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Several fixed point theorems in complete metric spaces

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Abstract. In this paper, we prove several fixed point theorems, which are generalizations of the Banach contraction principle and Kannan's fixed point theorem. Further we discuss a characterization of metric completeness.

1. Introduction

In 1922, Banach [1] proved the following famous fixed point theorem: Let X be a complete metric space with metric d and let T be a mapping from X into itself such that there exists $r\in[0,1)$ with $d(Tx,Ty)\leq rd(x,y)$ for every $x,y\in X$. Then T has a unique fixed point. This theorem called the Banach contraction principle is a very useful tool on nonlinear analysis. Later this theorem is generalized in several directions. For example, Takahashi [7] proved a nonconvex minimization theorem and Ćirić [2] proved a fixed point theorem for a quasi-contraction. Recently, Kada, Suzuki and Takahashi [3] introduced the concept of w-distance on a metric space and improved Takahashi's nonconvex minimization theorem, Ćirić's fixed point theorem and so on. Suzuki and Takahashi [6] also proved a fixed point theorem for a weakly contractive mapping, which is a generalization of the Banach contraction principle. On the other hand, Kannan [4] proved the following interesting fixed point theorem, which is not an extension of the Banach contraction principle: Let X be a complete metric space with metric d and let T be a mapping from X into itself such that there exists $\alpha\in\left[0,\frac{1}{2}\right)$ with $d(Tx,Ty)\leq\alpha d(x,Tx)+\alpha d(y,Ty)$ for every $x,y\in X$. Then T has a unique fixed point.

In this paper, we prove several fixed point theorems, which are generalizations of the Banach contraction principle and Kannan's fixed point theorem. Further we discuss a characterization of metric completeness.

2. w-distance

In this Section, we state the definition of w-distance which was introduced by Kada, Suzuki and Takahashi [3] and then give some Lemmas which are connected with w-distance.

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Definition ([3]) Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;

(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The metric d is a w-distance on X. Other examples of w-distance are stated in [3] and [6]. The following two Lemmas generalizing Lemma 1 in [3] are crucial in the proofs of our theorems.

Lemma 1 ([3]) *Let X be a metric space with metric d, let p be a w-distance on X, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in X and let $x, y, z \in X$. Then the following hold:*

(i) *If $p(x_n, y) \to 0$ and $p(x_n, z) \to 0$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y=z$;*

(ii) *if $p(x_n, y_n) \to 0$ and $p(x_n, z) \to 0$, then $\{y_n\}$ converges to z;*

(iii) *if $p(x_n, y_n) \to 0$ and $p(x_n, z_n) \to 0$, then $\{d(y_n, z_n)\}$ converges to 0.*

Proof. It is clear that (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i). So, to complete the proof, we prove (iii). Let $\varepsilon > 0$ be given. From the definition of w-distance, there exists $\delta > 0$ such that $p(u, v) \leq \delta$ and $p(u, w) \leq \delta$ imply $d(v, w) \leq \varepsilon$. Choose $n_0 \in \mathbb{N}$ such that $p(x_n, y_n) \leq \delta$ and $p(x_n, z_n) \leq \delta$ for every $n \geq n_0$. Then for any $n \geq n_0$, we have $d(y_n, z_n) \leq \varepsilon$. This implies (iii). This completes the proof.

Lemma 2 *Let X be a metric space with metric d, let p be a w-distance on X and let $\{x_n\}$ be a sequence in X. Suppose that*

$$\lim_{n \to \infty} \sup_{m > n} \min\{p(x_n, x_m), p(x_m, x_n)\} = 0.$$

Then $\{x_n\}$ is Cauchy. In particular, the following hold:

(i) *If $\lim_{n \to \infty} \sup_{m > n} p(x_n, x_m) = 0$, then $\{x_n\}$ is Cauchy;*

(ii) *if $\lim_{n \to \infty} \sup_{m > n} p(x_m, x_n) = 0$, then $\{x_n\}$ is Cauchy.*

Proof. Let $\varepsilon > 0$ be given. From the definition of w-distance, there exists $\delta > 0$ such that $p(u, v) \leq 3\delta$ and $p(u, w) \leq 3\delta$ imply $d(v, w) \leq \varepsilon$. Choose $n_0 \in \mathbb{N}$ such that $\sup_{m > n} \min\{p(x_n, x_m), p(x_m, x_n)\} \leq \delta$ for every $n \geq n_0$. Let i, j, k, ℓ be four distinct integers with $i, j, k, \ell \geq n_0$. Then there exists $m \in \{i, j, k, \ell\}$ such that $p(x_m, x_n) \leq 3\delta$ for every $n \in \{i, j, k, \ell\} \setminus \{m\}$. So, we have

$$\operatorname{diam}\{x_n : n \in \{i, j, k, \ell\} \setminus \{m\}\} \leq \varepsilon.$$

Therefore we have

$$\min\{\operatorname{diam}\{x_i, x_j, x_k\}, \operatorname{diam}\{x_i, x_j, x_\ell\}, \\ \operatorname{diam}\{x_i, x_k, x_\ell\}, \operatorname{diam}\{x_j, x_k, x_\ell\}\} \leq \varepsilon$$

for every four distinct integers i, j, k, ℓ with $i, j, k, \ell \geq n_0$. So, we have

$$\min\{\operatorname{diam}\{x_{n_0}, x_{n_0+1}, x_{n_0+2}\}, \operatorname{diam}\{x_{n_0}, x_{n_0+1}, x_{n_0+3}\}, \\ \operatorname{diam}\{x_{n_0}, x_{n_0+2}, x_{n_0+3}\}, \operatorname{diam}\{x_{n_0+1}, x_{n_0+2}, x_{n_0+3}\}\} \leq \varepsilon.$$

Without loss of generality, we may assume that $\operatorname{diam}\{x_{n_0}, x_{n_0+1}, x_{n_0+2}\} \leq \varepsilon$. Put

$$I = \{n \in \mathbb{N} : n \geq n_0 + 4, d(x_{n_0}, x_n) > 2\varepsilon\}.$$

Then I consists of at most one point. If not, then there exist $m, n \in I$ with $m \neq n$. Since

$$\min\{\operatorname{diam}\{x_{n_0}, x_{n_0+1}, x_m\}, \operatorname{diam}\{x_{n_0}, x_{n_0+1}, x_n\}, \\ \operatorname{diam}\{x_{n_0}, x_m, x_n\}, \operatorname{diam}\{x_{n_0+1}, x_m, x_n\}\} \leq \varepsilon,$$

we have

$$\min\{d(x_{n_0}, x_m), d(x_{n_0}, x_n), d(x_{n_0}, x_m), d(x_{n_0+1}, x_m)\} \leq \varepsilon$$

and hence $d(x_{n_0+1}, x_m) \leq \varepsilon$. On the other hand, we have

$$d(x_{n_0+1}, x_m) \geq d(x_{n_0}, x_m) - d(x_{n_0}, x_{n_0+1}) > 2\varepsilon - \varepsilon = \varepsilon.$$

This is a contradiction. Therefore we have the desired result.

3. Fixed Point Theorems

In this Section, we discuss some fixed point theorems in complete metric spaces. We first give the following Theorem, which is essentially proved in [3].

Theorem 1 **([3])** *Let X be a complete metric space and let p be a w-distance on X. Let T be a mapping from X into itself. Suppose that there exists $r \in [0, 1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$ for every $x \in X$. Assume that either of the following holds:*

(i) *If $y \neq Ty$, then $\inf\{p(x, Tx) + p(x, y) : x \in X\} > 0$;*

(ii) *if $\{x_n\}$ and $\{Tx_n\}$ converge to y, then $y = Ty$;*

(iii) *T is continuous.*

Then there exists $x_0 \in X$ such that $x_0 = Tx_0$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

Proof. In the case of (i), it is proved in [3]. Let us prove (ii) $\Rightarrow$ (i). Suppose that $\inf\{p(x,Tx)+p(x,y) : x \in X\} = 0$. Then there exists $\{z_n\}$ such that $p(z_n,Tz_n) \to 0$ and $p(z_n,y) \to 0$. By Lemma 1, we have $Tz_n \to y$. Since

$$\begin{aligned} p(z_n,T^2z_n) &\leq p(z_n,Tz_n)+p(Tz_n,T^2z_n) \\ &\leq (1+r)p(z_n,Tz_n) \to 0, \end{aligned}$$

by Lemma 1 we have $T^2z_n \to y$ again. Put $x_n = Tz_n$. Then both $\{x_n\}$ and $\{Tx_n\}$ converge to y. So we have $y = Ty$ by (ii). This implies (ii) $\Rightarrow$ (i). Finally, we show (iii) $\Rightarrow$ (ii). Let T be continuous. Further assume that $\{x_n\}$ and $\{Tx_n\}$ converge to y. Then we have

$$Ty = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = y.$$

This completes the proof.

From Theorem 1, we have the following.

Corollary 1 *Let X be a complete metric space and let p be a w-distance on X. Let T be a mapping from X into itself. Suppose that there exists $r \in [0,1)$ such that either (a) or (b) holds:*

(a) $\max\{p(T^2x,Tx),p(Tx,T^2x)\} \leq r\max\{p(Tx,x),p(x,Tx)\}$ *for every* $x \in X$*;*

(b) $p(T^2x,Tx)+p(Tx,T^2x) \leq rp(Tx,x)+rp(x,Tx)$ *for every* $x \in X$.

Further assume that either of the following holds:

(i) *If* $y \neq Ty$*, then* $\inf\{p(x,Tx)+p(Tx,x)+p(x,y) : x \in X\} > 0$*;*

(ii) *if* $\{x_n\}$ *and* $\{Tx_n\}$ *converge to* y*, then* $y = Ty$*;*

(iii) T *is continuous.*

Then there exists $x_0 \in X$ *such that* $x_0 = Tx_0$. *Moreover, if* $v = Tv$*, then* $p(v,v) = 0$.

Before proving it, we prove the following Lemma.

Lemma 3 ([3]) *Let X be a metric space with metric d, let p be a w-distance on X and let α be a function from X into $[0,\infty)$. Then two functions on $X \times X$ defined as follows are w-distances on X:*

(i) $q(x,y) = \max\{\alpha(x),p(x,y)\}$ *for every* $x,y \in X$*;*

(ii) $q(x,y) = \alpha(x)+p(x,y)$ *for every* $x,y \in X$.

Proof. In the case of (i), it is proved in [3]. In the case of (ii), for every $x,y,z \in X$, we have

$$\begin{aligned} q(x,z) &= \alpha(x)+p(x,z) \\ &\leq \alpha(x)+\alpha(y)+p(x,y)+p(y,z) \\ &= q(x,y)+q(y,z). \end{aligned}$$

Therefore (1) is satisfied. (2) is obvious. We show (3). Let $\varepsilon > 0$ be fixed. Then since p is a w-distance on X, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \varepsilon$. So, assume $q(z,x) \le \delta$ and $q(z,y) \le \delta$. Then $p(z,x) \le \delta$ and $p(z,y) \le \delta$. Therefore $d(x,y) \le \varepsilon$.

Proof of Corollary 1 In the case of (a), we define $q_1 : X \times X \to [0,\infty)$ by $q_1(x,y) = \max\{p(Tx,x), p(x,y)\}$. In the case of (b), we define $q_2 : X \times X \to [0,\infty)$ by $q_2(x,y) = p(Tx,x) + p(x,y)$. These two functions q_1 and q_2 are w-distances by Lemma 3. Further we have that $q_i(Tx,T^2x) \le rq_i(x,Tx)$ for every $x \in X$ and $i = 1,2$. The conditions (ii) and (iii) are not connected with w-distance p. In the case of (i), let $y \in X$ be an element with $y \ne Ty$. Then we have, for all $i = 1,2$,

$$\begin{aligned} 0 &< \frac{1}{2}\inf\{p(x,Tx) + p(Tx,x) + p(x,y) : x \in X\} \\ &\le \inf\{q_i(x,Tx) + q_i(x,y) : x \in X\}. \end{aligned}$$

From Theorem 1, we have that there exists $x_0 \in X$ with $x_0 = Tx_0$. If $v = Tv$, then for all $i = 1,2$, $q_i(v,v) = 0$ from Theorem 1. This implies $p(v,v) = 0$.

In general, a w-distance p on X does not satisfy that $p(x,y) = p(y,x)$ for every $x, y \in X$. Hence, $p(T^2x,Tx) \le rp(Tx,x)$ differs from $p(Tx,T^2x) \le rp(x,Tx)$. So, the following Theorem is different from Theorem 1.

Theorem 2 *Let X be a complete metric space with metric d and let p be a w-distance on X. Let T be a mapping from X into itself. Suppose that there exists $r \in [0,1)$ such that $p(T^2x,Tx) \le rp(Tx,x)$ for every $x \in X$. Assume that either of the following holds:*

(i) *If $\{x_n\}$ converges to y and $\{p(Tx_n,x_n)\}$ converges to 0, then $p(Ty,y) = 0$;*

(ii) *if $\{x_n\}$ and $\{Tx_n\}$ converge to y, then $y = Ty$;*

(iii) *T is continuous.*

Then there exists $x_0 \in X$ such that $x_0 = Tx_0$. Moreover, if $v = Tv$, then $p(v,v) = 0$.

Proof. First, we shall show that $p(Ty,y) = 0$ is equivalent to $Ty = y$. If $p(Ty,y) = 0$, we have

$$p(T^2y,Ty) \le rp(Ty,y) = 0$$

and

$$p(T^2y,y) \le p(T^2y,Ty) + p(Ty,y) = 0.$$

So, we obtain $Ty = y$ by Lemma 1. If $Ty = y$, we have

$$p(Ty,y) = p(T^2y,Ty) \le rp(Ty,y)$$

and hence $p(Ty,y) = 0$. Next, we shall show (ii) $\Rightarrow$ (i). Let $\{x_n\}$ be a sequence in X such that $x_n \to y$ and $p(Tx_n,x_n) \to 0$. Then we have

$$p(T^2x_n,Tx_n) \le rp(Tx_n,x_n) \to 0$$

and hence

$$p(T^2x_n,x_n) \le p(T^2x_n,Tx_n) + p(Tx_n,x_n) \to 0.$$

By Lemma 1, we have $d(Tx_n, x_n) \to 0$. From $x_n \to y$, $\{Tx_n\}$ also converges to y. So, from (ii), $y = Ty$. This implies (ii) $\Rightarrow$ (i). We have (iii) $\Rightarrow$ (ii) from the proof of Theorem 1. So, we prove that T has a fixed point in the case of (i). Let $u \in X$ and put $u_n = T^n u$ for every $n \in \mathbb{N}$. Then we have

$$p(u_{n+1}, u_n) \leq rp(u_n, u_{n-1}) \leq \cdots \leq r^n p(u_1, u)$$

for every $n \in \mathbb{N}$. So, if $m > n$,

$$\begin{aligned} p(u_m, u_n) &\leq p(u_m, u_{m-1}) + \cdots + p(u_{n+1}, u_n) \\ &\leq r^{m-1} p(u_1, u) + \cdots + r^n p(u_1, u) \\ &\leq \frac{r^n}{1-r} p(u_1, u). \end{aligned}$$

By Lemma 2, $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $x_0 \in X$. We also have

$$p(Tu_n, u_n) \leq r^n p(u_1, u) \to 0.$$

So, by (i), we have $p(Tx_0, x_0) = 0$. Therefore x_0 is a fixed point of T. This completes the proof.

The final result of this Section is a generalization of Meir-Keeler's fixed point theorem [5].

Theorem 3 *Let X be a complete metric space, let p be a w-distance on X and let T be a mapping from X into itself. Suppose that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in X$, $p(x, y) < \varepsilon + \delta$ implies $p(Tx, Ty) < \varepsilon$. Then T has a unique fixed point in X.*

Proof. We first show $p(Tx, Ty) \leq p(x, y)$ for every $x, y \in X$. If not, there exist $x, y \in X$ and $\varepsilon > 0$ such that

$$p(Tx, Ty) > \varepsilon > p(x, y).$$

By the assumption, there exists $\delta > 0$ such that for every $z, w \in X$, $p(z, w) < \varepsilon + \delta$ implies $p(Tz, Tw) < \varepsilon$. So, we obtain $p(Tx, Ty) < \varepsilon$. This is a contradiction. We next show

$$\lim_{n \to \infty} p(T^n x, T^n y) = 0 \quad \text{for every} \quad x, y \in X. \tag{3.1}$$

In fact, $\{p(T^n x, T^n y)\}$ is nonincreasing and hence converges to some real number r. Assume $r > 0$. Then there exists $\delta > 0$ such that for every $z, w \in X$, $p(z, w) < r + \delta$ implies $p(Tz, Tw) < r$. For such δ, we can choose $m \in \mathbb{N}$ such that $p(T^m x, T^m y) < r + \delta$. So, we have $p(T^{m+1} x, T^{m+1} y) < r$. This is a contradiction and hence (3.1) holds. Let $u \in X$ and put $u_n = T^n u$ for every $n \in \mathbb{N}$. From (3.1) we have $\lim_{n \to \infty} p(u_n, u_{n+1}) = 0$. We shall show that

$$\lim_{n \to \infty} \sup_{n < m} p(u_n, u_m) = 0. \tag{3.2}$$

Let $\varepsilon > 0$ be arbitrary. Then without loss of generality, there exists $\delta \in (0, \varepsilon)$ such that for every $z, w \in X$, $p(z, w) < \varepsilon + \delta$ implies $p(Tz, Tw) < \varepsilon$. For such δ, there exists $n_0 \in \mathbb{N}$ such that $p(u_n, u_{n+1}) < \delta$ for every $n \geq n_0$. Assume that there exists $m > \ell \geq n_0$ such that $p(u_\ell, u_m) > 2\varepsilon$. Since

$$p(u_\ell, u_{\ell+1}) < \varepsilon + \delta < p(u_\ell, u_m),$$

there exists $k \in \mathbb{N}$ with $\ell < k < m$ such that

$$p(u_\ell, u_k) < \varepsilon + \delta \leq p(u_\ell, u_{k+1}).$$

Then since $p(u_\ell, u_k) < \varepsilon + \delta$, we have $p(u_{\ell+1}, u_{k+1}) < \varepsilon$. On the other hand, we have

$$\begin{aligned} p(u_{\ell+1}, u_{k+1}) &\geq p(u_\ell, u_{k+1}) - p(u_\ell, u_{\ell+1}) \\ &> \varepsilon + \delta - \delta \\ &= \varepsilon. \end{aligned}$$

This is a contradiction. Therefore $m > n \geq n_0$ implies $p(u_n, u_m) \leq 2\varepsilon$ and hence (3.2) holds. From Lemma 2 , $\{u_n\}$ is Cauchy and hence there exists $x_0 \in X$ such that $\{u_n\}$ converges to x_0. Since for $x \in X$, $p(x, \cdot)$ is lower semicontinuous, we have

$$\begin{aligned} \limsup_{n\to\infty} p(u_n, x_0) &\leq \limsup_{n\to\infty} \liminf_{m\to\infty} p(u_n, u_m) \\ &\leq \lim_{n\to\infty} \sup_{n<m} p(u_n, u_m) = 0. \end{aligned}$$

So,

$$\limsup_{n\to\infty} p(u_n, Tx_0) \leq \lim_{n\to\infty} p(u_{n-1}, x_0) = 0.$$

By Lemma 1 we have $Tx_0 = x_0$. From (3.1), we obtain

$$p(x_0, x_0) = \lim_{n\to\infty} p(T^n x_0, T^n x_0) = 0.$$

If $z = Tz$, then

$$p(x_0, z) = \lim_{n\to\infty} p(T^n x_0, T^n z) = 0.$$

So, from Lemma 1 , $x_0 = z$. Therefore a fixed point of T is unique. This completes the proof.

4. Kannan Mappings

In this Section, we shall discuss fixed point theorems for Kannan mappings with respect to a w-distance p. Let X be a metric space and let T be a mapping from X into itself. Then T is called weakly Kannan or p-Kannan if there exist a w-distance p on X and $\alpha \in \left[0, \dfrac{1}{2}\right)$ such that either (a) or (b) holds:

(a) $p(Tx, Ty) \leq \alpha p(Tx, x) + \alpha p(Ty, y)$ for every $x, y \in X$;

(b) $p(Tx, Ty) \leq \alpha p(Tx, x) + \alpha p(y, Ty)$ for every $x, y \in X$.

Theorem 4 *Let X be a complete metric space. If a mapping T from X into itself is p-Kannan, then T has a unique fixed point $x_0 \in X$. Further such x_0 satisfies $p(x_0, x_0) = 0$.*

Proof. In the case of (a), there are a w-distance p and and $\alpha \in \left[0, \frac{1}{2}\right)$ such that $p(Tx, Ty) \leq \alpha p(Tx, x) + \alpha p(Ty, y)$ for every $x, y \in X$. Putting $r = \dfrac{\alpha}{1-\alpha} \in [0, 1)$, we have $p(T^2x, Tx) \leq rp(Tx, x)$ for every $x \in X$. Assume that $x_n \to y$ and $p(Tx_n, x_n) \to 0$. Then we have

$$\begin{aligned} p(Ty, y) &\leq \liminf_{n\to\infty} p(Ty, x_n) \\ &\leq \liminf_{n\to\infty} \{p(Ty, Tx_n) + p(Tx_n, x_n)\} \\ &\leq \liminf_{n\to\infty} \{\alpha p(Ty, y) + \alpha p(Tx_n, x_n) + p(Tx_n, x_n)\} \\ &= \alpha p(Ty, y) \end{aligned}$$

and hence $p(Ty, y) = 0$. By Theorem 2, there exists $x_0 \in X$ such that $x_0 = Tx_0$ and $p(x_0, x_0) = 0$. Further a fixed point of T is unique. In fact, if $z = Tz$, then $p(z, z) = 0$ by Theorem 2. So, we have

$$\begin{aligned} p(x_0, z) &= p(Tx_0, Tz) \leq \alpha p(Tx_0, x_0) + \alpha p(Tz, z) \\ &= \alpha p(x_0, x_0) + \alpha p(z, z) = 0. \end{aligned}$$

From Lemma 1 we have $x_0 = z$. In the case of (b), putting $r = \dfrac{\alpha}{1-\alpha} \in [0, 1)$, we have $p(Tx, T^2x) \leq rp(Tx, x)$ and $p(T^2x, Tx) \leq rp(x, Tx)$ for every $x \in X$. So,

$$p(T^2x, Tx) + p(Tx, T^2x) \leq rp(Tx, x) + rp(x, Tx)$$

for every $x \in X$. Assume that $p(x_n, Tx_n) \to 0$ and $p(x_n, y) \to 0$. Then $\{Tx_n\}$ converges to y by Lemma 1. So, we have

$$\begin{aligned} p(Ty, y) &\leq \liminf_{n\to\infty} p(Ty, Tx_n) \\ &\leq \liminf_{n\to\infty} \{\alpha p(Ty, y) + \alpha p(x_n, Tx_n)\} \\ &= \alpha p(Ty, y) \end{aligned}$$

and hence $p(Ty, y) = 0$. Since $p(Ty, T^2y) \leq rp(Ty, y) = 0$, we have $y = T^2y$ by Lemma 1. So, $p(y, Ty) = p(T^2y, Ty) \leq rp(y, Ty)$ and hence $p(y, Ty) = 0$. We also have $p(y, y) \leq p(y, Ty) + p(Ty, y) = 0$. So, we have $y = Ty$ from Lemma 1. Therefore $y \neq Ty$ implies that

$$\begin{aligned} 0 &< \inf\{p(x, Tx) + p(x, y) : x \in X\} \\ &\leq \inf\{p(x, Tx) + p(Tx, x) + p(x, y) : x \in X\}. \end{aligned}$$

By Corollary 1, there exists $x_0 \in X$ such that $x_0 = Tx_0$ and $p(x_0, x_0) = 0$. As in the case of (a), we obtain that a fixed point of T is unique.

A mapping T from a metric space X into itself is called weakly contractive [6] if there exist a w-distance p on X and $r \in [0, 1)$ such that $p(Tx, Ty) \leq rp(x, y)$ for every $x, y \in X$. We obtain the following.

Proposition *Let X be a metric space with metric d and let T be a weakly contractive mapping from X into itself. Then T is weakly Kannan.*

Proof. Since T is weakly contractive, there exist a w-distance p on X and $r \in [0, 1)$ such that $p(Tx, Ty) \leq rp(x, y)$ for every $x, y \in X$. We first show

$$p(T^n x, x) \leq \frac{1}{1-r} p(Tx, x) \quad \text{and} \quad p(x, T^n x) \leq \frac{1}{1-r} p(x, Tx)$$

for every $x \in X$ and $n \in \mathbb{N}$. In fact,

$$\begin{aligned} p(T^n x, x) &\leq p(T^n x, T^{n-1} x) + p(T^{n-1} x, T^{n-2} x) + \cdots + p(Tx, x) \\ &\leq r^{n-1} p(Tx, x) + r^{n-2} p(Tx, x) + \cdots + p(Tx, x) \\ &\leq \frac{1}{1-r} p(Tx, x). \end{aligned}$$

Similarly we have

$$\begin{aligned} p(x, T^n x) &\leq p(x, Tx) + p(Tx, T^2 x) + \cdots + p(T^{n-1} x, T^n x) \\ &\leq p(x, Tx) + rp(x, Tx) + \cdots r^{n-1} p(x, Tx) \\ &\leq \frac{1}{1-r} p(x, Tx). \end{aligned}$$

We next prove a function β from X into $[0, \infty)$ defined by $\beta(x) = \lim_{k \to \infty} p(T^k x, x)$ is well-defined and lower semicontinuous. Let $x \in X$ be fixed. Take $m, n \in \mathbb{N}$ with $m > n$. Then since $p(T^m x, x) \leq p(T^m x, T^n x) + p(T^n x, x)$ and $p(T^n x, x) \leq p(T^n x, T^m x) + p(T^m x, x)$, we have

$$\begin{aligned} |p(T^m x, x) - p(T^n x, x)| &\leq \max\{p(T^m x, T^n x), p(T^n x, T^m x)\} \\ &\leq r^n \max\{p(T^{m-n} x, x), p(x, T^{m-n} x)\} \\ &\leq \frac{r^n}{1-r} \max\{p(Tx, x), p(x, Tx)\}. \end{aligned}$$

So, $\{p(T^n x, x)\}$ is a Cauchy sequence and hence $\beta(x)$ is well-defined for every $x \in X$. Let $y \in X$ be fixed. Take a sequence $\{x_n\}$ such that $\{x_n\}$ converges to y and $\{\beta(x_n)\}$ converges to some $t \in [0, \infty)$. Then $\{p(y, x_n)\}$ is bounded. In fact, from

$$\begin{aligned} p(y, x_n) &\leq p(y, T^k y) + p(T^k y, T^k x_n) + p(T^k x_n, x_n) \\ &\leq \frac{p(y, Ty)}{1-r} + r^k p(y, x_n) + p(T^k x_n, x_n) \end{aligned}$$

for every $n, k \in \mathbb{N}$, we have $p(y, x_n) \leq \dfrac{p(y, Ty)}{1-r} + \beta(x_n)$ for every $n \in \mathbb{N}$ and hence $\{p(y, x_n)\}$ is bounded. Let $\varepsilon > 0$ be arbitrary. Then there exists $k_0 \in \mathbb{N}$ which satisfies $p(T^{k_0} y, y) \geq \beta(y) - \varepsilon$, $\dfrac{r^{k_0}}{1-r} p(y, Ty) \leq \varepsilon$ and $r^{k_0} p(y, x_n) \leq \varepsilon$ for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed. Then there exists $k_1 \in \mathbb{N}$ such that $k_1 > k_0$ and $p(T^{k_1} x_n, x_n) \leq \beta(x_n) + \varepsilon$. We obtain that

$$p(T^{k_0} y, x_n) \leq p(T^{k_0} y, T^{k_1} y) + p(T^{k_1} y, T^{k_1} x_n) + p(T^{k_1} x_n, x_n)$$

$$
\begin{aligned}
&\le \frac{r^{k_0}}{1-r}p(y,Ty) + r^{k_1}p(y,x_n) + \beta(x_n) + \varepsilon \\
&\le \varepsilon + \varepsilon + \beta(x_n) + \varepsilon \\
&= \beta(x_n) + 3\varepsilon.
\end{aligned}
$$

So we have

$$\beta(y) \le p(T^{k_0}y, y) + \varepsilon \le \liminf_{n\to\infty} p(T^{k_0}y, x_n) + \varepsilon \le \lim_{n\to\infty} \beta(x_n) + 4\varepsilon = t + 4\varepsilon.$$

Since ε is arbitrary, we have $\beta(y) \le t$. Therefore β is lower semicontinuous. Define a function q from $X \times X$ into $[0,\infty)$ by $q(x,y) = \beta(x) + \beta(y)$. Let us prove that q is a w-distance on X. (1) and (2) are obvious. To show (3), we let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $p(z,x) \le 3\delta$ and $p(z,y) \le 3\delta$ imply $d(x,y) \le \varepsilon$. Assume that $q(z,x) \le \delta$ and $q(z,y) \le \delta$. Then $\beta(x) \le \delta$ and $\beta(y) \le \delta$. We take $k_2 \in \mathbb{N}$ which satisfies $p(T^{k_2}x, x) \le \beta(x) + \delta$, $p(T^{k_2}y, y) \le \beta(y) + \delta$ and $r^{k_2}p(x,y) \le \delta$. Then we have

$$
\begin{aligned}
p(T^{k_2}x, x) &\le \beta(x) + \delta \le 3\delta \\
&\text{and} \\
p(T^{k_2}x, y) &\le p(T^{k_2}x, T^{k_2}y) + p(T^{k_2}y, y) \\
&\le r^{k_2}p(x,y) + \beta(y) + \delta \\
&\le 3\delta
\end{aligned}
$$

and hence $d(x,y) \le \varepsilon$. This implies (3). So, we obtain that q is a w-distance on X. Finally, we prove that T is q-Kannan. Put $\alpha = \dfrac{r}{1+r} \in \left[0, \dfrac{1}{2}\right)$. Since

$$\beta(Tx) = \lim_{k\to\infty} p(T^kTx, Tx) \le r \lim_{k\to\infty} p(T^kx, x) = r\beta(x)$$

for every $x \in X$, we have

$$
\begin{aligned}
q(Tx,Ty) &= \beta(Tx) + \beta(Ty) \\
&= \frac{r}{1+r}\beta(Tx) + \frac{1}{1+r}\beta(Tx) + \frac{r}{1+r}\beta(Ty) + \frac{1}{1+r}\beta(Ty) \\
&\le \frac{r}{1+r}\beta(Tx) + \frac{r}{1+r}\beta(x) + \frac{r}{1+r}\beta(Ty) + \frac{r}{1+r}\beta(y) \\
&= \alpha q(Tx,x) + \alpha q(Ty,y)
\end{aligned}
$$

for every $x, y \in X$. This completes the proof.

As a direct consequence of Proposition, we obtain the following characterization of metric completeness.

Corollary 2 *Let X be a metric space. Then the following are equivalent:*

(i) *X is complete;*

(ii) *every weakly contractive mapping from X into itself has a fixed point in X;*

(iii) *every weakly Kannan mapping from X into itself has a fixed point in X.*

Proof. In [6], we have that (i) and (ii) are equivalent. From Theorem 4 , we have that (i) implies (iii). From Proposition, we have that (iii) implies (ii). This completes the proof.

5. Appendix

In general, a w-distance p does not necessarily satisfy $p(x,y) = p(y,x)$. So, in our definition, a mapping T is not necessarily called weakly Kannan even if there exist a w-distance p and $\alpha \in \left[0, \frac{1}{2}\right)$ such that either (c) or (d) holds:

(c) $p(Tx,Ty) \le \alpha p(x,Tx) + \alpha p(Ty,y)$ for every $x, y \in X$;

(d) $p(Tx,Ty) \le \alpha p(x,Tx) + \alpha p(y,Ty)$ for every $x, y \in X$.

We know the following Example.

Example *Let $X = [0,1] \subset \mathbb{R}$ be a metric space with the usual metric. Define a w-distance p on X by*

$$p(x,y) = \begin{cases} 9, & \text{if } x = 0, \\ y - x, & \text{if } 0 < x \le y, \\ 3x - 3y, & \text{if } x > y \end{cases}$$

and a mapping T from X into itself by

$$Tx = \begin{cases} 1, & \text{if } x = 0, \\ x/10, & \text{if } x \neq 0. \end{cases}$$

Then (c) and (d) hold in the case of $\alpha = \frac{1}{3}$. But T has not a fixed point.

Proof. Since a function $q : X \times X \to [0,\infty)$ defined by

$$q(x,y) = \begin{cases} y - x, & \text{if } x \le y, \\ 3x - 3y, & \text{if } x > y \end{cases}$$

is a w-distance on X, p is also a w-distance on X from Lemma 3. For every $x, y \in X$, we have

$$\begin{aligned} p(T0,Ty) &= 3 - 3Ty \le 3 = \frac{1}{3}p(0,T0) \le \frac{1}{3}p(0,T0) + \frac{1}{3}p(Ty,y) \quad \text{and} \\ p(Tx,T0) &= 1 - Tx \le 1 = \frac{1}{3}p(T0,0) \le \frac{1}{3}p(x,Tx) + \frac{1}{3}p(T0,0). \end{aligned}$$

If $x \neq 0$ and $y \neq 0$, then

$$\begin{aligned} p(Tx,Ty) &= \frac{1}{10}p(x,y) \le \frac{3}{10}|x-y| \le \frac{3}{10}x + \frac{3}{10}y \\ &\le \frac{1}{3}p(x,Tx) + \frac{1}{3}p(Ty,y). \end{aligned}$$

We also have $p(Tx,x) \le p(x,Tx)$ for every $x \in X$. Therefore

$$p(Tx,Ty) \le \frac{1}{3}p(x,Tx) + \frac{1}{3}p(Ty,y) \le \frac{1}{3}p(x,Tx) + \frac{1}{3}p(y,Ty)$$

for every $x, y \in X$ and hence (c) and (d) hold. Clearly, T has not a fixed point. This completes the proof.

However, we have the following.

Theorem 5 *Let X be a complete metric space and let T be a continuous mapping from X into itself. Suppose that there exist a w-distance p on X and $\alpha \in \left[0, \frac{1}{2}\right)$ such that either (c) or (d) holds Then there exists a unique fixed point $x_0 \in X$ of T. Moreover, such x_0 satisfies $p(x_0, x_0) = 0$.*

Proof. In the case of (c), putting $r = \frac{\alpha}{1-\alpha} \in [0, 1)$, from $p(Tx, T^2x) \leq \alpha p(x, Tx) + \alpha p(T^2x, Tx)$ and $p(T^2x, Tx) \leq \alpha p(Tx, T^2x) + \alpha p(Tx, x)$, we have

$$p(T^2x, Tx) + p(Tx, T^2x) \leq rp(Tx, x) + rp(x, Tx)$$

for every $x \in X$. So, from Corollary 1, we prove the desired result. In the case of (d), we have $p(Tx, T^2x) \leq rp(x, Tx)$ for every $x \in X$. Therefore from Theorem 1, we prove the desired result. This completes the proof.

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