ together with a matrix order $\{M_n(V)^+\}$ such that $(V, \{M_n(V)\})$ is positively generated and $\{\| \cdot \|_n\}$ is a matrix Riesz (semi) norm on V.

2.2. Let $(V, \{M_n(V)\})$ be a matrix ordered space. For $A \subset V$ we define

$$S(A) = \left\{ v \in V : \begin{pmatrix} u & v \\ v^* & u \end{pmatrix} \in M_2(V)^+ \text{for some } u \in A \cap V^+ \right\}.$$

If $A \cap V^+ \neq \emptyset$ then $S(A)$ is circled and self-adjoint subset of V. If, in addition, A is convex then so is $S(A)$. Note that if V^+ is generating $S(V^+) = V$.

We say, $A \subset V$ is solid if $S(A) = A$. In this terminology we have the following result.

2.3. Lemma. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. If V^+ is generating and $\| \cdot \|$ is a Riesz (semi) norm on V, then the open unit ball of $(V, \| \cdot \|)$ is solid. Conversely, if $\| \cdot \|$ is a (semi) norm on V and the open unit ball of $(V, \| \cdot \|)$ is solid, then V^+ is generating and $\| \cdot \|$ is a Riesz (semi) norm on V.*

Proof. First, let V^+ be generating and $\| \cdot \|$ a Riesz (semi) norm on V. Let U denote the open unit ball of $(V, \| \cdot \|)$. We show that $S(U) = U$.

Let $v \in S(U)$. Then, there is $u \in U \cap V^+$ such that $\begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+$. Thus, by the definition of a Riesz (semi) norm, we have $\| v \| \leq \| u \| < 1$ so that $v \in U$. If $v \in U$, we can find $\epsilon > 0$ such that $\| v \| < 1 - \epsilon$. Then there is $u \in V^+$ with $\| u \| < 1 - \epsilon/2$ such that $\begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+$. Thus $v \in S(U)$. Hence, $S(U) = U$.

Next, we assume that $S(U) = U$. We show that V^+ is generating and $\| \cdot \|$ is a Riesz (semi) norm on V. Let $v \in V$. Then for any $\epsilon > 0$, $(\| v \| + \epsilon)^{-1} v \in U = S(U)$, so there is $u \in U \cap V^+$ such that

$$\begin{pmatrix} u & (\| v \| + \epsilon)^{-1} v \\ (\| v \| + \epsilon)^{-1} v^\star & u \end{pmatrix} \in M_2(V)^+$$

or equivalently

$$\begin{pmatrix} (\| v \| + \epsilon) u & v \\ v^\star & (\| u \| + \epsilon) u \end{pmatrix} \in M_2(V)^+,$$

whence V^+ is generating. Let $u_1 = (\| v \| + \epsilon) u \in V^+$. We get $\begin{pmatrix} u_1 & v \\ v^\star & u_1 \end{pmatrix} \in M_2(V)^+$ with $\| u_1 \| < \| v \| + \epsilon$. Thus

$$\| v \| \geq \inf \left\{ \| u \| : u \in V^+ \text{ and } \begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+ \right\}.$$

Also if $\begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+$ with for any $u \in V^+$ then for given $\epsilon > 0$,

$$u_1 = (\| u \| + \epsilon)^{-1} u \in U \cap V^+$$

and

$$\begin{pmatrix} u_1 & (\| u \| +\epsilon)^{-1}v \\ (\| u \| +\epsilon)^{-1}v^\star & u_1 \end{pmatrix} \in M_2(V)^+$$

so that $(\| u \| +\epsilon)^{-1}v \in S(U) = U$ or equivalently, $\| v \| < \| u \| +\epsilon$. Since $\epsilon > 0$ is arbitrary $\begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+$ impies $\| v \| \leq \| u \|$ whence

$$\| v \| \leq \inf \left\{ \| u \| : u \in V^+ \text{ and } \begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+ \right\}.$$

Therefore $\| \cdot \|$ is a Riesz (semi) norm on V, and the lemma is proved. □

2.4. Next theorem shows an important and usefull property of cones in $nR(s)n$ spaces. This is a generalization of Klee's theorem [5, 11] in the context of $mR(s)n$ spaces.

Theorem. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and suppose that $\| \cdot \|$ is a Riesz semi norm on V. If V^+ is $\| \cdot \|$-complete, then $(V, \| \cdot \|)$ is a Banach space.*

Proof. Let V^+ be $\| \cdot \|$-complete. Suppose $\{v_n\}$ be a Cauchy sequence in V. Then

$$\| v_n^\star - v_m^\star \| = \| v_n - v_m \|$$

for all $n, m \in \mathsf{N}$. Thus $\{v_n^\star\}$ and consequently $\{\Re v_n\}$, $\{\Im v_n\}$ are all Cauchy sequences in V. Hence we may assume that $v_n = v_n^\star$ for every $n \in \mathsf{N}$. For each $k \in \mathsf{N}$ we can find v_{n_k} in $\{v_n\}$ such that $n_1 \leq n_2 \leq \cdots$ and

$$\left\| v_{n_{k+1}} - v_{n_k} \right\| < \frac{1}{2^k}$$

for all $k \in \mathsf{N}$.

Let U denote the open unit ball of $(V, \| \cdot \|)$. Then

$$v_{n_{k+1}} - v_{n_k} \in \frac{1}{2^k}U = S\left(\frac{1}{2^k}U\right)$$

for all k. Thus, for each $k \in \mathsf{N}$ there is $u_k \in 2^{-k}U \cap V^+$ such that

$$\begin{pmatrix} u_k & v_{n_{k+1}} - v_{n_k} \\ v_{n_{k+1}} - v_{n_k} & u_k \end{pmatrix} \in M_2(V)^+$$

or equivalently $u_k \pm (v_{n_{k+1}} - v_{n_k}) \in V^+$ for all k. Put

$$x_p = \sum_{k=1}^{p} u_k \in V^+$$

for all $p \in \mathsf{N}$. Then

$$\| x_{p+r} - x_p \| = \| u_{p+r} + \cdots + u_{p+1} \| < \frac{1}{2^p}$$

for all $r \in \mathsf{N}$ so that $\{x_p\}$ is Cauchy in V^+. Since V^+ is complete $\{x_p\}$ converges in V^+ so that $u = \sum_{k=1}^{\infty} u_k$ exists in V^+. Also

$$\left(\sum_{k=1}^{p} u_k\right) \pm \left(v_{n_{p+1}} - v_{n_1}\right) \in V^+$$

for all p. Hence, in particular,

$$\left\{u + v_{n_{p+1}} - v_{n_1}\right\}_{p=1}^{\infty} \subset V^+.$$

Since

$$\left(u + v_{n_{p+1}} - v_{n_1}\right) - \left(u + v_{n_{q+1}} - v_{n_1}\right) = v_{n_{p+1}} - v_{n_{q+1}},$$

$\{u + v_{n_{p+1}} - v_{n_1}\}$ is Cauchy in V^+. Again, using the completeness of V^+

$$u + v_{n_{p+1}} - v_{n_1} \to u' \in V^+$$

and hence we get $v \in V_{sa}$ such that $v_{n_{p+1}} \to v$ for some $v \in V_{sa}$ or equivalently $v_n \to v$. Hence $(V, \|\cdot\|)$ is complete. $\square$

2.5. Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. An increasing net $\{e_l\}_{l\in D}$ in V^+ is called an *approximate order unit* for V if given $v \in V$ there are $l \in D$ and $\alpha > 0$ such that $\begin{pmatrix} \alpha e_l & v \\ v^* & \alpha e_l \end{pmatrix} \in M_2(V)^+$. If $v = v^*$ then by 1.8 (i) $\begin{pmatrix} \alpha e_l & v \\ v & \alpha e_l \end{pmatrix} \in M_2(V)^+$ if and only if $\alpha e_l \pm v \in V^+$. Thus our definition generalizes the definition of an approximate order unit for real ordered vector spaces [11].

Further, note that if V has an approximate order unit, then it is positively generated. We have more to say,

2.6. Lemma. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. If V has an approximate order unit, then $M_n(V)$ has an approximate order unit for every $n \in \mathsf{N}$.*

Proof. Let $\{e_l\}_{l\in D}$ be an approximate order unit for V. We show that $\{e_l^n\}_{l\in D}$ is an approximate order unit for $M_n(V)$ for all $n \in \mathsf{N}$, where $e_l^n = \begin{pmatrix} e_l & v \\ v & e_l \end{pmatrix}$, $l \in D$. Let $\delta_i \in M_{1,n}$ with 1 at $(1,i)$th place and 0 elsewhere. Then

$$e_l^n = \sum_{i=1}^n \delta_i^\star e_l \delta_i \;\in\; M_n(V)^+$$

for every l and $\{e_l^n\}$ is an increasing net in $M_n(V)^+$. Let $v = [v_{i,j}] \in M_n(V)$. Then for each pair (i,j) there are $l_{i,j} \in D$ and $\alpha_{i,j} > 0$ such that

$$\begin{pmatrix} \alpha_{ij} e_{lij} & v_{ij} \\ v_{ij}^\star & \alpha_{ij} e_{lij} \end{pmatrix} \in M_2(V)^+.$$

Since D is directed, there is $l \in D$ such that $l_{ij} \le l$ for all $i,j = 1,\cdots,n$. Thus

$$\begin{pmatrix} \alpha_{ij} e_l & v_{ij} \\ v_{ij}^\star & \alpha_{ij} e_l \end{pmatrix} \in M_2(V)^+ \quad \text{for all } i,j = 1,\cdots n.$$

Then using techniques of the proof of 1.8 (v) and putting

$$\alpha \;=\; \sum_{i=1}^n \alpha_{ij} + \sum_{j=1}^n \alpha_{ij} \;>\; 0$$

we get

$$\begin{pmatrix} \alpha e_l^n & v \\ v^\star & \alpha e_l^n \end{pmatrix} \in M_{2n}(V)^+.$$

Hence $\{e_l^n\}$ is an approximate order unit for $M_n(V)$. Since $n \in \mathsf{N}$ is arbitrary, the proof is complete. □

2.7. Let $(V, \{M_n(V)^+\})$ be a matrix ordered space, and suppose that V has an approximate order unit $\{e_l\}_{l\in D}$. For each $v \in V$, we define

$$\| v \|^a = \inf\left\{\alpha > 0 : \begin{pmatrix} \alpha e_l^n & v \\ v^\star & \alpha e_l^n \end{pmatrix} \in M_2(V)^+ \text{ for some } l \in D\right\}.$$

Then, $\|\cdot\|^a$ is a semi norm on V. We show that $\|\cdot\|^a$ is a Riesz semi norm on V.

First we note that $\| e_l \|^a \le 1$ for all l.

Now, let $v \in V$. If $\begin{pmatrix} u & v \\ v^\star & u \end{pmatrix} \in M_2(V)^+$. Then for any $\epsilon > 0$ there exists $l \in D$ such that

$$\begin{pmatrix} (\| u \|^a + \epsilon) e_l & u \\ u & (\| u \|^a + \epsilon) e_l \end{pmatrix} \in M_2(V)^+$$

or equivalently $(\| u \|^a + \epsilon)e_l - u \in V^+$ by 1.8 (i). Thus

$$\begin{pmatrix} (\| u \|^a + \epsilon)e_l & v \\ v^* & (\| u \|^a + \epsilon)e_l \end{pmatrix} \in M_2(V)^+$$

whence $\| v \|^a \leq \| u \|^a + \epsilon$. Since $\epsilon > 0$ is arbitrary,

$$\| v \|^a \leq \inf \left\{ \| u \|^a : \; u \in V^+ \text{ and } \begin{pmatrix} u & v \\ v^* & u \end{pmatrix} \in M_2(V)^+ \right\}.$$

Next, let $k > \| v \|^a$. Then, there is $l \in D$ such that $\begin{pmatrix} ke_l & v \\ v^* & ke_l \end{pmatrix} \in M_2(V)^+$. Since $ke_l \in V^+$, $\| ke_l \|^a \leq k$ and $k > \| v \|^a$ is arbitrary, we have

$$\| v \|^a \geq \inf \left\{ \| u \|^a : \; u \in V^+ \text{ and } \begin{pmatrix} u & v \\ v^* & u \end{pmatrix} \in M_2(V)^+ \right\}.$$

Hence $\| \cdot \|^a$ is a Riesz (semi) norm on V.

$\| \cdot \|^a$ is called the *approximate order unit semi norm* on V determined by $\{e_l\}_{l \in D}$. Thus, an approximate order unit (semi) norm on V is a Riesz (semi) norm on V. Conversely, we have the following lemma.

2.8. Lemma. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and suppose $\| \cdot \|$ be a Riesz (semi) norm on V. Then $\| \cdot \|$ is an approximate order unit (semi) norm on V if and only if the positive part U^+ of the open unit ball of $(V, \| \cdot \|)$ is directed upwards. In this case, U^+ is an approximate order for V. (For real case* [11] *also see* [1] *and* [6].)

Proof. First, let $\| \cdot \|$ be an approximate order unit norm on V determined by an approximate order unit $\{e_l\}_{l \in D}$ of V. We show that U^+ is directed upwards. Let $u_1, u_2 \in U^+$. Choose $\epsilon > 0$ such that $\| u_k \| < 1 - \epsilon$, $k = 1, 2$. Therefore, there exists $\alpha_k > 0$ with $\| u_k \| + \epsilon > \alpha_k$ and $l_k \in D$ $k = 1, 2$ such that $\begin{pmatrix} \alpha_k e_{l_k} & u_k \\ u_k & \alpha_k e_{l_k} \end{pmatrix} \in M_2(V)^+$ or equvalently $\alpha_k e_{l_k} - u_k \in M_2(V)^+$, for $k = 1, 2$. Since D is directed, there is $l \in D$ such that $l_1 \leq l$, $l_2 \leq l$. Put $u = \alpha e_l$, where $\alpha = \max\{\alpha_1, \alpha_2\}$. Then, $u \in U^+$ and $u_1 \leq u, u_2 \leq u$. Hence U^+ is directed upwards. Then $\{u\}_{u \in U^+}$ is an increasing net in V^+. Since $\| \cdot \|$ is a Riesz semi norm we have $S(U^+) = S(U) = U$. Thus given $v \in V$ and $\epsilon > 0$ $(\| v \| + \epsilon)^{-1} v \in U$ so that we get a $u \in U^+$ such that

$$\begin{pmatrix} u & (\| v \| + \epsilon)^{-1} v \\ (\| v \| + \epsilon)^{-1} v^* & u \end{pmatrix} \in M_2(V)^+.$$

Hence, U^+ becomes an approximate order unit for V. Also, then, we get that $\| v \|^a \leq \| v \|^\epsilon$ if $\| \cdot \|^a$ is the approximate order unit semi norm corresponding to U^+. Since $\epsilon > 0$ is arbitrary, we get $\| v \|^a \leq \| v \|$.

Next, let $\| v \|^a < k$ then there is $u \in U^+$ and $\| v \|^a < k_1 < k$ such that $\begin{pmatrix} k_1 u & v \\ v^\star & k_1 u \end{pmatrix} \in M_2(V)^+$. Thus by definition of a Riesz semi norm $\| v \|$ semi norm $\| v \| \leq \| k_1 u \| < k$. Taking infimum over $\| v \|^a < k$ we have $\| v \| \leq \| v \|^a$ for all $v \in V$. Therefore, $\| \cdot \|^a = \| \cdot \|$ and consequently $\| \cdot \|$ is an approximate order unit semi norm. □

Remark. U^+ will be called the *canonical approximate order unit* for V.

2.9. A (semi) approximate matrix order unit space ((s) amou space) is an L^∞-$mR(s)n$ space $(V, \{\| \cdot \|_n\}, \{M_n(V)^+\})$ in which $\| \cdot \|_n$ is an approximate order unit (semi) norm on $M_n(V)$ for every $n \in \mathbb{N}$.

2.10. Example. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space with an approximate order unit. Then V is a (s) amou space.*

Proof. Let $\{e_l\}_{l\in D}$ be an approximate order unit for V. Then by lemma 2.6, $\{e_l^n\}$ is an approximate order unit for $M_n(V)^+$ for every $n \in \mathbb{N}$. So by 2.7 we get matrix Riesz semi norm $\{\| \cdot \|_n^a\}$ on V such that $\| \cdot \|_n$ is an approximate order unit semi norm on $M_n(V)$ for every $n \in \mathbb{N}$. Thus it only remains to show that $(V, \{\| \cdot \|_n^a\}, \{M_n(V)^+\})$ is an L^∞-$m(s)n$ space. For this, fix $n, m \in \mathbb{N}$ and let $v \in M_n(V)$, $w \in M_m(V)$ and $\alpha > \| v \|_n^a$ and $\beta > \| w \|_m^a$. Then there exist $k \leq \max\{\alpha, \beta\}$ and $l \in D$ such that

$$\begin{pmatrix} ke_l^n & v \\ v^\star & ke_l^n \end{pmatrix} \in M_{2n}(V)^+ \quad \text{and} \quad \begin{pmatrix} ke_l^m & w \\ w^\star & ke_l^m \end{pmatrix} \in M_{2m}(V)^+.$$

Then

$$\begin{aligned}\begin{pmatrix} ke_l^{n+m} & v \oplus w \\ v^\star \oplus w^\star & ke_l^{n+m} \end{pmatrix} &= \xi_{n,m}^\star \begin{pmatrix} ke_l^n & v \\ v^\star & ke_l^n \end{pmatrix} \xi_{n,m} + \eta_{n,m}^\star \begin{pmatrix} ke_l^m & w \\ w^\star & ke_l^m \end{pmatrix} \eta_{n,m} \\ &\in M_{2n+2m}(V)^+\end{aligned}$$

where

$$\xi_{n,m} = \begin{pmatrix} I_n & 0_{n,m} & 0_n & 0_{n,m} \\ 0_n & 0_{n,m} & I_n & 0_{n,m} \end{pmatrix} \in M_{2n,2n+2m}(V)^+$$

and

$$\eta_{n,m} = \begin{pmatrix} 0_{m,n} & I_m & 0_{m,n} & 0_m \\ 0_{m,n} & 0_m & 0_{m,n} & I_m \end{pmatrix} \in M_{2m,2n+2m}(V)^+.$$

Thus

$$\| v \oplus w \|_{n+m}^a \leq k \leq \max\{\alpha, \beta\}.$$

Taking infimum over α and β we get

$$\| v \oplus w \|^a_{n+m} \leq \max\{\| v \|^a_n, \ \| w \|^a_m\}.$$

Conversely, let $k > \| v \oplus w \|^a_{n+m}$. Then, there are $l \in D$ and $k > k_1 > \| v \oplus w \|^a_{n+m}$ such that

$$\begin{pmatrix} ke^{n+m}_l & v \oplus w \\ v^\star \oplus w^\star & ke^{n+m}_l \end{pmatrix} \in M_{2n+2m}(V)^+.$$

Thus

$$\begin{pmatrix} ke^n_l & v \\ v^\star & ke^n_l \end{pmatrix} = \xi_{n,m} \begin{pmatrix} ke^{n+m}_l & v \oplus w \\ v^\star \oplus w^\star & ke^{n+m}_l \end{pmatrix} \xi^\star_{n,m} \in M_{2n}(V)^+$$

and similarly

$$\begin{pmatrix} ke^m_l & w \\ w^\star & ke^m_l \end{pmatrix} \in M_{2m}(V)^+$$

Therefore

$$\max\{\| v \|^a_n, \| w \|^a_m\} \ \leq \ k_1 \ < \ k$$

and taking infimum over $k > \| v \oplus w \|^a_{n+m}$ we get

$$\| v \oplus w \|^a_{n+m} \ = \ \max\{\| v \|^a_n, \| w \|^a_m\}.$$

Next, let $\alpha \in M_n$. First assume that $\| \alpha \| \leq 1$ so that $\alpha\alpha^\star \leq I_n$. Now, if $k > \| v \|^a_n$, then there is $l \in D$ such that $\begin{pmatrix} ke^n_l & v \\ v^\star & ke^n_l \end{pmatrix} \in M_{2n}(V)^+$. Thus

$$\begin{aligned} \begin{pmatrix} k\alpha\alpha^\star e^n_l & \alpha v \\ v^\star\alpha^\star & ke^n_l \end{pmatrix} &= \begin{pmatrix} k\alpha e^n_l \alpha^\star & \alpha v \\ v^\star\alpha^\star & ke^n_l \end{pmatrix} \\ &= \begin{pmatrix} \alpha^\star & 0 \\ 0 & I_n \end{pmatrix}^\star \begin{pmatrix} ke^n_l & v \\ v^\star & ke^n_l \end{pmatrix} \begin{pmatrix} \alpha^\star & 0 \\ 0 & I_n \end{pmatrix} \in M_{2n}(V)^+, \end{aligned}$$

for $e^n_l\alpha^\star = \alpha^\star e^n_l$. Hence, $\begin{pmatrix} ke^n_l & \alpha v \\ (\alpha v)^\star & ke^n_l \end{pmatrix} \in M_{2n}(V)^+$ so that $\| \alpha v \|_n \leq k$. Taking infimum over $k > \| v \|^a_n$, we get $\| \alpha v \|^a_n \leq \| v \|_n$.

Next, let $\alpha \in M_n$, $\alpha \neq 0$ and put $\beta = \| \alpha \|^{-1} \alpha$. Then $\| \beta v \|^a_n \leq \| v \|^a_n$ whence $\| \alpha v \|^a_n \leq \| \alpha \| \| v \|^a_n$ for every $v \in M_n(V)$, $\alpha \in M_n$ as $\alpha = 0$ is a trivial case. Since, by the definition of $\| \cdot \|^a_n$, $\| v^\star \|^a_n = \| v \|^a_n$ for all $v \in M_n(V)$ so that

$$\| v\alpha \|^a_n = \| \alpha^\star v^\star \|^a_n \leq \| \alpha^\star \| \| v^\star \|^a_n = \| \alpha \| \| v^\star \|^a_n .$$

Therefore by the arbitrariness of $v \in M_n(V)$, $w \in M_n(V)$, $\alpha \in M_n$ and $n, m \in \mathbb{N}$ we conclude that $(V, \{\| \cdot \|_n^a\}, \{M_n(V)^+\})$ is an L^∞-$nR(s)n$ space. □

Remark. If $\| \cdot \|_1^a$ is a norm, then $(V, \{\| \cdot \|_n^a\}, \{M_n(V)^+\})$ is an amou space following 1.1.

2.11. Now, we come to the main result of this paper. We start with a historical problem: *given a $C^\star$-algebra, what can we say about its norm in terms of its order structure ?* Three partial ansers of this question are already known. In 1951, Kadison proved in [4] that the $C^\star$-norm on the self-adjoint part of any unital $C^\star$-algebra is an order unit norm. Later in 1969 Ng proved in [6] that the $C^\star$-norm on the self-adjoint part of any $C^\star$-algebra is an approximate order unit norm. Order theoretic characterization of the $C^\star$-norm on the non-self-adjoint part of a $C^\star$-algebra first appeared in 1977 when Choi and Effros defined a matrix order unit space [2]. Knowing the fact that every $C^\star$-algebra has a matrix norm [9]; one can deduce, from Choi-Effros' characterization for matrix order unit spaces, that the *matrix $C^\star$-norm* on a unital $C^\star$-algebra is a matrix order unit norm. We now prove that the matrix $C^\star$-norm of any $C^\star$-algebra is an approximate matrix order unit norm.

2.12. Theorem. *Every $C^\star$-algebra is an A-$mR(s)n$ space.*

Proof. Let A be a $C^\star$-algebra. Then for each $n \in \mathbb{N}$, $M_n(A)$ is a $C^\star$-algebra and with the corresponding matrix $C^\star$-norm, A is an L^∞ $m(s)n$ space, [9]. Moreover, noting the fact that the positive part is generating and that the positive part of the open unit ball of any $C^\star$-algebra is directed upwards [8], the theorem follows from Lemma 2.8, if we show that the $C^\star$-norm is a Riesz norm.

Let $a \in A$ with $\| a \| < 1$. First assume that $a \in A_{sa}$ and let $a = a^+ - a^-$ where a^+ and a^- are positive and negative parts of a, so that $\| a \| = \max\{\| a^+ \|, \| a^- \|\}$. Thus $a^+, a^- \in U^+$ where U^+ is the positive parts of the open unit ball of A. Since U^+ is directed upwards, we get $u \in U^+$ with $a^+ \leq u$, $a^- \leq u$. Then $u \pm a \in A^+$.

Next, let $a \in A$ be arbitrary, with $\| a \| < 1$ then $\begin{pmatrix} 0 & a \\ a^\star & 0 \end{pmatrix} \in M_2(A)_{sa}$. Furthermore

$$1 > \| a \| = \left\| \begin{pmatrix} a & 0 \\ 0 & a^\star \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} 0 & a \\ a^\star & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} 0 & a \\ a^\star & 0 \end{pmatrix} \right\|_2,$$

by 1.1. Thus as above there is $\begin{pmatrix} u_1 & x \\ x^\star & u_2 \end{pmatrix} \in M_2(A)^+$ with $\left\| \begin{pmatrix} u_1 & x \\ x^\star & u_2 \end{pmatrix} \right\|_2 < 1$ such that

$$\begin{pmatrix} u_1 & x \\ x^\star & u_2 \end{pmatrix} \pm \begin{pmatrix} 0 & a \\ a^\star & 0 \end{pmatrix} \in M_2(A)^+.$$

Now, $\left\| \begin{pmatrix} u_1 & x \\ x^\star & u_2 \end{pmatrix} \right\|_2 < 1$ so by 1.1, $\| u_k \| < 1$, $k = 1, 2$ and $\begin{pmatrix} u_1 & x \\ x^\star & u_2 \end{pmatrix} \in M_2(A)^+$ implies that $u_1, u_2 \in A^+$, so that there is $u \in U^+$ such that $u_1 \leq u$, $u_2 \leq u$. Hence

$$\begin{pmatrix} u & x \\ x^\star & u \end{pmatrix} \pm \begin{pmatrix} 0 & a \\ a^\star & 0 \end{pmatrix} = \begin{pmatrix} u & x \pm a \\ x^\star \pm a^\star & u \end{pmatrix} \in M_2(A)^+.$$

so by 1.8 (i)

$$\begin{pmatrix} u & -x \pm a \\ -x^\star \pm a^\star & u \end{pmatrix} = \begin{pmatrix} u & e^{i\pi}(x - a) \\ e^{-i\pi}(x^\star - a^\star) & u \end{pmatrix} \in M_2(A)^+.$$

Therfore

$$\begin{pmatrix} u & a \\ a^\star & u \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u & x + a \\ x^\star + a^\star & u \end{pmatrix} + \frac{1}{2} \begin{pmatrix} u & -x + a \\ -x^\star + a^\star & u \end{pmatrix} \in M_2(A)^+.$$

This implies $U \subset S(U)$, where U is the open unit ball of A.

We know from [7] that if $u \in A^+$ and $a \in A$ with $\begin{pmatrix} u & a \\ a^\star & u \end{pmatrix} \in M_2(A)^+$ then $\| a \| \leq \| u \|$. Now, let $A \in S(U)$ then there is $u \in U^+$ such that $\begin{pmatrix} u & a \\ a^\star & u \end{pmatrix} \in M_2(A)^+$. Hence $\| a \| \leq \| u \| < 1$ so that $a \in U$ and consequently $S(U) \subset U$. Hence $\| \cdot \|$ is a Riesz norm on A, by Lemma 2.3. □

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Department of Mathematics
Univesity of Delhi
South Campaus
Benito Juarez Road
New Delhi 110021, India