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Totally real submanifolds of a complex space form with nonzero parallel mean curvature vector

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Abstract. Any pseudo–umbilical submanifold of a non–flat complex space form with nonzero parallel mean curvature vector is a totally real submanifold.

1. Introduction

The aim of this paper is to show that the class of totally real submanifolds of a non–flat complex space form contains all pseudo–umbilical submanifolds with nonzero parallel mean curvature vector.

Our result is as follows:

Theorem 1 *Let M^n be an n–dimensional pseudo–umbilical submanifold of a complex m–dimensional complex space form $\tilde{M}^m(\tilde{c})(\tilde{c}\neq 0)$ with nonzero parallel mean curvature vector. Then $2m>n$ and M is a totally real submanifold of $\tilde{M}^m(\tilde{c})$.*

As an immediate consequence of Theorem 1, we see the following fact; The mean curvature vector of a non–minimal, pseudo–umbilical submanifold of $\tilde{M}^m(\tilde{c})(\tilde{c}\neq 0)$ is not parallel unless it is totally real.

Moreover, we get the following

Corllary 1 *Let M^n be a pseudo–umbilical submanifold of $\tilde{M}^m(\tilde{c})(\tilde{c}\neq 0)$ with nonzero parallel mean curvature vector. Then the scalar curvature ρ of M satisfies the inequality:*

$$\rho\leq n(n-1)(\frac{\tilde{c}}{4}+g(H,H)), \tag{1.1}$$

where H is the mean curvature vector of M in $\tilde{M}(\tilde{c})$.

If the equality sign of (1.1) holds, then M is a real space form immersed in $\tilde{M}(\tilde{c})$

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as a totally real and totally umbilical submanifold.

This case occurs only when $2m > n$.

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2. Preliminaries

Let f be an isometric immersion of a Riemannian n–manifold M^n into a Kaehlerian m–manifold $\tilde{M}^m$ with complex structure J. We denote by g the metric tensor of $\tilde{M}$ as well as that induced on M. For all local formulas we regard f as an imbedding and thus identify $x \in M$ with $f(x) \in \tilde{M}$. The tangent space $T_x(M)$ is identified with a subspace of $T_x(\tilde{M})$. The normal space $T_x^{\perp}(M)$ is a subspace of $T_x(\tilde{M})$ consisting of all $X \in T_x(M)$ which are orthogonal to $T_x(M)$ with respect to the Riemannian metric g. Let $\tilde{\nabla}$(resp. ∇) be the Riemannian connection on $\tilde{M}$(resp. M). Moreover, we denote by σ the second fundamental form of M in $\tilde{M}$. Then the Gauss formula and the Weingarten formula are given respectively by

$$\begin{aligned} \sigma(X,Y) &= \tilde{\nabla}_X Y - \nabla_X Y, && \text{for} \quad X,Y \in T_x(M), \\ \tilde{\nabla}_X \xi &= -A_\xi X + D_X \xi, && \text{for} \quad \xi \in T_x^{\perp}(M), \end{aligned}$$

where $-A_\xi X$(resp. $D_X\xi$) denotes the tangential (resp. normal) component of $\tilde{\nabla}_X\xi$. A normal vector field ξ is said to be parallel if $D_X\xi = 0$ for all $X \in T_x(M)$. Let $H = \frac{1}{n}$ trace σ be the mean curvature vector of M in $\tilde{M}$. If the second fundamental form $\sigma(X,Y) = g(X,Y)H$, then M is said to be totally umbilical. M is said to be pseudo–umbilical if the second fundamental form σ is of the form $g(\sigma(X,Y),H) = g(X,Y)g(H,H)$, for $X,Y \in T_x(M)$, or equivalently, $A_H = ||H||^2 I$.

Let $\tilde{R}$(resp. R) be the Riemannian curvature for $\tilde{\nabla}$(resp. ∇). Then the Gauss equation is given by

$$\begin{aligned} g(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + g(\sigma(X,Z),\sigma(Y,W)) \\ &- g(\sigma(Y,Z),\sigma(X,W)), \end{aligned}$$

for $X,Y,Z,W \in T_x(M)$.

A Kaehlerian manifold of constant holomorphic sectional curvature $\tilde{c}$ is called a complex space form and will be denoted by $\tilde{M}(\tilde{c})$, and a Riemannian manifold of constant sectional curvature is called a real space form. The Riemmannian curvature $\tilde{R}$ of $\tilde{M}(\tilde{c})$ is given by

$$\begin{aligned} &\tilde{R}(X,Y)Z \\ = &\frac{\tilde{c}}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\}, \end{aligned} \tag{2.1}$$

for all vector fields X, Y, Z on $\tilde{M}(\tilde{c})$.

The submanifold M is called a totally real submanifold of $\tilde{M}$ if $J(T_x(M))$ $T_x^{\perp}(M)$ (see, [1][2]).

3. Proof of Theorem

Proof of Theorem 1 . Let M^n be a pseudo–umbilical submanifold of $\tilde{M}(\tilde{c})(\tilde{c} \neq o)$ with nonzero parallel mean curvature vector H. Then we get

$$\tilde{\nabla}_X H = -A_H X = -\|H\|^2 \quad \text{,for all} \quad X, Y \in T_x(M),$$

where $\|H\|$ is a constant. Therefore we have

$$\begin{aligned} \tilde{R}(X,Y)H &= \tilde{\nabla}_X\tilde{\nabla}_Y H - \tilde{\nabla}_Y\tilde{\nabla}_X H - \tilde{\nabla}_{[X,Y]}H \\ &= \|H\|^2(\tilde{\nabla}_Y X - \tilde{\nabla}_X Y + [X,Y]) \\ &= 0 \end{aligned} \tag{3.1}$$

for all $X, Y \in T_x(M)$.
On the other hand, from (2.1) we get

$$g(\tilde{R}(X,Y)H, JH) = \frac{\tilde{c}}{2}g(X,JY)g(H,H). \tag{3.2}$$

Since $\tilde{c} \neq 0$ and $g(H,H) \neq 0$, it follows from (3.1) and (3.2) that M is totally real in $\tilde{M}$.

Moreover, from (2.1) we get

$$g(\tilde{R}(X,Y)H, JX) = \frac{\tilde{c}}{4}g(g(X,X)JY - g(X,Y)JX, H). \tag{3.3}$$

Choose X in such a way that $g(JX,H) = 0$. Since $\tilde{c} \neq 0$, from (3.1) and (3.3) we have

$$g(JY,H) = 0 \quad \text{for all} \quad Y \in T_x(M). \tag{3.4}$$

Since M is a totally real submanifold of $\tilde{M}$, the normal space $T_x^{\perp}(M)$ is decomposed in the following way: $T_x^{\perp}(M) = JT_x(M) \oplus \nu_x$ at each point x of M, where ν_x denotes the orthogonal complement of $JT_x(M)$ in $T_x^{\perp}(M)$. Thus it follows from (3.4) that $H \in \nu_x$. This implies $2m > n$. Q.E.D.

Proof of Corollary 1. Let M^n be a pseudo–umbilical submanifold of $\tilde{M}(\tilde{c})(\tilde{c} \neq 0)$ with nonzero parallel mean curvature vector H. Let $\{E_i\}_{i=1,\cdots,n}$ be orthonormal tangent vector fields and $\{\xi_\alpha\}_{\alpha=1,\cdots,2m-n}$ orthonormal normal vector fields such that the nonzero mean curvature vector H is equal to $\|H\|\xi_1$. Theorem 1 implies that M is totally real in $\tilde{M}$. By the Gauss equation the scalar curvature ρ is given by

$$\rho = n(n-1)(\frac{\tilde{c}}{4} + g(H,H)) - \sum_{i,j=1}^{n} \sum_{\alpha=2}^{2m-n} g(\sigma(E_i,E_j),\xi_\alpha)^2. \tag{3.5}$$

Thus we get (1.1). If the equality sign of (1.1) holds, it follows from (3.5) that M is totally umbilical in $\tilde{M}$. By the Gauss equation, we easily see that any totally real and totally umbilical submanifold in $\tilde{M}(\tilde{c})$ also has constant curvature $\frac{\tilde{c}}{4} + g(H,H)$. Thus we obtain M is a real space form immersed in $\tilde{M}$. Q.E.D.

References

[1] B. Y. Chen and K. Ogiue, On totally real submanifolds, *Trans. A.M.S.*,**193** (1974), 257–266.

[2] ———, Two theorems on Kaehler Manifolds, *Michigan M. J.*,**21** (1974), 225–229.

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