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On the Kervaire classes of tangential normal maps of Lens spaces

By

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Abstract. We study the Kervaire classes of tangential normal maps for the lens space when the order of the fundamental group is a multiple of four. We obtain the vanishing of the Kervaire classes upto certain dimension. The method used here is the relation of the secondary cohomology operation corresponding to the element $h_0^2h_2(=h_1^3)$ in the cohomology of the mod 2 Steenrod algebra.

1. Introduction and statement of results

Let M be a smooth manifold and let $f: M \to F/O$ be a normal map. The surgery obstruction for $M = CP^n$ was studied by Stolz [7]. His main result was to show that when n is odd, the Kevaire surgery obstruction of a tangential normal map $f: CP^n \to SF$ vanishes. The analogous result for a real projective space RP^n was given in [4].

We shall denote by $L^{2n-1}(4m; q_1, q_2, \ldots, q_n)$ the lens space which is the quotient of the $(2n-1)$-dimensional sphere $S^{2n-1} = \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n | \sum |z_k|^2 = 1\}$ under the free periodic action of order $4m$,$(z_1, z_2, \ldots, z_n) \mapsto (\omega^{q_1}z_1, \omega^{q_2}z_2, \ldots, \omega^{q_n}z_n)$, where $\omega = \exp(\pi\sqrt{-1}/2m)$ and $(q_k, 4m) = 1$ $(k = 1, 2, \ldots, n)$. We shall simply write $L^{2n-1}(4m)$ instead of $L^{2n-1}(4m; q_1, q_2, \ldots, q_n)$ since the choice of q_k's is not important in this paper.

A tangential normal map with target space $L^{2n-1}(4m)$ is represented by a map $f: L^{2n-1}(4m) \to SF$ where SF is the H-space of base point preserving degree one maps $S^\infty \to S^\infty$. We shall denote by $k_{2^{i+1}-2}$ $(i \geq 1)$ the universal smooth Kervaire class in $H^{2^{i+1}-2}(F/O)$. Here and throughout this paper, all cohomology coefficients are $\mathbf{Z}/2$ and will be omitted.These classes are characterized by Sullivan's *characteristic variety formula* of the Kervaire invariant for a normal map $g: M \to F/O$ in general

$$c(g) = (V(M)^2 \sum_{i\geq 1} g^*(k_{2^{i+1}-2}))[M],$$

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where $V(M)$ is the total Wu class of M.

Let $\lambda : SF \to F/O$ be the natural projection. Our main results are the following.

Theorem I. Let $f : L^{2n-1}(4m) \to SF$ be a tangential normal map $n \geq 3$. Then $f^*\lambda^* k_{2^{i+1}-2}$ vanishes if $3 \cdot 2^{i-1} \leq n$ holds.

Theorem II. For any tangential normal map of $L^{2n-1}(4m)$ $(n \geq 3)$, its surgery obstruction vanishes unless n is a power of 2.

2. Universal Kervaire classes and secondary cohomology operations

Let us recall some facts concerning the characterization of universal smooth Kervaire classes from the viewpoint of secondary cohomology operations. First we shall explain the spaces and maps that appear in the diagram below.

$$\begin{array}{ccccc} & & \widetilde{BSF} & \longrightarrow & PK \\ & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow \\ \Sigma SF & \xrightarrow{\;h\;} & BSF & \xrightarrow{\;w\;} & K \\ \downarrow{\scriptstyle \Sigma\lambda} & & & & \\ \Sigma(F/O) & & & & \end{array}$$

Here K is the product of Eilenberg-McLane spaces $\prod_{i \geq 2} K(\mathbf{Z}/2, i)$ and w classifies the Stiefel-Whitney class w_i for each $i \geq 2$. The fibration $\tilde{p} : \widetilde{BSF} \to BSF$ is the pull-back of the canonical path fibration $PK \to K$ via w. The map $h : \Sigma SF \to BSF$ classifies the fibration over ΣSF with characteristic map $id : SF \to SF$. Let $\tilde{\gamma} = \tilde{p}^*\gamma$ be the pull-back of the universal spherical fibration γ over BSF. The Thom class of $\tilde{\gamma}$ will be denoted by $U(\tilde{\gamma})$. The main results of [3] can be summarized as follows (see also [6]).

Theorem ([3]) a) Let η be a stable spherical fibration with vanishing Stiefel-Whitney classes. Then $\phi_{r,r}(U(\eta))$ is defined with total inderminacy zero. In particular, there exists a unique element e_{r+1} in $H^{2^{r+1}-1}(\widetilde{BSF})$ with $e_{r+1} \cup U(\tilde{\gamma}) = \phi_{r,r}(U(\tilde{\gamma}))$. Here, $\phi_{r,r}$ is the Adams' secondary cohomology operation ([1]) dual to h_r^2.
b) There exists a unique primitive element $\varepsilon_{r+1} \in H^{2^{r+1}-1}(BSF)$ with the properties $\tilde{p}^*(\varepsilon_{r+1}) = e_{r+1}$ and $h^*(\varepsilon_{r+1}) = \Sigma\lambda^*(\Sigma k_{2^{r+1}-2})$.

3. Proof of Theorems

We begin with the following lemma.

Lemma. $Sq^1 H^4(\widetilde{BSF}) = 0$.

Proof. Recall that $H^*(\widetilde{BSF}) \cong C \otimes H^*(\widetilde{BSO})$, where $C = \mathbf{Z}/2\Box_{H^*(BSO)} H^*(BSF)$ is the exterior algebra ([2],[3],[5]), and $\widetilde{BSO}$ is the fiber of the map $w : BSO \longrightarrow K$, which classifies $\sum_{i\geq 2} w_i$. In dimensions less than 5, C is additively generated by$e_{0,1}$ and $e_{1,1} = Sq^1 e_{0,1}$ with $\dim e_{0,1} = 3$. We know that $H^*(\widetilde{BSO}) = \mathbf{Z}/2[u_{n,I} | n \geq 2,\ I$: admissible $, 0 < e(I) < n]$, with $\dim u_{n,I} = |I| + n - 1$. Hence the generators of $H^4(\widetilde{BSF})$ are $Sq^1 e_{0,1} \otimes 1$, $1 \otimes u_{2,(1)}^2$, $1 \otimes u_{3,(2)}$ and $1 \otimes u_{4,(1)}$. Since the inclusion of the fiber $i : \Omega K \longrightarrow \widetilde{BSO}$ induces an injective map $i^* : H^*(\widetilde{BSO}) \longrightarrow H^*(\Omega K)$,with $i^*(u_{n,I}) = Sq^I \iota_{n-1}$, these generators are annihilated by Sq^1. This completes the proof of Lemma.

The main point of Theorem I is based on the following proposition. Recall that the Adem relation $Sq^2 Sq^2 + Sq^3 Sq^1 = 0$ defines the secondary cohomology operation $\phi_{1,1}$.

Proposition. Let e_2 be the secondary characteristic class in $H^3(\widetilde{BSF})$ associated to the Adem relation $Sq^2 Sq^2 + Sq^3 Sq^1 = 0$. Then we have $Sq^2 e_2 = 0$ mod zero indeterminacy.

Proof. Consider the bar resolution $B(A)$ of $\mathbf{Z}/2$ over the mod 2 Steenrod algebra A([1])

$$\cdots \xrightarrow{d_{s+1}} B(A)_s \xrightarrow{d_s} \cdots \xrightarrow{d_3} B(A)_2 \xrightarrow{d_2} B(A)_1 \xrightarrow{d_1} B(A)_0 = A \xrightarrow{\varepsilon} \mathbf{Z}/2 \to 0\ ,$$

where $B(A)_s = A \otimes \bar{B}(A)_s$, $\bar{B}(A)_s$ is the s-fold tensor product of $I(A) = \mathrm{Ker}(\varepsilon : A \to \mathbf{Z}/2)$ and the differential is given by

$$d_s(a[a_1|a_2|\ldots|a_s]) = aa_1[a_2|\ldots|a_s] + \sum_{i=1}^{s-1} a[a_1|\ldots|a_i a_{i+1}|\ldots|a_s]\ .$$

Adams defined a map $\theta : \mathrm{Ker}\, d_s \to \mathrm{Tor}^A_{s+1,*}(\mathbf{Z}/2, \mathbf{Z}/2)$ by $\theta(z) = [1 \otimes_A z']$, where $z' \in B(A)_{s+1}$ satisfies $d_{s+1}(z') = z$. Take the following three elements in Ker d_1

$$\begin{aligned} z_1 &= Sq^1[Sq^1] \\ z_2 &= Sq^2[Sq^2] + Sq^3[Sq^1] \\ z_3 &= Sq^1[Sq^4] + Sq^4[Sq^1] + Sq^2Sq^1[Sq^2]\ . \end{aligned}$$

They have inverse images in $B(A)_2$:

$$\begin{aligned} z_1' &= [Sq^1|Sq^1] \\ z_2' &= [Sq^2|Sq^2] + [Sq^3|Sq^1] \\ z_3' &= [Sq^1|Sq^4] + [Sq^4|Sq^1] + [Sq^2Sq^1|Sq^2]\ . \end{aligned}$$

It is easy to verify that the elements z_1, z_2, and z_3 define secondary cohomology operations $\phi_{0,0}$, $\phi_{1,1}$ and $\phi_{0,2}$ respectively. These elementsenjoy the relation

$$Sq^4\, z_1 + Sq^2\, z_2 + Sq^1\, z_3 = 0\ .$$

In fact if we put $z = Sq^4 z_1' + Sq^2 z_2' + Sq^1 z_3'$, z is in Ker d_2 and $\theta(z)$ is dual to $h_0^2 h_2 = h_1^3$. By Theorems 3.7.1 and 3.7.2 of [1], we get the relation

$$Sq^4\,\phi_{0,0}(U(\tilde{\gamma})) + Sq^2\,\phi_{1,1}(U(\tilde{\gamma})) + Sq^1\,\phi_{0,2}(U(\tilde{\gamma})) \equiv 0$$

modulo indeterminacy $Q = Sq^4 Q_1 + Sq^2 Q_2 + Sq^1 Q_3$ where Q_i is the total indeterminacy of the secondary operation defined by z_i $(i = 1, 2, 3)$. The first two indeterminacies Q_1 and Q_2 vanish since z_1 and z_2 are of the form $\sum Sq^{a_j}[Sq^{b_j}]$ with $a_j \geq b_j$. Q_3 is contained in $\mathrm{Im}\, Sq^1 + \mathrm{Im}\, Sq^4 + \mathrm{Im}\, Sq^2 Sq^1$. Therefore Q is contained in $\mathrm{Im}\, Sq^5 + \mathrm{Im}\, Sq^3 Sq^1$. But since $Sq^5(U(\tilde{\gamma})) = 0$ and BSF is 1-connected, Q vanishes. In the above relation, it is clear that $Sq^4\phi_{0,0}(U(\tilde{\gamma})) = 0$, and by Lemma $Sq^1\phi_{0,2}(U(\tilde{\gamma}))$ vanishes.These arguments show that$Sq^2\phi_{1,1}(U(\tilde{\gamma})) = 0$ with zero indeterminacy. This is equivalent to $Sq^2(e_2 \cup U(\tilde{\gamma})) = 0$ and we get our assertion.

Proof of Theorem I. Let $f : L^{2n-1}(4m) \to SF$ be a tangential normal map and $\pi : RP^{2n-1} \to L^{2n-1}(4m)$ be the canonical $2m$-fold covering. Then we have $\Sigma(f \circ \pi)^* h^*(w_{2i}) = 0$ since the map $\pi^* : H^*(L^{2n-1}(4m)) \to H^*(RP^{2n-1})$ is zero in odd dimensions. Odd dimensional Stiefel-Whitney classes also vanish since $w_{2i+1} = Sq^1 w_{2i}$ for oriented fibrations. Therefore we have a lifting $g : \Sigma(RP^{2n-1}) \to \widetilde{BSF}$ of $h \circ \Sigma(f \circ \pi)$. We have

$$Sq^2\Sigma\pi^*\Sigma f^*\Sigma\lambda^*(k_2) = Sq^2\Sigma\pi^*\Sigma f^* h^*(\varepsilon_2) = Sq^2 g^*(e_2) = g^*(Sq^2 e_2) = 0\,,$$

by our Proposition. This shows that $Sq^2\pi^* f^*\lambda^*(k_2) = 0$ and since Sq^2 is an isomorphism on $H^2(RP^{2n-1})$, and π^* is an isomorphism in even dimensions, we have $f^*\lambda^*(k_2) = 0$. The vanishing of $f^*\lambda^*(k_{2^{i+1}-2})$ for $3 \cdot 2^{i-1} \leq n$ can be deduced from Theorem B of [4] by considering the class $\pi^* f^*\lambda^*(k_{2^{i+1}-2})$. This proves Theorem I.

Proof of Theorem II. If n is odd, the surgery obstruction for $f : L^{2n-1}(4m) \to SF$ vanishes since the surgery obstruction group $L_{2n-1}(\mathbf{Z}/4m) = 0$. We assume that n is even. It is known that the surgery obstruction coincides with the surgery obstruction for $(f \circ \pi)|RP^{2n-2} : RP^{2n-2} \to SF$([8], Chap.14).By the characteristic variety formula, it is not hard to see that this obstruction vanishes if and only if$(f \circ \pi)^* k_{2^{\nu_2(n)+1}-2}$ vanishes, where $\nu_2(n)$ denotes the 2-order of n. If n is not a power of 2, we have $2^{\nu_2(n)} \leq n/3$. Hence by Theorem I, $(f \circ \pi)^* k_{2^{\nu_2(n)+1}-2} = 0$. This completes the proof of Theorem II.

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