Yokohama Mathematical
Journal Vol. 44,1997

The Lamperti transformation for self-similar processes

(Dedicated to the memory of Stamatis Cambanis)

By

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(Received December 12,1995; Revised May 15,1996)

Abstract. In this paper we establish the uniqueness of the Lamperti transformation leading from self-similar to stationary processes, and conversely. We discuss α-stable processes, which allow to understand better the difference between the Gaussian and non-Gaussian cases. As a by-product we get a natural construction of two distinct α-stable Ornstein–Uhlenbeck processes via the Lamperti transformation for $0<\alpha<2$. Also a new class of mixed linear fractional α-stable motions is introduced.

1. Introduction

Self-similar (*ss*) processes, introduced by Lamperti [6], are the ones that are invariant under suitable translations of time and scale (Definition 1. 1 below). In the past few years there has been an explosive growth in the study of self-similar processes, cf. e.g. Taqqu [11], Maejima [7], Samorodnitsky and Taqqu [9], and Willinger et al. [12]. This caused that various examples of such processes have been found and relationships with distinct types of processes have been established.

Lamperti has defined a transformation which changes stationary processes to the corresponding self-similar ones in the following way:

Proposition 1.1 (Lamperti [6]) *If* $Y=(Y(t))_{t\in R}$ *is a stationary process and if for some* $H>0$

$$X(t)=t^{H}Y(\log t),\quad for\ \ t>0,\quad X(0)=0,$$

then $X=(X(t))$ *is* $H-ss$. *Conversely, every non-trivial ss-process with* $X(0)=0$ *is obtained in this way from some stationary process* Y.

1991 Mathematics Subject Classification: 60G18.
Key words and phrases: Self-similar process,Lamperti transformation,stationary processes,stable Ornstein-Uhlenbeck processes.

In this context the question arises whether the transformations proposed by Lamperti are unique. In this paper we search for functions ϕ, ψ, ζ and η such that

$$X(t) = \phi(t)Y(\psi(t)) \text{ is } H-ss \text{ for a non-trivial stationary process } Y$$

and

$$Y(t) = \zeta(t)X(\eta(t)) \text{ is stationary for a non-trivial } H-ss \text{ process } X.$$

There are two theorems presented in Section 3 which lead to the conclusion that essentially $\phi(t) = t^H$, $\psi(t) = a \log t$, $\zeta(t) = e^{-bHt}$ and $\eta(t) = e^{bt}$ for some $a, b \in R$ according to our convention (see Definition 1. 4). In Section 2, a computer visualization of the Lamperti transformation is provided. Section 4 is devoted to the study of the influence of various a's and b's on distributions of these processes. This is illustrated by four processes chosen to express a difference between the Gaussian and non-Gaussian case. As a result of this investigation, we construct, in a natural way, a pair of distinct α-stable Ornstein–Uhlenbeck processes for $\alpha < 2$, already known in the literature (Adler et al. [1]). This supports the conjecture that there are only two such processes. In the last section (Section 5), we discuss a new class of self-similar stable processes whose corresponding stationary processes Y through the Lamperti transformation are stable mixed moving average.

We start with some basic definitions. $X(t) \stackrel{d}{=} Y(t)$ denotes the equality of all finite-dimensional distributions and $X(t) \stackrel{d}{\sim} Y(t)$ means the equality of one-dimensional distributions for fixed t.

Definition 1. 1 (Lamperti [6]) *A process $X = (X(t))_{t\geq 0}$ is self-similar (ss) if for some $H \in R$, $X(ct) \stackrel{d}{=} c^H X(t)$ for every $c > 0$.*

We call this X an $H-ss$ *process*. X is said to be trivial if $X(t) = t^H X(1)$ *a.e.*, $t \geq 0$.

Definition 1. 2 *A process $Y = (Y(t))_{t\in R}$ is stationary if $Y(t+\sigma) \stackrel{d}{=} Y(t)$ for every $\sigma \in R$.*

Y is said to be trivial if $Y(t) = Y(0)$ *a.e.*, $t \in R$.

Definition 1. 3 *For $\alpha \in (0,2]$, a process $(X(t))_{t\in R}$ is called symmetric α-stable (which will be referred to as $S\alpha S$) if for arbitrary $n \in N$, $a_1,\dots,a_n \in R$, $t_1,\dots,t_n \in R$ a random variable $\sum_{i=1}^{n} a_i X(t_i)$ has an $S\alpha S$ distribution. An $S\alpha S$ process $(X(t))_{t\in R}$ is called an $S\alpha S$ Lévy motion if it has stationary and independent increments, is continuous in probability and $X(0) = 0$ a.e. We denote it by $Z_\alpha = (Z_\alpha(t))_{t\in R}$.*

Definition 1.4 *When for two stochastic processes $X = (X(t))$ and $Y = (Y(t))$, $X(t) \stackrel{d}{=} aY(t)$ for some $a \in R\backslash\{0\}$, we say that X and Y are essentially equivalent.*

Henceforth we will not distinguish between such processes. Furthermore, we will assume that all considered processes throughout this paper are stochastically continuous.

2. Computer visualization of the Lamperti transformation

We find it interesting to illustrate the Lamperti transformation by demonstrating graphically self-similar processes and corresponding stationary ones. We generate the fractional stable motion with parameters H and α, applying its integral representation, that is,

$$X(t) = \int_{-\infty}^{0} (|t-u|^{H-\frac{1}{\alpha}} - |u|^{H-\frac{1}{\alpha}}) Z_\alpha(du) + \int_{0}^{t} |t-u|^{H-\frac{1}{\alpha}} Z_\alpha(du), \qquad (2.1)$$

which is well defined for $0 < H < 1$ and $0 < \alpha \leq 2$.

In order to approximate the integral, we use the method introduced by Mandelbrot and Wallis [8] replacing a sequence of Gaussian with α-stable random variables. In Fig.1 we can see four trajectories of the process (thin lines) for $\alpha = 1.8$ and $H = 0.7$. To give the insight view on the nature of the process, we follow Janicki and Weron [4]. We evaluate a large number of realizations of the process and compute quantiles in the points of discretization for some fixed p ($0 < p < 0.5$), i.e. we compute $F^{-1}(p)$ and $F^{-1}(1-p)$, where F is the distribution function. Fig.1 and Fig.2 have the same graphical form of output. The number of considered realizations is 4000. The thin lines represent four sample trajectories of the process. The thick lines stand for quantile lines, the bottom one for $p = 0.1$ and the top one for $1-p = 0.9$. The lines determine the subdomain of R^2 to which the trajectories of the approximated process should belong with probabilities 0.8 at any fixed moment of time. In Fig. 2 we can see the corresponding process transformed by the Lamperti transformation for the parameter $H = 0.7$. We can see that the quantile lines are "parallel". This means they are time invariant, demonstrating the stationarity of the process.

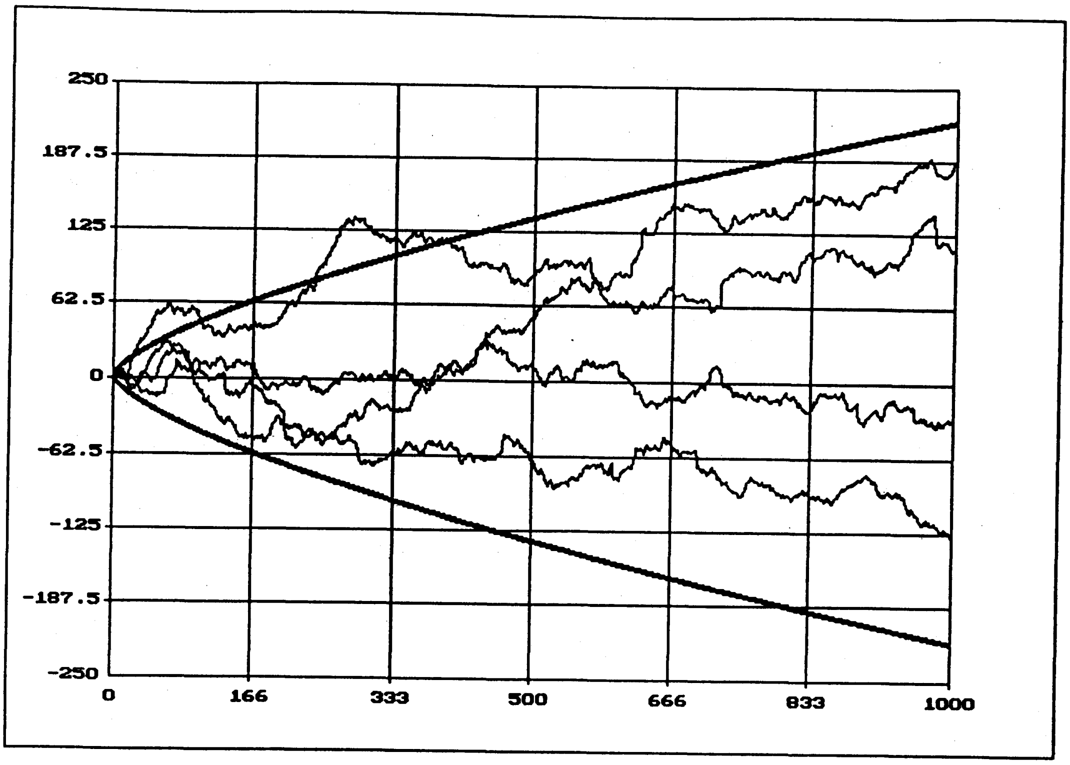

Figure 1: Visualization of the fractional stable motion for $H = 0.7$ and $\alpha = 1.8$.

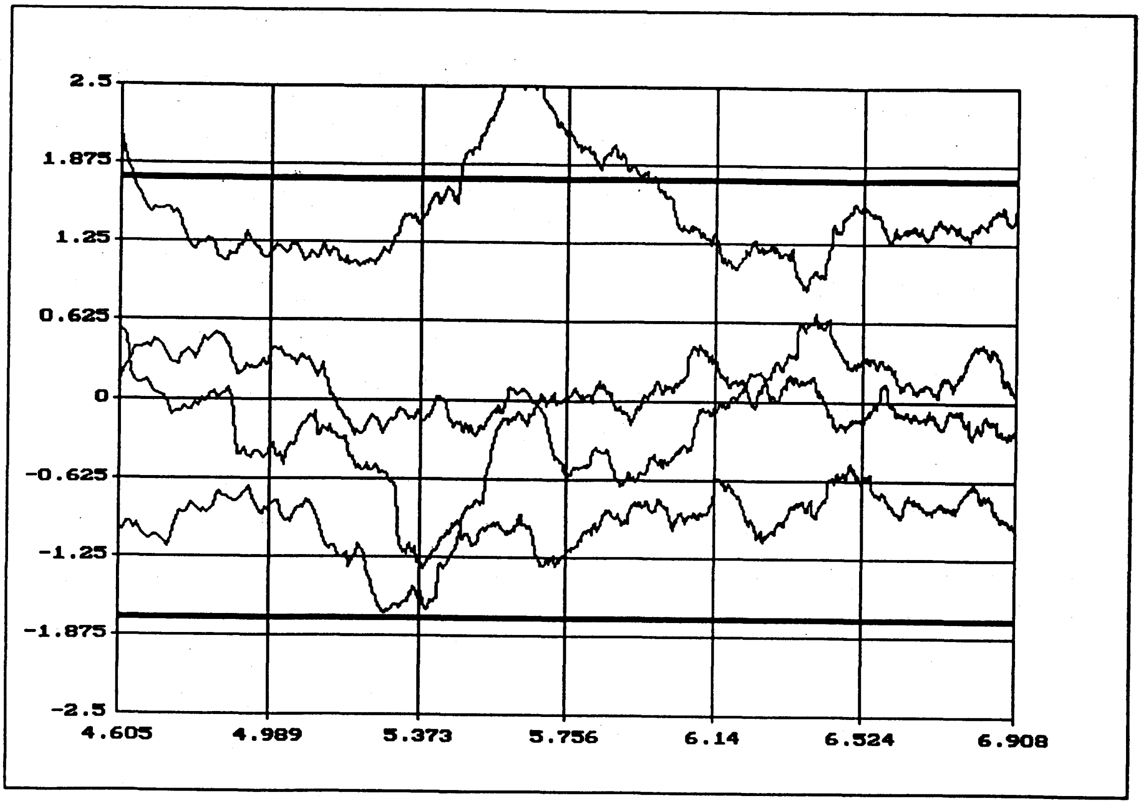

Figure 2: Visualization of the stationary process obtained from the fractional stable motion by the Lamperti transformation.

3. General results

In this section we establish the uniqueness of the Lamperti transformations leading from stationary to self-similar processes, and conversely. The following lemma on stationary processes makes a technical argument used in the proof of Theorem 3.1 (ii).

Lemma 3. 1 *Let $(Y(t))_{t\in R}$ be a non-trivial stochastically continuous stationary process and let $f : R \to R$ be a continuous monotone increasing function. If*

$$Y(f(t)) \stackrel{d}{=} Y(t), \tag{3.1}$$

then $f(t) = t + h$ for some $h \in R$.

Proof. Suppose that the conclusion is not true. Then (i) there exist an interval $[a, b]$ and $\theta \in (0, 1)$ such that for every $t \in [a, b]$, $0 \le f(t) - f(a) < \theta(t - a)$, or (ii) there exist an interval $[a, b]$ and $\theta > 1$ such that for every $t \in [a, b]$, $f(t) - f(a) > \theta(t - a)$. Note that since f is continuous and monotone increasing, it follows from (3.1) that $Y(f^{-1}(t)) \stackrel{d}{=} Y(t)$. Thus without loss of generality, we suppose (i).

For any $t_0 \in (0, b - a]$, define $t_1 = f(a + t_0) - f(a)$. Then $0 \le t_1 < \theta t_0$. From the assumption and the stationarity of Y, we have

$$(Y(0), Y(t_0)) \stackrel{d}{=} (Y(a), Y(a + t_0)) \stackrel{d}{=} (Y(f(a)), Y(f(a + t_0))) \stackrel{d}{=} (Y(0), Y(t_1)).$$

For every $n = 2, 3, \ldots$, define $t_n = f(a + t_{n-1}) - f(a)$. Then $0 \le t_n < \theta t_{n-1}$ and by the same argument as above, we have

$$(Y(0), Y(t_0)) \stackrel{d}{=} (Y(0), Y(t_n)).$$

Since $t_n \to 0$ as $n \to \infty$, it follows from the stochastic continuity of Y that

$$(Y(0), Y(t_0)) \stackrel{d}{=} (Y(0), Y(0)).$$

Namely

$$Y(t_0) = Y(0) \quad a.s.$$

Since $t_0 \in (0, b - a]$ was taken arbitrary, this together with the stationarity of Y gives us that

$$Y(t) = Y(0) \quad a.s \; for \; every \; t \in R,$$

which is an contradiction to that Y is non-trivial. Therefore it must be that for some $h \in R$

$$f(t) = t + h \quad for \; any \; t \in R \quad \square$$

Theorem 3. 1 *Let* $0 < H < \infty$.

(i) If $(Y(t))_{t\in R}$ *is a stationary process and* $a \in R$, *then*

$$X(t) = \begin{cases} t^H Y(a \log t), & for\ t > 0 \\ 0, & for\ t = 0 \end{cases}$$

is $H - ss$.

(ii) Conversely, if for some continuous functions ϕ, ψ *on* $(0, \infty)$ *and for non-trivial stationary process* $Y = (Y(t))_{t\in R}$,

$$X(t) = \begin{cases} \phi(t)Y(\psi(t)), & for\ t > 0 \\ 0, & for\ t = 0 \end{cases} \tag{3.2}$$

is $H - ss$, *then* $\phi(t) = t^H$ *and* $\psi(t) = a \log t$ *for some* $a \in R$.

Proof. (i) Note that

$$X(ct) = c^H t^H Y(a \log t + a \log c) \stackrel{d}{=} c^H X(t),$$

hence we conclude that $(X(t))_{t\geq 0}$ is $H - ss$.

(ii) Since $(X(t))_{t\geq 0}$ in (3.2) is $H - ss$, we have

$$\phi(ct)Y(\psi(ct)) \stackrel{d}{=} c^H \phi(t)Y(\psi(t)) \quad for\ every\ c > 0, \tag{3.3}$$

which leads to

$$\phi(ct) = c^H \phi(t) \quad for\ every\ t > 0\ and\ c > 0,$$

since (3.3) must agree with respect to marginal distributions as well. Consequently, $\phi(t) = t^H \phi(1), t > 0$. The constant $\phi(1)$ is of no importance by Definition 1. 4 , thus we consider $\phi(t)$ only of the form $\phi(t) = t^H, t > 0$. Now (3.3) can be phrased as

$$c^H t^H Y(\psi(ct)) \stackrel{d}{=} c^H t^H Y(\psi(t)),$$

namely

$$Y(\psi(ct)) \stackrel{d}{=} Y(\psi(t)) \quad for\ every\ c > 0. \tag{3.4}$$

This yields that ψ is monotone on $(0, \infty)$. In order to see it, suppose a'contrario that $\psi(t_1) = \psi(t_2)$ for some $t_1 \neq t_2$. Since

$$(Y(\psi(ct_1)), Y(\psi(ct_2))) \stackrel{d}{=} (Y(\psi(t_1)), Y(\psi(t_2))) \stackrel{d}{=} (Y(0), Y(0))$$

for every $c > 0$, $\psi(ct_1) - \psi(ct_2)$ is continuous with respect to variable c and Y is stationary, we infer that $Y(t) = Y(0)$ a.s. for every $t \in R$. Thus Y is trivial.

Therefore ψ must be monotone on $(0, \infty)$. Furthermore ψ takes every value in R. One can see this from (3.4) letting $c \to 0$ and $c \longrightarrow \infty$.

Taking

$$f_c(t) = \psi(c\psi^{-1}(t)), \tag{3.5}$$

we obtain by Lemma 3. 1 that for some $h \in R$

$$\psi(c\psi^{-1}(t)) = t + h, \quad for\ every\ t \in R. \tag{3.6}$$

Notice that ψ can be either decreasing or increasing. Nevertheless f_c defined by (3.5) is always increasing. Clearly, (3.6) can be rewritten as

$$\psi(ct) = \psi(t) + h(c), \quad for\ any\ t > 0\ and\ c > 0,$$

where $h(c)$ is a function depending only on c. From this and Definition 1. 4 , one can easily see that for some $a \in R$

$$\psi(t) = a \log t, \quad t > 0. \quad \square$$

Theorem 3. 2 *Let* $0 < H < \infty$.

(i) If $(X(t))_{t \geq 0}$ *is an* $H - ss$ *process and* $b \in R$*, then*

$$Y(t) = e^{-bHt} X(e^{bt}), \quad t \in R,$$

is stationary.

(ii) Conversely, if for some continuous functions ζ, η, *where* η *is invertible, and for a non-trivial* $H - ss$ *process* $(X(t))$,

$$Y(t) = \zeta(t) X(\eta(t)), \quad t \in R,$$

is stationary, then

$$\zeta(t) = e^{-bHt} \ and\ \eta(t) = e^{bt} \ for\ some\ b \in R.$$

Proof. (i) We have

$$Y(t+\sigma) = e^{-bH(t+\sigma)} X(e^{b(t+\sigma)}) \stackrel{d}{=} e^{-bH(t+\sigma)} e^{bH\sigma} X(e^{bt}) = Y(t).$$

Thus we conclude that Y is stationary.

(ii) Since $Y(t) = \zeta(t) X(\eta(t))$ is stationary and η is invertible, one can easily claim that the process

$$\frac{1}{\zeta(\eta^{-1}(t))} Y(\eta^{-1}(t)) = X(t)$$

is $H-ss$. Thus, by Theorem 3. 1 we obtain

$$\eta^{-1}(t)=a\log t \quad \textit{for some } a\in R\backslash\{0\}.$$

This is equivalent to

$$\eta(t)=e^{bt}, \quad \textit{some } b\in R.$$

Using the same arguments for ζ, we have $\zeta(a\log t)=t^{-H}$. This yields $\zeta(t)=e^{-bHt}$. □

Remarks.

- Marginal distributions do not depend on the choice of a and b, that is,

  $$X(t)=t^{H}Y(a\log t)\overset{d}{\sim}t^{H}Y(1)$$

  since Y is stationary, and

  $$Y(t)=e^{-bHt}X(e^{bt})\overset{d}{\sim}X(1)$$

  since X is $H-ss$.

- The parameters a and b are meaningful when considering finite-dimensional distributions. The influence of a and b will be discussed in the sequel.

4. Finite-dimensional distributions

We want to establish the influence of a's and b's on distributions of the corresponding processes. To this end we need the following lemma.

Lemma 4. 1 *If* $Y=(Y(t))_{t\in R}$ *is a non-trivial stationary stochastic process and if*

$$Y(ct)\overset{d}{=}Y(t), \quad \textit{for some } c\in R\backslash\{0\}, \tag{4.1}$$

then either $c=-1$ *or* $c=1$.

Proof. It is enough to prove that if Y satisfies (4.1) for some c with $0<|c|<1$, then Y is trivial. Since

$$(Y(t_1),\ldots,Y(t_m))\overset{d}{=}(Y(c^n t_1),\ldots,Y(c^n t_m))$$

for $0\le t_1<\ldots<t_m$, and $n\ge 1$, it follows from the stochastic continuity that

$$(Y(t_1),\ldots,Y(t_m))\overset{d}{=}(Y(0),\ldots,Y(0)) \quad □$$

The following theorem is a direct consequence of Lemma 4. 1 .

Theorem 4.1 *Let* $0 < H < \infty$.

(i) If $Y = (Y(t))_{t\in R}$ *is a non-trivial stationary process and if for some* $a, a' \in R\backslash\{0\}$

$$t^H Y(a \log t) \stackrel{d}{=} t^H Y(a' \log t),$$

then either $a = a'$ *or* $a = -a'$.

(ii) If $X = (X(t))_{t\geq 0}$ *is a non-trivial* $H - ss$ *process and if for some* $b, b' \in R\backslash\{0\}$

$$e^{-bHt} X(e^{bt}) \stackrel{d}{=} e^{-b'Ht} X(e^{b't}),$$

then either $b = b'$ *or* $b = -b'$.

Proof. Part (i) follows directly from Lemma 4.1. In order to prove (ii) it is enough to apply Lemma 4.1 to $Y(t) = e^{-Ht}X(e^t)$. □

Up to now we have considered processes merely assuming that they are stochastically continuous. In order to gain insight into the influence of different a's and b's on finite-dimensional distributions of corresponding processes we are to concentrate on $\alpha-$stable processes. We will study Gaussian and non-Gaussian examples to take a different view of the foregoing results.

Note that for Gaussian stationary processes $Y(t) \stackrel{d}{=} Y(-t)$. Hence if Y is Gaussian, then the statement (i) in Theorem 4.1 can be replaced by that $t^H Y(a \log t) \stackrel{d}{=} t^H Y(a' \log t)$ if and only if $a = \pm a'$, and if X is Gaussian, then (ii) can be replaced by that $e^{-bHt} X(e^{bt}) \stackrel{d}{=} e^{-b'Ht} X(e^{b't})$ if and only if $b = \pm b'$. Therefore we have the following.

Example 4.1 *Let* $0 < H < \infty$ *and* $(Y_\lambda(t))_{t\in R}$ *be a Gaussian Ornstein-Uhlenbeck process, namely*

$$Y_\lambda(t) = \int_{-\infty}^{t} e^{-\lambda(t-x)} B(dx), \ t \in R,$$

where $(B(t))$ *is a standard Brownian motion. Then*

$$t^H Y_\lambda(a \log t) \stackrel{d}{=} t^H Y_\lambda(a' \log t), \quad t > 0$$

if and only if $a = \pm a'$.

Example 4.2 *Let* $(X(t))_{t\geq 0}$ *be a Gaussian* $H - ss$ *process and* $0 < H < 1$. *(If, in addition, it has stationary increments, it is the fractional Brownian motion defined by the stochastic integral with* $\alpha = 2$ *in (2.1)). Then*

$$e^{-bHt} X(e^{bt}) \stackrel{d}{=} e^{-b'Ht} X(e^{b't}), \quad t \in R,$$

if and only if $b = \pm b'$.

Remarks.

- Let us recall that the Gaussian Ornstein–Uhlenbeck process can be obtained by transforming the Brownian motion by the Lamperti transformation and there exists only one such a process (this was observed by Doob [2] and Itô [3]). How does this fact match the above theorems and examples? Comparing the covariance functions of the transformed Brownian motion and the Gaussian Ornstein–Uhlenbeck process (characterized by parameter λ) leads to the conclusion

  Brownian motion $\qquad\qquad$ G.O.U. process

  $B(t) \quad \overset{Lamp.tr.\ with\ a}{\Longrightarrow} \quad Y_\lambda(at)$

  (where $\lambda = \frac{1}{2}$).

  $Y_\lambda(at)$ and $Y_\lambda(a't)$ are different processes when $a \neq \pm a'$ (with respect to finite-dimensional distributions) but nevertheless they are still in the same class of processes because $Y_\lambda(at) \overset{d}{=} \sqrt{a} Y_{a\lambda}(t)$, (see Example 4. 1).

- Due to the above generalization of the Lamperti theorem we are able to obtain the complete class of Ornstein–Uhlenbeck processes from the standard Brownian motion.

- Using the generalized Lamperti transformation with different a's, one can generate the entire class of $H - ss$ Gaussian Markov processes starting from the standard Ornstein–Uhlenbeck process with $\lambda = 1$, (see Example 4. 1). They are given by the covariance function in the following way:

$$\begin{aligned} E[X(t)X(s)] &= t^H s^H E[Y_1(a \log t) Y_1(a \log s)] \\ &= t^H s^H e^{-a(\log t - \log s)} = t^{H-a} s^{H+a}, \end{aligned}$$

  where $a > 0$ and $s < t$.

We proceed to non-Gaussian stable cases.

Example 4. 3 *Let* $0 < H < \infty$ *and* $(Y_\lambda(t))_{t \in R}$ *be an* $S\alpha S$ *Ornstein–Uhlenbeck process, namely*

$$Y_\lambda(t) = \int_{-\infty}^{t} e^{-\lambda(t-x)} Z_\alpha(dx), \ t \in R$$

where $0 < \alpha < 2$. *Then*

$$t^H Y_\lambda(a \log t) \overset{d}{=} t^H Y_\lambda(a' \log t), \qquad t > 0, \tag{4.2}$$

if and only if $a = a'$.

Proof. We compute the characteristic function of vector $(Y_\lambda(as), Y_\lambda(at))$. Fixing $s < t$ and $a > 0$, we have the following equations :

$$\begin{aligned}
& E\exp\{i(\theta_1 Y_\lambda(as) + \theta_2 Y_\lambda(at))\} \\
= \;& E\exp\{i([\theta_1 + \theta_2 e^{-\lambda a(t-s)}]Y_\lambda(as) + \theta_2[Y_\lambda(at) - e^{-\lambda a(t-s)}Y_\lambda(as)])\} \\
= \;& E\exp\{i(\theta_1 + \theta_2 e^{-\lambda a(t-s)}) \int_{-\infty}^{as} e^{-\lambda(as-x)} Z_\alpha(dx)\} \\
\cdot \;& E\exp\{i\theta_2 \int_{as}^{at} e^{-\lambda(at-x)} Z_\alpha(dx)\} \\
= \;& \exp\{-(|\theta_1 + \theta_2 e^{-\lambda a(t-s)}|^\alpha \int_{-\infty}^{as} e^{-\alpha\lambda(as-x)}dx + |\theta_2|^\alpha \int_{as}^{at} e^{-\alpha\lambda(at-x)}dx)\} \\
= \;& \exp\{-\frac{1}{\alpha\lambda}[(1 - e^{-\alpha\lambda a(t-s)})|\theta_2|^\alpha \\
& + |1 + e^{-2\lambda a(t-s)}|^{\alpha/2} \cdot \left| \frac{\theta_1}{|1 + e^{-2\lambda a(t-s)}|^{1/2}} + \frac{\theta_2 e^{-\lambda a(t-s)}}{|1 + e^{-2\lambda a(t-s)}|^{1/2}} \right|^\alpha]\}.
\end{aligned}$$

Thus the spectral measure of vector $(Y_\lambda(as), Y_\lambda(at))$ is given by the formula

$$\begin{aligned}
\Gamma \;=\;& \frac{1}{2\alpha\lambda}[(1 - e^{-\alpha\lambda a(t-s)})(\delta(0,1) + \delta(0,-1)) \\
+ \;& (1 + e^{-2\lambda a(t-s)})^{\alpha/2}(\delta(c,d) + \delta(-c,-d))],
\end{aligned}$$

where

$$c = \frac{1}{(1 + e^{-2\lambda a(t-s)})^{1/2}}, \quad d = \frac{e^{-\lambda a(t-s)}}{(1 + e^{-2\lambda a(t-s)})^{1/2}}$$

and $\delta(p,q)$ is the delta measure at $(p,q) \in R^2$. Similarly, when $a < 0$ the spectral measure of vector $(Y_\lambda(as), Y_\lambda(at))$ is given by

$$\begin{aligned}
\Gamma \;=\;& \frac{1}{2\alpha\lambda}[(1 - e^{-\alpha\lambda a(s-t)})(\delta(1,0) + \delta(-1,0)) \\
+ \;& (1 + e^{-2\lambda a(s-t)})^{\alpha/2}(\delta(d,c) + \delta(-d,-c))].
\end{aligned}$$

Because of the uniqueness of the spectral measure Γ, formula (4.2) (as concerns bivariate distributions) holds only if $a = a'$. This completes the proof. □

Example 4. 4 *Let* $0 < \alpha < 2$, $H = \frac{1}{\alpha}$ *and* $(Z_\alpha(t))_{t\geq 0}$ *be an* $S\alpha S$ *Lévy motion. Then*

$$e^{-bHt}Z_\alpha(e^{bt}) \stackrel{d}{=} e^{-b'Ht}Z_\alpha(e^{b't}), \quad t \in R$$

if and only if $b = b'$.

Proof. By Theorem 4.1 it is enough to show that

$$e^{-Ht}Z_\alpha(e^t) \stackrel{d}{\neq} e^{Ht}Z_\alpha(e^{-t}),$$

which is equivalent to

$$Z_\alpha(t) \overset{d}{\neq} t^{2H} Z_\alpha(t^{-1}).$$

For that, we show that the process on the right hand side does not have independent increments. To this end, it suffices to represent the process by a stable integral $t^{2H} \int_0^{t^{-1}} dZ_\alpha(u)$ and to check its increments. Use the fact that two non-Gaussian stable random variables $\int f dZ_\alpha$ and $\int g dZ_\alpha$ are independent if and only if $f \cdot g = 0$ *a.e.* □

Remarks.

- As in the Gaussian case there is a correspondence between the $S\alpha S$ Lévy motion (characterized by the parameter α) and the $S\alpha S$ Ornstein-Uhlenbeck process (determined by α and λ) through the Lamperti transformation:

  $S\alpha S$ Lévy motion $\qquad\qquad\qquad$ $S\alpha S$ O.U. process

  $$Z_\alpha(t) \overset{\text{Lamp.tr. with } a}{\Longrightarrow} Y_\lambda(at)$$

  (where $\lambda = \frac{1}{\alpha}$).

  (See Adler et al. [1], Theorem 5.1 for $1 < \alpha < 2$ and for general $0 < \alpha < 2$ compute and compare the characteristic functions of processes $\{e^{-at/\alpha} Z_\alpha(e^{at})\}$ and $\{Y_{1/\alpha}(at)\}$, which can be calculated in a way similar to the above proof of Example 4.3.)

- Contrary to the Gaussian case, $Y_\lambda(at)$ defines distinct processes for a and for $-a$ (see Example 4. 3). For example, $a = 1$ and $a' = -1$ produce the $S\alpha S$ Ornstein–Uhlenbeck and the reverse $S\alpha S$ Ornstein–Uhlenbeck process, respectively (which are different when $0 < \alpha < 2$), (see Adler et al. [1]). Since $Y_\lambda(at) \overset{d}{=} a^{1/\alpha} Y_{a\lambda}(t)$, so we can construct only two different Ornstein–Uhlenbeck processes.

5. Mixed linear fractional α-stable motions

In the paper, Surgailis et al. [10], a new class of stationary non-Gaussian $S\alpha S$ processes, namely stable mixed moving averages, is introduced. This includes the well-studied class of moving averages. In this section, we discuss the self-similar stable processes whose corresponding stationary processes $(Y(t))$ through the Lamperti transformation are stable mixed moving averages.

Although more general class is introduced in Surgailis et al. [10], we focus here only on the following type of stable mixed moving averages (which are sums of independent usual moving average):

$$Y(t) = \sum_{k=1}^{K} \int_{-\infty}^{\infty} f_k(t-v) Z_\alpha^{(k)}(dv), \quad t \in R, \tag{5.1}$$

where the $Z_\alpha^{(k)}$'s are independent $S\alpha S$ Lévy motions, $f_k \in L^\alpha(-\infty, \infty)$ and where the $f_k's$ are not "equivalent" in the sense that for $k \neq \ell$, there do not exist c and τ such that $f_k(\cdot) = cf_\ell(\cdot - \tau)$. We call the process (5.1) the K-sum stable moving average. It is observed in Surgailis et al. [10] that K-sum stable moving average with $K \geq 2$ is different in law from the ordinary moving average.

We remark here that (5.1) is a special case of stable mixed moving averages introduced in Surgailis et al. [10], but finite sums of independent $S\alpha S$ moving averages as in (5.1) are dense in the class of stable mixed moving averages.

In the following, we give examples of self-similar processes with stationary increments, whose corresponding stationary processes are K-sum stable moving averages.

Definition 5. 1 *Let* $0 < H < 1, 0 < \alpha < 2, H \neq \frac{1}{\alpha}$, *and*

$$\begin{aligned} X(t) &= \sum_{n=1}^{N} \int_{-\infty}^{\infty} \Big\{ p_n[(t-u)_+^{H-\frac{1}{\alpha}} - (-u)_+^{H-\frac{1}{\alpha}}] \\ &+ q_n[(t-u)_-^{H-\frac{1}{\alpha}} - (-u)_-^{H-\frac{1}{\alpha}}] \Big\} Z_\alpha^{(n)}(du), \end{aligned} \tag{5.2}$$

where a_+ *and* a_- *stand for* $\max\{a, 0\}$ *and* $\max\{-a, 0\}$, *respectively. The process* $(X(t))$ *is called mixed linear fractional stable motion.*

It is easy to check that $(X(t))$ is H-self-similar and has stationary increments. When $N = 1$ and $p_n = 1, q_n = 1$, it is a linear fractional stable motion in (2.1). The distribution of $(X(t))$ is distinct for different collection of $\{p_n, q_n, n = 1, \cdots, N\}$ unless $p_n = p, q_n = q$ for all n.

In the following, we restrict ourselves to the stationary process $Y_+(t) = e^{-Ht}X(e^t)$. However, as we pointed out in Section 4, $(Y_+(t))$ is distinct from $(Y_-(t))$, where $Y_-(t) = e^{Ht}X(e^{-t})$, since we are dealing with non-Gaussian stable case. As to $(Y_-(t))$, we have a similar argument. We shall write below $Y(t)$ for $Y_+(t)$ and $\beta = H - \frac{1}{\alpha}$ for the notational simplicity.

Theorem 5. 1 *The mixed linear fractional stable motion* $X(t)$ *given by (5.2) corresponds via the Lamperti transformation to a* K*-sum stable moving average for some* $K \leq 2N$.

Proof. From (5.2), we have

$$\begin{aligned}
Y(t) &= e^{-Ht}X(e^t) \\
&= \sum_{n=1}^{N} e^{-Ht}\int_{-\infty}^{\infty}\Big\{p_n[(e^t-u)_+^{\beta}-(-u)_+^{\beta}] \\
&\qquad +q_n[(e^t-u)_-^{\beta}-(-u)_-^{\beta}]\Big\} Z_\alpha^{(n)}(du) \\
&= \sum_{n=1}^{N} e^{-Ht}\Big\{p_n\int_{-\infty}^{0}[(e^t-u)^{\beta}-(-u)^{\beta}]Z_\alpha^{(n)}(du) \\
&\qquad +\int_0^{e^t}[p_n(e^t-u)^{\beta}-q_nu^{\beta}]Z_\alpha^{(n)}(du) \\
&\qquad +q_n\int_{e^t}^{\infty}[(u-e^t)^{\beta}-u^{\beta}]Z_\alpha^{(n)}(du)\Big\} \\
&= \sum_{n=1}^{N} e^{-Ht}\Big\{\int_{-\infty}^{0}p_n[(e^t-u)^{\beta}-(-u)^{\beta}]Z_\alpha^{(n)}(du) \\
&\qquad +\int_0^{\infty}\Big(I[0<u<e^t][p_n(e^t-u)^{\beta}-q_nu^{\beta}] \\
&\qquad\qquad +I[e^t<u]q_n[(u-e^t)^{\beta}-u^{\beta}]\Big)Z_\alpha^{(n)}(du)\Big\}.
\end{aligned}$$

Thus, for $c_j \in R$,

$$\begin{aligned}
&-\log E[\exp\{i\sum_j c_jY(t_j)\}] \\
&= \sum_{n=1}^{N}\Big\{\int_{-\infty}^{0}\Big|\sum_j c_je^{-Ht_j}p_n[(e^{t_j}-u)^{\beta}-(-u)^{\beta}]\Big|^{\alpha}du \\
&\qquad +\int_0^{\infty}\Big|\sum_j c_je^{-Ht_j}\{I[0<u<e^{t_j}][p_n(e^{t_j}-u)^{\beta}-q_nu^{\beta}] \\
&\qquad\qquad +I[e^{t_j}<u]q_n[(u-e^{t_j})^{\beta}-u^{\beta}]\}\Big|^{\alpha}du\Big\}
\end{aligned}$$

by the change of variables $|u|=e^v$,

$$\begin{aligned}
&= \sum_{n=1}^{N}\Big\{\int_{-\infty}^{\infty}\Big|\sum_j c_je^{-Ht_j}p_n[(e^{t_j}+e^v)^{\beta}-e^{\beta v}]\Big|^{\alpha}e^v dv \\
&\qquad +\int_{-\infty}^{\infty}\Big|\sum_j c_je^{-Ht_j}\{I[v<t_j][p_n(e^{t_j}-e^v)^{\beta}-q_ne^{\beta v}] \\
&\qquad\qquad +I[t_j<v]q_n[(e^v-e^{t_j})^{\beta}-e^{\beta v}]\Big|^{\alpha}e^v dv\Big\}
\end{aligned}$$

$$= \sum_{n=1}^{N}\Big\{\int_{-\infty}^{\infty}\Big|\sum_j c_j e^{-Ht_j+\beta v} p_n[(e^{t_j-v}+1)^\beta-1]\Big|^\alpha e^v dv$$
$$+\int_{-\infty}^{\infty}\Big|\sum_j c_j e^{-Ht_j+\beta v}\{I[t_j-v>0][p_n(e^{t_j-v}-1)^\beta-q_n]$$
$$+I[t_j-v<0]q_n[(1-e^{t_j-v})^\beta-1]\}\Big|^\alpha e^v dv\Big\}$$
$$= \sum_{n=1}^{N}\Big\{\int_{-\infty}^{\infty}\Big|\sum_j c_j e^{-H(t_j-v)} p_n[(e^{t_j-v}+1)^\beta-1]\Big|^\alpha dv$$
$$+\int_{-\infty}^{\infty}\Big|\sum_j c_j e^{-H(t_j-v)}\{I[t_j-v<0]q_n[(1-e^{t_j-v})^\beta-1]$$
$$+I[t_j-v>0][p_n(e^{t_j-v}-1)^\beta-q_n]\}\Big|^\alpha dv\Big\}$$
$$= \sum_{n=1}^{N}\Big\{\int_{-\infty}^{\infty}\Big|\sum_j c_j f_n(t_j-v)\Big|^\alpha dv+\int_{-\infty}^{\infty}\Big|\sum_j c_j g_n(t_j-v)\Big|^\alpha dv\Big\},$$

where

$$\begin{aligned} f_n(t) &= e^{-Ht}p_n[(e^t+1)^\beta-1] \\ g_n(t) &= e^{-Ht}\{I[t<0]q_n[(1-e^t)^\beta-1]+I[t>0][p_n(e^t-1)^\beta-q_n]\}. \end{aligned}$$

Thus we have

$$Y(t) \stackrel{d}{=} \sum_{n=1}^{N}\int_{-\infty}^{\infty} f_n(t-v)Z_\alpha^{(n)}(dv) + \sum_{n=1}^{N}\int_{-\infty}^{\infty} g_n(t-v)Z_\alpha^{(N+n)}(dv),$$

where $Z_\alpha^{(n)}, n=1,2,\cdots,2N$ are independent stable motions. □

Example 5.1 *If $N=1, p_1=0, q_1\neq 0$, then*

$$Y(t) \stackrel{d}{=} \int_{-\infty}^{\infty} g_1(t-v)Z_\alpha(dv),$$

and hence $K=1$. The linear fractional stable motion corresponds to a stable moving average.

Example 5.2 *If $N=1, p_1\neq 0$ (whatever q_1 is), then $f_1(\cdot)=\pm c g_1(\cdot+\tau)$ is not true. Hence*

$$Y(t) \stackrel{d}{=} \int_{-\infty}^{\infty} f_1(t-v)Z_\alpha^{(1)}(dv)+\int_{-\infty}^{\infty} g_1(t-v)Z_\alpha^{(2)}(dv),$$

which is 2-sum stable moving average. Thus, the linear fractional stable motion can also correspond to a stable mixed moving average.

Example 5.3 *Let $K \geq 3$ and choose N such that $2N \geq K$. Then by choosing p_n and q_n, zero or non-zero suitably, we can construct K-sum stable moving average from the mixed linear fractional stable motion.*

Next we consider the case of $H = \frac{1}{\alpha}$.

Example 5.4 *Let $0 < \alpha < 2$, $H = \frac{1}{\alpha}$ and $X(t) = Z_\alpha(t)$. Then*

$$\begin{aligned} Y(t) &= e^{-\frac{1}{\alpha}t} X(e^t) = e^{-\frac{1}{\alpha}t} Z_\alpha(e^t) \\ &= e^{-\frac{1}{\alpha}t} \int_0^{e^t} Z_\alpha(du) = e^{-\frac{1}{\alpha}t} \int_{-\infty}^{s} Z_\alpha(e^v dv) \\ &\stackrel{d}{=} e^{-\frac{1}{\alpha}t} \int_{-\infty}^{s} e^{\frac{1}{\alpha}v} Z_\alpha(dv) = \int_{-\infty}^{s} e^{-\frac{1}{\alpha}(t-v)} Z_\alpha(dv) \\ &= \int_{-\infty}^{\infty} f(t-v) Z_\alpha(du), \end{aligned}$$

where

$$f(t) = e^{-\frac{1}{\alpha}t} I[t > 0].$$

Example 5.5 *Let $1 < \alpha < 2, H = \frac{1}{\alpha}$ and*

$$X(t) = \int_{-\infty}^{\infty} \log \left| \frac{t-u}{u} \right| Z_\alpha(du).$$

This $(X(t))$ is called a log-fractional stable motion. (See Kasahara et al. [5].) Then

$$\begin{aligned} Y(t) &= e^{-\frac{1}{\alpha}t} X(e^t) \\ &= e^{-\frac{1}{\alpha}t} \int_{-\infty}^{0} \log \left| \frac{e^t - u}{u} \right| Z_\alpha(du) + e^{-\frac{1}{\alpha}t} \int_{0}^{\infty} \log \left| \frac{e^t - u}{u} \right| Z_\alpha(du) \\ &\stackrel{d}{=} e^{-\frac{1}{\alpha}t} \int_{-\infty}^{\infty} \log \left| \frac{e^t - e^v}{e^v} \right| Z_\alpha^{(1)}(-e^v dv) + e^{-\frac{1}{\alpha}t} \int_{-\infty}^{\infty} \log \left| \frac{e^t - e^v}{e^v} \right| Z_\alpha^{(2)}(e^v dv) \\ &\stackrel{d}{=} e^{-\frac{1}{\alpha}t} \int_{-\infty}^{\infty} \log \left| \frac{e^t - e^v}{e^v} \right| e^{\frac{1}{\alpha}v} \left(Z_\alpha^{(1)}(dv) + Z_\alpha^{(2)}(dv) \right) \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{\alpha}(t-v)} \log |e^{t-v} - 1| \left(Z_\alpha^{(1)}(dv) + Z_\alpha^{(2)}(dv) \right) \\ &= \int_{-\infty}^{\infty} f(t-v) \left(Z_\alpha^{(1)}(dv) + Z_\alpha^{(2)}(dv) \right), \end{aligned}$$

where

$$f(t) = e^{-\frac{1}{\alpha}t} \log |e^t - 1|.$$

Thus, the log-fractional stable motion also corresponds to a 2-sum moving average as in the case of the linear fractional stable motion in Example 5.2.

Acknowledgement

The second and the third authors would like to acknowledge many fruitful discussions on the subject of this paper with the late Stamatis Cambanis. Especially, the results in Section 5 are based on an unpublished project with him. Also the authors are grateful to the referee, who inspired some of the changes in the original version of the paper.

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