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Busemann functions on Alexandrov surfaces

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1. Introduction

A well known theorem due to Cohn-Vossen states that total curvature of a connected, finitely connected, complete and nocompact Riemannian 2- manifold M without boundary dose not exceed $2\pi\chi(M)$. Here $\chi(M)$ is the Euler characteristic of M. Busemann discussed the Cohn-Vossen theorem on Busemann G-surfaces on which he introduced the notion of the Busemann total excess. He proved in [B: (43.3)] the fundamental relation between the excess and Euler characteristic of Busemann G-surfaces. The Busemann total excess was recently discussed on Alexandrov surfaces in [M: Theorem 1.8] with necessary change for the notion of total excess and used to define the Gaussian curvature on Alexandrov surfaces. Here the fundamental relation between the total excess and Euler characteristic of on Alexandrov surfaces is important.

Since the geometric meaning of total curvature on complete open Riemannian 2-manifold is investigated in many aspects , it is interesting to study the geometric meaning of total excess on complete noncompact Alexandrov surfaces. The total excess of Alexandrov surfaces will play the same role as the total curvature of Riemannian 2-manifolds.

Throughout this note let X be a connected, complete, finitely connected and noncompact Alexandrov surface without boundary whose curvature is bounded below by $k > -\infty$. We shall study the behavior of Busemann functions on X. As is seen in Riemannian case [S: Main Theorem] the behavior of Busemann function is controled by the total curvature. It was proved in [ST: Theorem A] that set of all copoints of an arbitrary fixed ray γ on X can be viewed as the cut locus to the point $\gamma(+\infty)$ at infinity and has the structure of a local tree whose interior consists of a countable union of rectifiable Jordan arcs. Moreover the set $Crit(F_\gamma)$ of all critical points of the Busemann function F_γ for γ is contained in the set of all copoints of γ. The structure of cut loci on X was also discussed in [ST].

Our results are stated as follows.

Theorem A Assume that X has one end. If the total excess $c(X)$ of X satisfies

$$c(X) > (2\chi(X) - 1)\pi,$$

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then every Busemann function on X is exhaustion.

In due course of the proof of Theorem A, we obtain the following Corollary. The proof is essentially the same as that of Theorem A, and omitted there.

Corollary to Theorem A Under the same assumptions in Theorem A, $Crit(F_\gamma)$ for every Busemann function F_γ is bounded.

Theorem B Assume that X has one end. If

$$c(X) < (2\chi(X) - 1)\pi,$$

then every Busemann function on X is non-exhaustion. Note that if $c(X) = (2\chi(X) - 1)\pi$ then such an X admits both exhaustion and non-exhaustion Busemann functions. Such an example is seen in [S: Example 3].

2. Definitions and notations

At a point $p \in X$ the space of direction is denoted by Σ_p. The Σ_p equipped with the angle distance is a circle of length at most 2π. A point $p \in X$ is by definition *singular* if and only if the length $L(\Sigma_p)$ of Σ_p is less than 2π. It is shown in [OS: Theorem A] that the set of all singuler points on X is countable. We consider a bounded domain $D \subset X$ such that the boundary ∂D of D consists of finite union of geodesic polygons. The class of all such domains is denoted by $\mathcal{D}(X)$. We decompose a $D \in \mathcal{D}(X)$ by finitely many geodesic triangles $\triangle = \{\triangle_i\}_{i=1,\dots,N}$ each triangle is bounding a disk. For such a small geodesic triangle $\triangle(abc)$, we set $\varepsilon_0(\triangle(abc)) = A + B + C - \pi$, where A, B, C are the angles. Setting $\varepsilon(D; \triangle) := \sum_{i=1}^{N} \varepsilon_0(\triangle_i)$, the funamental relation between the excess and the Euler characteristic $\chi(D)$ of D is expressed by [M: Theorem 1.8],

$$\varepsilon(D; \triangle) + \sum_{j=1}^{J}(2\pi - L(\Sigma_{q_j})) = 2\pi\chi(D) - \sum_{k=1}^{l}(\pi - \omega_k). \quad (1)$$

Here $q_1, \dots, q_J$ are all the vertices of $\triangle_i$'s lying in D , and $\omega_1, \dots, \omega_l$ are all the inner angles at corners of ∂D. Then Machigashira [M;Theorem 2.0] proves that the first term on the left hand side of (1) is bounded below by $k \cdot Area(D)$. Let $\Phi_\delta(D)$ for $\delta > 0$ be the class of all the simplicial decompositions of D by geodesic triangles such that the maximum circumference of triangles of each $\triangle \in \Phi_\delta(D)$ is less than δ. We then see that $\lim_{\delta \to 0, \triangle \in \Phi_\delta(D)} \varepsilon(D; \triangle)$ and $\lim_{\delta \to 0, \triangle \in \Phi_\delta(D)} \sum(2\pi - L(\Sigma_{q_j}))$ exist, and therefore we define the *excess* $c(D)$ of D as follows.

$$c(D) := \lim_{\delta \to 0, \triangle \in \Phi_\delta(D)} (\varepsilon(D; \triangle) + \sum(2\pi - L(\Sigma_{q_j})).$$

Let $\{D_k\}_k$ be a monotone increasing sequence in $\mathcal{D}(X)$ such that $\cup D_k = X$. We then have a sequence $\{c(D_k)\}_k$ of real numbers. The *total excess* $c(X)$ of X is defined by

$$c(X) := \lim_{k \to \infty} c(D_k),$$

where the limit is independent of the choice of $\{D_k\}$. Let $\mathcal{T} := \{\{\triangle_i\}_{i=1}^{\infty}; \triangle_i \cap \triangle_j = \phi \text{ for } i \neq j\}$ be the set of all small geodesic triangles of X. Then the total excess of X exists if and only if $\sup_{\mathcal{T}} \sum_{i=1}^{\infty} \varepsilon(\triangle_i) + \sum_{x \in X}(2\pi - L(\Sigma_x)) < +\infty$.

For a finitely connected, complete and noncompact X with m ends a compact set $C \subset X$ is called a *core* of X if $\overline{M \backslash C}$ consists of m unbounded components each of which is homeomorphic to a *tube* $S^1 \times [0, \infty)$.

Let $\gamma : [0, \infty) \to X$ be a ray. The *Busemann function* $F_\gamma : X \to \mathbf{R}$ for γ is defined by $F_\gamma(x) := \lim_{t \to \infty}(t - d(x, \gamma(t))), x \in X$. The right hand side is monotone non-decreasing in t and bounded above by $d(\gamma(0), x)$. Clealy $| F_\gamma(x) - F_\gamma(y) | \leq d(x, y)$, and hence F_γ is Lipshitz continuous. A ray $\sigma : [0, \infty) \to X$ is by definition *asymptotic* to γ iff there exist a monotone divergent sequence $\{t_j\}$ and points $\{q_j\}$ converging to $\sigma(0)$ such that geodesics joining q_j to $\gamma(t_j)$ converges to σ. If σ is asymptotic to γ, then $F_\gamma \circ \sigma(t) = F_\gamma \circ \sigma(0) + t, t \geq 0$, and the converce is true. A point $q \in X$ is by definition a *copoint* of γ iff it is the starting point of a maximal asymptotic ray to γ. A function $h : X \to \mathbf{R}$ is said to be an *exhaustion* if $h^{-1}((-\infty, a])$ is compact for all $a \in \mathbf{R}$. By definition h is a *nonexhaustion* if it is not exhaustion. A point $p \in X$ is a *critical point* of F_γ if and only if there exists for every direction $\xi \in \Sigma_p$ a ray σ asymptotic to γ such that $\sigma(0) = p$ and $\angle(\xi, \dot\sigma(0)) \leq \pi/2$. A point $p \in X$ is called *non-critical* of F_γ if it is not a critical point of F_γ. If $p \in X$ has the property $L(\Sigma_p) \leq \pi$, then p is a critical point of every Busemann function.

Now let X have one end and $\gamma : [0, \infty) \to X$ a ray. We may choose a core C of X such that $\gamma(0) \in \partial C$. Let $U = X - int(C)$ be a tube relative to C. Let $\pi : \tilde U \to U$ be the universal covering of U and π the covering projection,and let $\hat U \subset \tilde U$ be the fundamental domain whose boundary consists of two rays $\hat\gamma_1, \hat\gamma_2 : [0, \infty) \to \tilde U$ and a broken geodesic $\hat P$ such that $\pi \circ \hat\gamma_1 = \pi \circ \hat\gamma_2 = \gamma$ and $\pi(\hat P) = P = \partial U$. Let $\hat d$ be the distance function on $\hat U$. Any two points in $\hat U$ can be joined by a $\hat d$-segment whose length realizes the $\hat d$-distance between them. Let $\hat\Gamma(t)$ for every $t \geq 0$ be the set of all $\hat d$- segment in $\hat U$ joining $\hat\gamma_1(t)$ to $\hat\gamma_2(t)$. Namely, each $\hat P(t) \in \hat\Gamma(t)$ has length $L(\hat P(t)) = \hat d(\hat\gamma_1(t), \hat\gamma_2(t))$. It follows from the $\hat d$-distance minimizing property of $\hat P_t$ that if $\hat P_t \cap \hat P = \phi$ for some $t > 0$ and some $\hat P_t \in \hat\Gamma_t$, then $\hat P_{t'} \cap \hat P = \phi$ for all $t' > t$ and for all $\hat P_{t'} \in \hat\Gamma_{t'}$. Also, if $\hat P_t \cap \hat P \neq \phi$ for some $t \geq 0$ and some $\hat P_t \in \hat\Gamma_t$, then $\hat P_{t'} \cap \hat P \neq \phi$ for all $t' < t$ and for all $\hat P_{t'} \in \hat\Gamma_{t'}$. If $\hat P_t \cap \hat P = \phi$, then $P_t := \pi(\hat P_t)$ is a simple geodesic loop on X at $\gamma(t)$. Note also that P_t is freely homotopic to $P = \pi(\hat P)$ for any $\hat P_t$. Now we prepare the following lemmas as same as in Riemannian case [S: Theorem 4.2 and Theorem 4.3].

Lemma 1 If $\hat P(t) \cap \hat P \neq \phi$ holds for every $\hat P(t) \in \hat\Gamma(t)$ and for every $t \geq 0$, then $c(X) \leq (2\chi(X) - 1)\pi$.

Proof. Let $\{t_j\}$ be a monotone divergent sequence and D_j be a disk domain surrounded by $\pi(\hat P(t_j))$. If ω_j is the inner angle at $\gamma(t_j)$ of D_j, then from (1) we get $c(D_j) \leq 2\pi\chi(D_j) - (\pi - \omega_j)$. Let $\{C_j\}$ be an increasing sequence of cores of X such that $\cup_{j=1}^{\infty} C_j = X$. For an arbitrary given $\varepsilon > 0$ we find a subsequence $\{t_k\}$ of $\{t_j\}$ such that $\omega_k < \varepsilon$ and such that $\{D_k\}$ satisfies $C_1 := C, C_k \subset D_k \subset C_{k+1} \subset D_{k+1}$. From the assumption that the total excess exists, we get $c(X) \leq (2\chi(X) - 1)\pi$. □

Lemma 2 Assume that there exists a $t_0 > 0$ such that $\hat{P}(t) \cap \hat{P} = \phi$ for all $\hat{P}(t) \in \hat{\Gamma}(t)$ and for all $t > t_0$. If $g_\gamma(t) := t - \frac{1}{2}L(\hat{P}(t))$ is bounded above on $[0, \infty)$, then $c(X) \leq (2\chi(X) - 1)\pi$.

Proof. Let $\hat{D}(t) \subset \hat{U}$ be the disk domain bounded by $\hat{P}, \hat{\gamma}_1[0,t] \cup \hat{P}(t) \cup \hat{\gamma}_2[0,t]$. Let $\hat{\alpha}_1(t), \hat{\alpha}_2(t)$ be the angle at $\hat{\gamma}_1(t), \hat{\gamma}_2(t)$ of $\hat{D}(t)$. Because g_γ is Lipschitz continuous and increasing, the first variation formula [OS: Theorem 3.5] implies for almost all $t > 0$,

$$\frac{d}{dt}g_\gamma(t) = 1 - \frac{1}{2}(\cos\hat{\alpha}_1(t) + \cos\hat{\alpha}_2(t)).$$

From the boundness of g_γ there exists a divergent sequence $\{t_j\}$ such that g_γ is differentiable at t_j and

$$\lim_{j\to\infty} \hat{\alpha}_1(t_j) = \lim_{j\to\infty} \hat{\alpha}_2(t_j) = 0. \qquad (2)$$

Clearly $\{\hat{D}(t)\}_{t>0}$ is monotone increasing in t. Assume tht $\cup_{t>0}\hat{D}(t)$ is a proper subset of $\hat{U}$. Then $\{\hat{P}(t)\}_{t>0}$ converges to a $\hat{d}$-stright line $\hat{P}_\infty$ in $\hat{U}$. Let $\hat{H} \subset \hat{U}$ be the half plane bounded by $\hat{P}_\infty$. Then $c(\hat{H}) \leq 0$ is clear from the discussion in [CV: Satz 2(p142)]. Setting $H := \pi(\hat{H})$ and $D(t) := C \cup \pi(\hat{D}(t)) \subset X$, that $D(t)$ has only one corner at $\gamma(t)$, and from (1) and (2)

$$\lim_{t\to\infty} c(D(t)) = 2\pi\chi(X) - \pi.$$

In particular

$$c(X) = c(H) + c(X \backslash H) \leq (2\chi(X) - 1)\pi.$$

If $\cup_{t>0}\hat{D}(t) = \hat{U}$, then the proof in this case is essentially contained the previous one. This complete the proof of Lemma 2. □

3. The behavior of Busemann functions

For an arbitrary fixed γ we choose a core C of X and $U := X \backslash int(C)$ as before. For the first statement of Theorem A we suppose that $c(X) > (2\chi(X) - 1)\pi$ and that F_γ is nonexhaustion. The function g_γ plays an important role. The contrapositive of Lemmas 1 and 2 then imply that there is a $t_0 > 0$ such that all the members of $\hat{\Gamma}(t)$ for all $t > t_0$ do not intersect $\hat{P}$ and that g_γ is unbounded on $[0, \infty)$. Setting $\Gamma(t) := \pi(\hat{\Gamma}(t))$ and $P(t) := \pi(\hat{P}(t))$, we see that the midpoint of $P(t)$ for all large t is the cut point to $\gamma(t)$ along $P(t)$.

Lemma 3 Assume that there exists a $t_0 > 0$ such that all the segment in $\hat{\Gamma}(t)$ for all $t > t_0$ do not meet $\hat{P}$. Then we have for every point $x \in P(t)$ and for every $P(t) \in \Gamma(t)$,

$$F_\gamma(x) \geq g_\gamma(t), \quad \text{for all } t > t_0. \qquad (3)$$

Proof. Since F_γ is 1-Lipschitz, we have

$$| F_\gamma(x) - F_\gamma(\gamma(t)) | \leq d(x, \gamma(t)) \leq 1/2L(P_t).$$

Thus we get $F_\gamma(x) \geq t - 1/2L(P_t) = g_\gamma(t)$. □

Let $\hat{\Omega}(t) \subset \hat{U}$ for large t be the maximal disk domain bounded by two geodesics $\hat{P}^+, \hat{P}^- \in \hat{\Gamma}(t)$. Then $\hat{\Omega}(t)$ contains every $\hat{P} \in \hat{\Gamma}(t)$ except $\hat{P}^{\pm}(t)$. Let $\hat{m}^{\pm}(t) \in \hat{P}^{\pm}(t)$ be midpoint. Then $m^{\pm}(t) := \pi(\hat{m}^{\pm}(t))$ is the cut point to $\gamma(t)$ along $P^{\pm}(t)$. Set $\Omega(t) := \pi(\hat{\Omega}(t))$. From what we have supposed, there is a constant c and an unbounded sequence $\{q_j\}$ in U such that

$$F_\gamma(q_j) \leq c \quad \text{for } j = 1,2,... \qquad (4)$$

Proposition 4 Let $c(X) > (2\chi(X) - 1)\pi$. Suppose that X admits a nonexhaustion Busemann function F_γ satisfying (4).Then there exists a monotone divergent sequence $\{t_j\}_j$ such that $q_j \in \Omega(t_j)$. In particular F_γ has a local minimum (and of course a critical point of F_γ) in $\Omega(t_j)$.

Proof. From the assumption, for an unbounded sequence $\{q_j\}$, we choose a monotone divergent sequence $\{t_j\}_j$ such that $q_j \in \Omega(t_j)$. From Lemma1 and Lemma2, we may assume that $g_\gamma(t_j) > c$ for all j. By Lemma 3, F_γ takes value not less than $g_\gamma(t_j)$ on $\partial\Omega(t_j)$. Therfore F_γ takes a local minimum in $\Omega(t_j)$. □

The idea of the proof of Theorem A is as follows. Let F_γ takes a local minimum at $x_j \in \Omega(t_j)$ and $\hat{x}_j \in \hat{\Omega}(t_j)$ be such that $\pi(\hat{x}_j) = x_j$. We choose rays $\hat{\tau}_{j1}, \hat{\tau}_{j2} : [0, \infty) \to \hat{U}$ emanating from $\hat{x}_j$ such that $\pi(\hat{\tau}_{ji}) = \tau_{ji}$ for every $i = 1, 2$ is asymptotic to γ and such that the half plane $\hat{H} \subset \hat{U}$ bounded by $\hat{\tau}_{j1}[0, \infty) \cup \hat{\tau}_{j2}[0, \infty)$ contains all the rays from $\hat{x}_j$ and asymptotic to $\hat{\gamma}_1$ and $\hat{\gamma}_2$. Let φ_j be the angle at $\hat{x}_j$ of $\hat{H}$. Setting $\varepsilon := c(X) - (2\chi(X) - 1)\pi$, we derive a contradiction by constructing an infinite sequence $\{\triangle_j\}$ of disjoint geodesic triangles such that $c(\triangle_j) > \varepsilon/2$ for all j.

Lemma 5 Under th same assumptions as in Proposition 4, there exists a geodesic triangle $\triangle_j$ with vertices at $x_j, \hat{\gamma}_1(s_j), \hat{\gamma}_2(s_j)$ such that $c(\triangle_j) \geq \varepsilon/2$. Here $s_j > t_j$ is taken sufficiently large.

Proof. From the choice of $\hat{\tau}_{j1}$ and $\hat{\tau}_{j2}$ we observe that if $\hat{\tau}_{jit}$ for $i = 1, 2$ and for $t > t_j$ is a $\hat{d}$-segment in $\hat{U}$ joining $\hat{x}_j$ to $\hat{\gamma}_i(t)$, then $\lim_{t\to\infty} \hat{\tau}_{jit} = \hat{\tau}_{ji}$. If $\hat{\alpha}_{ji}(t)$ is the angle at $\hat{\gamma}_i(t)$ between $\hat{\gamma}_i(t)$ and $\hat{\tau}_{ji}$, then $\lim_{t\to\infty} \hat{\alpha}_{ji}(t) = 0$. If $\alpha(t)$ and $\beta(t)$ are the angles at $\gamma(t)$ and x_j of the core $C(t) \subset X$ bounded by geodesic biangle $\tau_{j1t} := \pi(\hat{\tau}_{j1t})$ and $\tau_{j2t} := \pi(\hat{\tau}_{j2t})$, then $\alpha(t) = \hat{\alpha}_{j1}(t) + \hat{\alpha}_{j2}(t)$ and $\lim_{t\to\infty} \alpha(t) = 0$ and $\lim_{t\to\infty} \beta(t) = L(\Sigma_{x_j}) - \varphi_j =: \beta$. We then have
$c(C(t)) + (\pi - \alpha(t)) + (\pi - \beta(t)) = 2\pi\chi(X)$. Setting $E := \cup_{t>0}(C(t))$,and $t \to \infty$, we have

$$c(E) + \pi - \beta = (2\chi(X) - 1)\pi \qquad (5)$$

To estimate the total excess of H we choose $\hat{d}$-segments $\hat{Q}_j(t)$ joining $\hat{\tau}_{j1}(t)$ to $\hat{\tau}_{j2}(t)$. Let $\hat{\theta}_{ji}(t)$ be the angle at $\hat{\tau}_{ji}(t)$ of geodesic triangle $\hat{\triangle}_j(t) \subset \hat{U}$ bounding a disk and its edges are $\hat{\tau}_{j1}[0, t]$, $\hat{\tau}_{j2}[0, t]$ and $\hat{Q}_j(t)$. Then we have $c(\hat{\triangle}_j(t)) = \hat{\theta}_{j1}(t) + \hat{\theta}_{j2} + \varphi_j - \pi$, and clearly $\hat{H} = \cup_{t>0}\hat{\triangle}_j(t)$. Setting $t \to \infty$ and $H := \pi(\hat{H})$, we observe $X = H \cup E$. In view of $c(H) = c(X) - c(E)$ and $L(\Sigma_{x_j}) = \beta + \varphi_j$, we have

$$c(H) = \varepsilon + \pi - \beta \geq \varepsilon. \qquad (6)$$

Note that x_j is a critical point of F_γ, and hence $\beta \leq \pi$. Thereore there is a large number $s_j > t_j$ such that $\hat{\triangle}_j(s_j)$ has its excess

$$c(\hat{\triangle}_j(s_j)) > \varepsilon/2.$$

This completes the proof of Lemma 5.□

Proof of Theorem A
Assume that $c(X) > (2\chi(X) - 1)\pi$. Suppose that F_γ is nonexhaustion . Then Proposition 4 implies that there exists a divergent sequence $\{x_j\}$ of critical points of F_γ. Lemma 5 implies that there is an infinite sequence $\{\triangle_k\}$ of geodesic triangles bounding disks such that they are all disjoint and $c(\triangle_k) > \varepsilon/2$ for all k. This contradicts to the assumption that X admits total excess.

The following Lemma 6 is useful for the proof of Theorem B.

Lemma 6 If F_γ is exhaustion, then there exists a $t_0 > 0$ such that every $\hat{P}(t) \in \hat{\Gamma}$ for every $t > t_0$ dose not intersect $\hat{P}$. Moreover if $\hat{D}(t)$ for $t > t_0$ is the domain in $\hat{U}$ bounded by $\hat{P}(t), \hat{\gamma}_1[0,t], \hat{\gamma}_2[0,t]$ and $\hat{P}$, then $\cup_{t>t_0}\hat{D}(t) = \hat{U}$.

Proof. From Theorem B in [ST] , there is a set $\mathcal{E} \subset [0,\infty)$ of measure zero such that if $t \notin \mathcal{E}$ then $F_\gamma^{-1}(t)$ consists of a finite disjoint union of circles each of which is rectifiable. So,we choose such a set $\mathcal{E}$ and $t \notin \mathcal{E}$ form now on. Denote by $B(p,r)$ and $S(p,R)$ the r-ball and metric r-sphere around p respectively. Because $\lim_{s\to\infty} S(\gamma(s+t),s) = F_\gamma^{-1}(t)$ and X has one end, we observe that for a sufficiently large $s >> t$ the s-ball $\bar{B}(\gamma(s+t),s)$ is homeomorphic to a cylinder $S^1 \times [0,1]$ and $S(\gamma(s+t),s)$ has two components passing through $\gamma(t)$ and $\gamma(2s+t)$. The component passing through $\gamma(2s+t)$ diverges as $s \to \infty$. Setting $\hat{F}_\gamma^{-1}(t) := \hat{U} \cap \pi^{-1}(F_\gamma^{-1}(t))$, $\hat{B}(\gamma(t+s),s) := \hat{U} \cap \pi^{-1}(B(\gamma(t+s),s))$ and $\hat{S}(\gamma(t+s),s) := \hat{U} \cap \pi^{-1}(S(\gamma(t+s),s))$, we observe that there are points $\hat{a}_1, \hat{a}_2$ on each component of $\partial\hat{B}(\gamma(t+s),s)$ such that $\hat{a}_1$ is close to $\hat{F}_\gamma^{-1}(t)$ and such that $\hat{d}(\hat{\gamma}_i(t+s),\hat{a}_k) = s$ for all $i = 1,2$ and for all $k = 1,2$. In view of TheoremB in [ST] we may consider that each component of $\partial B(\gamma(t+s),s)$ is rectifiable and that the geodesic joining $\hat{\gamma}_i(t+s)$ to $\hat{a}_k$ for all $i,k = 1,2$ is perpendicular to $\partial\hat{S}(\gamma(t+s),s)$. Clearly four $\hat{d}$-segments joining $\hat{a}_k$ to $\hat{\gamma}_i(t+s)$ forms a convex disk domain in $\hat{U}$. Thus we find a $\hat{d}$-segment $\hat{P}(t+s)$ containing in $\hat{B}(\gamma(t+s),s)$. This proves the first statement of Lemma 6. The rest of the proof is clear because $P(t+s) := \pi(\hat{P}(t+s)) \subset X\backslash F_\gamma^{-1}([0,t])$ for all sufficiently large t and s.□

Proof of Theorem B Suppose that X admits an F_γ which is exhaustion. Then Lemma 6 implies that there exists for all sufficientlly large t a geodesic loop $P(t)$ with base point at $\gamma(t)$ which is freely homotopoic to the boundary of a core C. Let $D(t) := C \cup \pi(\hat{D}(t))$ and $\alpha(t)$ the angle at $\gamma(t)$ of $D(t)$. Then we have $\cup_{t\geq t_0} D(t) = X$ and $c(D(t)) = (2\chi(X) - 1)\pi + \alpha(t)$. Therefore we get $c(X) = \lim_{t\to\infty} c(D(t)) \geq (2\chi(X) - 1)\pi$, a contradiction.

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