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IRREDUCIBLE QUADRANGULATIONS OF THE KLEIN BOTTLE

By

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Abstract. In this paper, we shall determine the complete list of irreducible quadrangulations of the Klein bottle. By this result, we can easily list all the minor-minimal 2-representative graphs on the Klein bottle. Moreover, we shall show that any two bipartite quadrangulations of the Klein bottle with at least 10 vertices are transformed into each other by a sequence of diagonal slides and a sequence of diagonal rotations, up to homeomorphism, if they have the same number of vertices.

1. Introduction

A *quadrangulation* G of a closed surface F^2 is a simple graph embedded in F^2 so that each face of G is quadrilateral. Let f be a face of G with the boundary cycle *abcd*. The *face contraction* of f at $\{b,d\}$ is to eliminate f as shown in Figure 1. However, if b and d are adjacent to each other or have a common neighbor $v \neq a,c$, then a face contraction of f at $\{b,d\}$ yields a loop or multiple edges. If a face contraction destroys the simpleness of G, then we do not apply this deformation. If we can apply a face contraction of f at $\{b,d\}$, then f is said to be *contractible* at $\{b,d\}$. There are two ways to contract a face since each face has two diagonal pairs of vertices. If G is obtained from a quadrangulation T by a sequence of face contractions, then T is said to be *contractible* to G. A quadrangulation G of F^2 is said to be *irreducible* if G has no contractible face.

Nakamoto and Ota have been shown that an irreducible quadrangulation of a closed surface F^2 has at most $186(2-\chi(F^2))-64$ vertices, where $\chi(F^2)$ denotes the Euler characteristic of F^2 [7]. This implies that for any closed surface, there exist finitely many irreducible quadrangulations, up to homeomorphism. For the sphere, the projective-plane and the torus, all the irreducible quadrangulations have been already determined in [8] and [6]. There exists only one irreducible quadrangulation of the sphere, which is the cycle of

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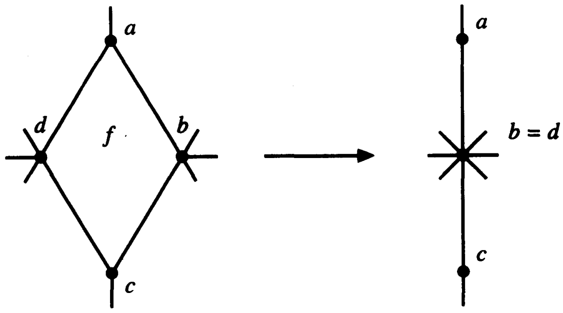

Figure 1 Face contraction at $\{b, d\}$

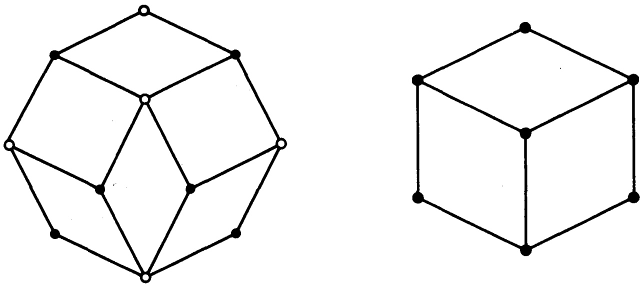

Figure 2 Irreducible quadrangulations of the projective plane

length 4. There exist two irreducible quadrangulations of the projective-plane as shown in Figure 2. (Both of the octagon and the hexagon represent the projective-planes by identifying each pair of antipodal points of them.) On the torus, there exist eight irreducible quadrangulations [6].

In this paper, we shall give the complete list of irreducible quadrangulations of the Klein bottle.

Theorem 1. *There exist precisely ten irreducible quadrangulations of the Klein bottle shown in Figures 3 and 4, up to homeomorphism.*

In Figures 3 and 4, the top and bottom of each rectangle should be identified in parallel and the pair of vertical sides in anti-parallel to represent the Klein bottle. Figure 3 shows the five irreducible bipartite quadrangulations of the Klein bottle while Figure 4 shows the five non-bipartite ones.

Our classification of irreducible quadrangulations of the Klein bottle has several applications. Sections 3 and 4 present some of those, as follows.

A graph G embedded in a closed surface F^2 is said to be *n-representative* if every non-trivial simple closed curve on F^2 which intersects no edge of G must

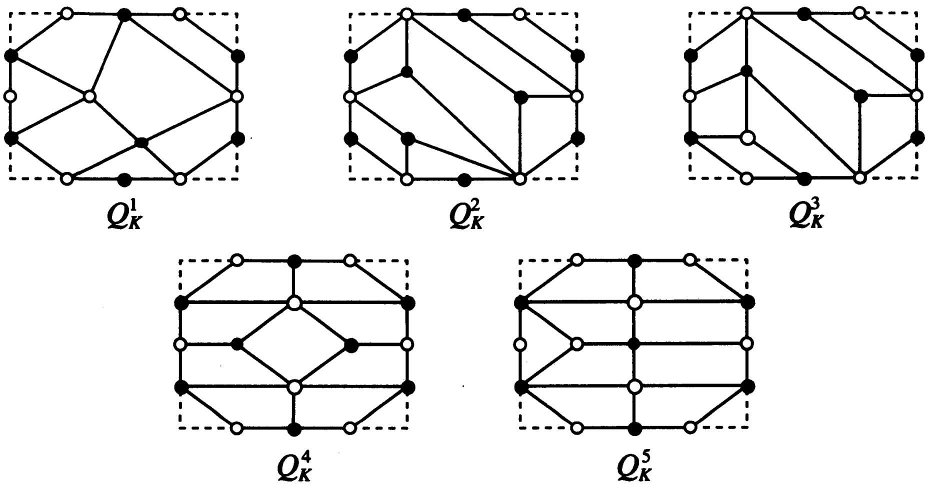

Figure 3 Irreducible bipartite quadrangulations of the Klein bottle

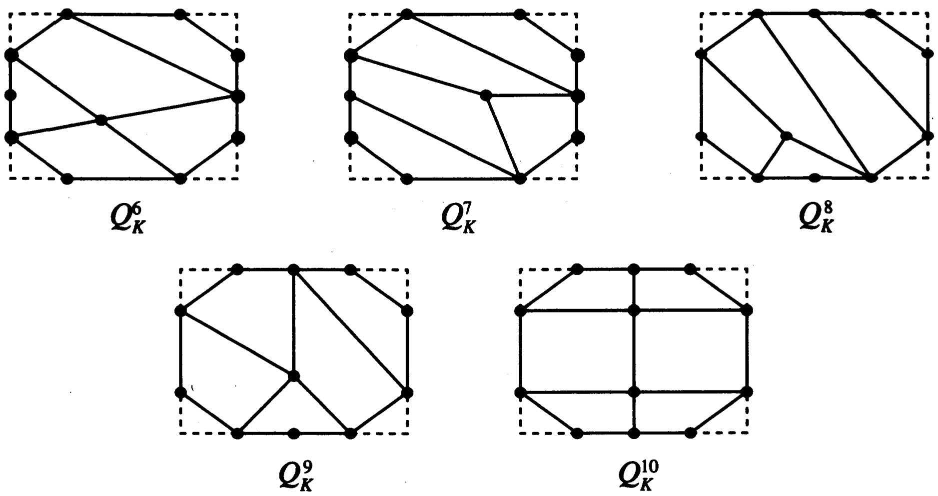

Figure 4 Irreducible non-bipartite quadrangulations of the Klein bottle

contain at least n vertices of G. The *contraction* of an edge e of G is to e and to identify its two endpoints. If G is obtained from a graph T by a sequence of deletions and contractions of edges, then G is said to be a *minor* of T. A graph G 2-cell embedded in a closed surface F^2 is said to be *minor-minimal n-representative* on F^2 if G is n-representative on F^2 and no proper minor of G is n-representative. In Section 3, we shall list all the minor-minimal 2-representative graphs on the Klein bottle. (These definition can also be found in [9].)

There have been defined the two transformations of quadrangulations in [3], called *the diagonal slide* and *the diagonal rotation*, as shown in Figure 5. If the graph obtained by a diagonal slide is not a simple graph, then we don't

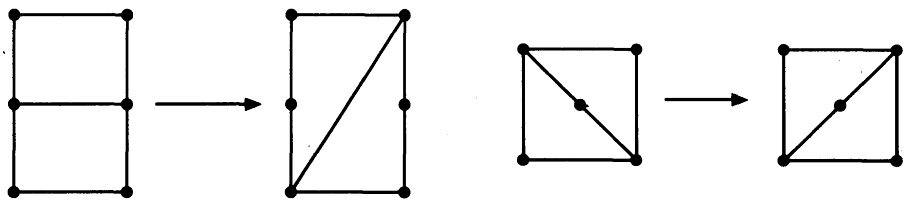

Figure 5 The diagonal slide and the diagonal rotation

carry out it. Two quadrangulations G and G' of a closed surface F^2 are said to be *equivalent* to each other (under diagonal slides and diagonal rotations), denoted by $G \approx G'$, if they are transformed into each other by a sequence of diagonal slides and diagonal rotations, up to homeomorphism. Observe that both of the two transformations preserve the bipartiteness of quadrangulations. Thus, a bipartite quadrangulation and a non-bipartite one are never equivalent to each other. In [3], the author has shown the following theorem:

Theorem 2. (A. Nakamoto [3]) *For any closed surface F^2, there exists a positive integer $N(F^2)$ such that if G_1 and G_2 are two bipartite quadrangulations of F^2 with $|V(G_1)| = |V(G_2)| \geq N(F^2)$, then G_1 and G_2 are equivalent to each other.*

It has been known that $N(S^2) = 4, N(P^2) = 7(8)$ and $N(T^2) = 10$ [6], where S^2, P^2 and T^2 denote the sphere, the projective plane and the torus. In Section 4, we shall determine $N(K^2)$ for the Klein bottle K^2.

Theorem 2 has been improved to hold for general quadrangulations [4]. Let G be a quadrangulation of a closed surface F^2 other than the sphere. It is easy to see that two homotopic cycles of G have the same parity of length. Thus there exists a unique map $\rho_G : \pi(F^2) \to Z_2$, which assigns the parity of length to each cycle in G. We call this map ρ_G the *cycle parity* of G. Observe that a diagonal slide and a diagonal rotation preserve the cycle parity. Thus, if two quadrangulations G_1 and G_2 are transformed into each other by diagonal slides and diagonal rotations, up to isotopy, then $\rho_{G_1} = \rho_{G_2}$. Two cycle parities ρ_{G_1} and ρ_{G_2} of G_1 and G_2 are said to be *congruent*, denoted by $\rho_{G_1} \equiv \rho_{G_2}$, if there is a homeomorphism $h: F^2 \to F^2$ such that $\rho_{G_1} = \rho_{h(G_2)}$. So, if G_1 and G_2 are transformed into each other by diagonal slides and diagonal rotations, up to homeomorphism, then $\varphi_{G_1} \equiv \rho_{G_2}$.

Theorem 3. (A. Nakamoto [4]) *For any closed surface F^2, there exists a positive integer $N'(F^2)$ such that G_1 and G_2 are two quadrangulations of F^2*

with $|V(G_1)|=|V(G_2)|\geq N'(F^2)$ *are equivalent to each other if and only if* $\rho_{G_1}\equiv\rho_{G_2}$.

Since a bipartite quadrangulation G can be characterized as one with the trivial cycle parity, Theorem 3 implies Theorem 2. The projective plane and the torus admit only one congruence class of non-bipartite quadrangulations, respectively. It follows that any two non-bipartite quadrangulations of the projective plane or the torus with the same number of vertices are equivalent to each other [8, 6]. However, in case of the Klein bottle, there are two different congruence classes of non-bipartite quadrangulations [3]. In Section 4, we shall also determine the value of $N'(K^2)$ for the Klein bottle K^2 in Theorem 3, checking the equivalence of quadrangulations in each congruence class.

2. Proof of Theorem 1

This section is devoted to prove Theorem 1, that is, we shall determine the complete list of irreducible quadrangulations of the Klein bottle.

We first give some general facts on irreducible quadrangulations. The following Lemmas 4, 5 and 6 have been shown in [8], [3] and [6].

Lemma 4. *Let G be a quadrangulation of a closed surface* F^2. *If there is a 2-cell region bounded by a cycle of length 4 in G which is not a face of G, then there is a contractible face in this 2-cell.*

Lemma 5. *Let G be an irreducible quadrangulation of a closed surface* F^2. *Then, any 2-cell region bounded by a closed walk of length 6 in G contains either a single edge or a single vertex of degree 3.*

Lemma 6. *Let G be an irreducible quadrangulation and v a vertex of degree 3 in G with neighbors* v_1, v_3 *and* v_5. *Then there is a subgraph H in G with vertex set* $\{v, v_1, v_2, v_3, v_4, v_5, v_6\}$ *such that:*

(i) $v_1 v_2 \cdots v_6$ forms a closed walk of length 6 which bounds the union of three faces of G incident to v.
(ii) *each pair of* v_i *and* $v_{i+3} (i=1,2,3)$ *are either adjacent or identical.*

Note that if $v_1,\cdots,v_6$ are all distinct, then H is isomorphic to $K_{3,4}$ with partite sets $\{v_1, v_3, v_5\}$ and $\{v, v_2, v_4, v_6\}$. This case happens actually when G is bipartite.

There are four types of simple closed curves on the Klein bottle [2]. Let l be a simple closed curve on the Klein bottle. If l bounds a 2-cell, then l is called *trivial*. Otherwise, l is called *essential*. If l separates the Klein bottle into two

Möbius bands, then l is called an *equator*. If cutting the Klein bottle along l yields one Möbius band, then l is called a *longitude*. In this case, a tubular neighborhood of l is homeomorphic to a Möbius band. If cutting the Klein bottle along l yields an annulus, then l is called a *meridian*.

If a quadrangulation of a closed surface has a vertex v of degree 2, then a face incident with v is contractible. Hence any irreducible quadrangulation has no vertex of degree less than 3. It can be easily obtained from the Euler's formula that the average degree of vertices in a quadrangulation of the Klein bottle is equal to 4. Thus, either an irreducible quadrangulation of the Klein bottle contains a vertex of degree 3 or it is 4-regular. We first consider irreducible quadrangulations of the Klein bottle containing a vertex of degree 3.

Lemma 7. *There exist precisely eight irreducible quadrangulations of the Klein bottle which contain a vertex of degree 3, up to homeomorphism.*

Proof. Let G be a quadrangulation of the Klein bottle with a vertex v of degree 3. Let H be the subgraph associated with v as given in Lemma 6 and R the hexagonal region bounded by the closed walk $v_1v_2v_3v_4v_5v_6$ in the Klein bottle. Then we have three essential simple closed curves l_1, l_2 and l_3 in the Klein bottle such that l_i runs across R through v_i, v, v_{i+3} in this order and along $v_{i+3}v_i$. Since they cross one another at one point, at most one of l_1, l_2 and l_3 is a meridian.

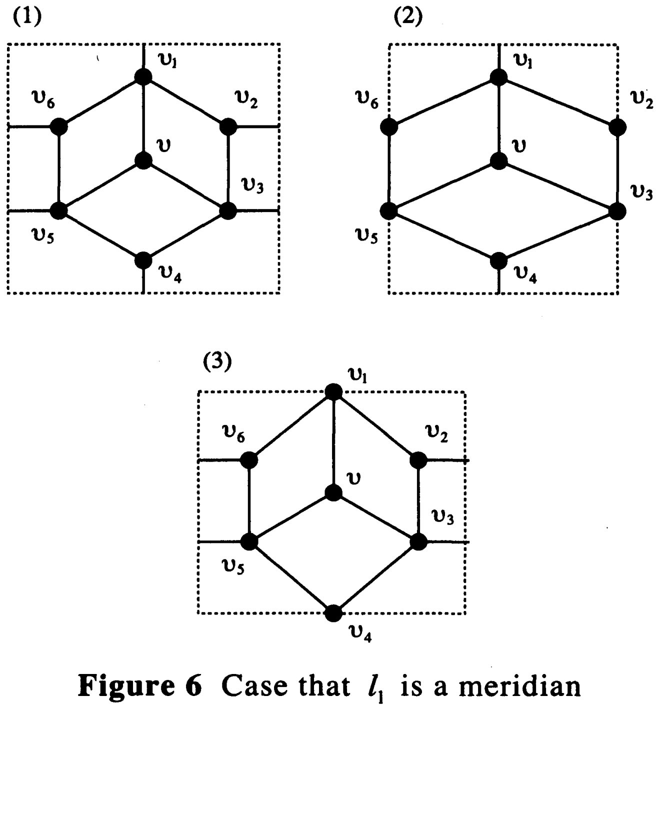

Figure 6 Case that l_1 is a meridian

We first consider the case when one of l_1, l_2 and l_3, say l_1, is a meridian. Then there are the three cases (1), (2) and (3) shown in Figure 6, up to homeomorphism. (Each rectangle represents the Klein bottle. Identify the top and bottom lines in parallel and the both vertical lines in anti-parallel.) Note that if $v_2 = v_5$ but $v_3 \neq v_6$ in Figure 6 (1), then the length of two homotopic cycles $v_2 v_3 v$ and $v_3 v v_1 v_6$ would have different parities. By Lemma 4, a 2-cell region bounded by a cycle of length 4 must be a face.

In Case (1), we have to quadrangulate the octagonal 2-cell region F bounded by $v_1 v_2 v_5 v_4 v_1 v_6 v_3 v_4$.

Claim 1. *There is a vertex of degree 3 in* Int$F$.

Proof. Since H is the complete bipartite graph, F includes no *diagonal*, that is, no edge joining two vertices on ∂F through Int$F$. Thus, there is at least one vertex in Int$F$. Let $u_1, \ldots, u_m$ be vertices in Int$F$ and K the graph induced by $\{u_1, \ldots, u_m\}$. We denote the number of edges between K and ∂F by e_1, and $|E(K)|$ by e_2.

Suppose that any vertex in Int$F$ has degree at least 4. By Euler's formula for a disk, we have that

$$m + 8 - (e_1 + e_2 + 8) + f = 1 \cdots\cdots (*)$$

where f denotes the number of faces of G in F. Since F is quadrangulated, we have that $4f = 2(e_1 + e_2) + 8$. Since $\deg(v_1) \geq 4$ for any i, we have $\sum_{i=1}^{m} \deg(v_1) = e_1 + 2e_2 \geq 4m$. Substituting these two into $(*)$, we obtain that $e_1 \leq 4$.

On the other hand, at least four faces are incident to ∂F since F contains no diagonal. This implies that $e_1 \leq 4$ and that if $e_1 = 4$, then Int$F$ contains only one vertex which has degree 4. However, G would have multiple edges in such a case, a contradiction. Thus, the claim follows. ■

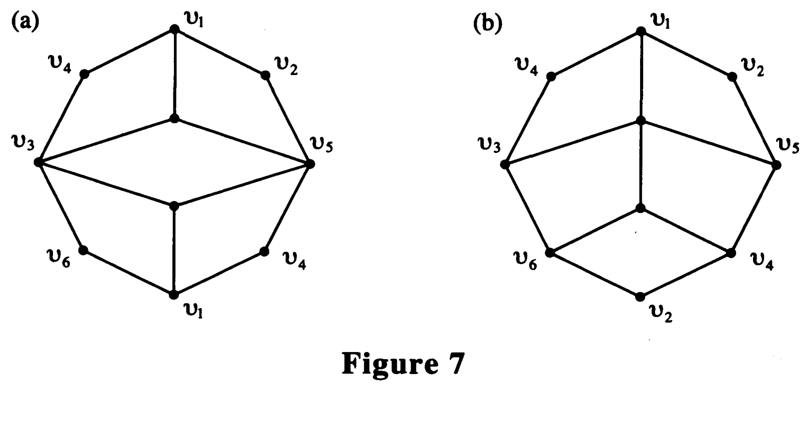

Figure 7

Claim 2. *There are two ways to quadrangulate F, up to symmetry, as shown in Figure 7.*

Proof. Let x be a vertex of degree 3 in Int$F$, which exists actually by Claim 1. Then we can take the subgraph given in Lemma 6 for x with vertices $u_1,\cdots,u_6$, corresponding to $v_1,\cdots,v_6$. Now, we shall consider how the hexagonal region bounded by $u_1u_2u_3u_4u_5u_6$, denoted by R', lies on F. It is easy to see that $\partial R' \cap \partial F \neq \emptyset$.

Since F contains no diagonal, we have that $|\partial F \cap \{u_1,\cdots,u_6\}| \leq 5$ and hence at least one of $u_1,\cdots,u_6$, say u_1, is contained in Int$F$. Since there is an edge u_1u_4 outside R' and since the closed curve through u_1, x, u_4 and along the edge u_1u_4 is essential, u_1 is joined to the vertices arising twice on ∂F, say v_1, and u_4 coincides with the antipodal v_1. (See Figure 8 (1).)

Since F contains no diagonal and no 2-cell region bounded by an odd cycle and since there must be edges u_2u_5 and u_3u_6 outside R', we can conclude that $v_5 = u_2$, $v_4 = u_3$, $v_6 = u_5$ and $v_3 = u_6$. Thus, we can obtain Figure 8 (2), up to symmetry. Finally, there are two ways to add a vertex of degree 3 into R'. Thus, we obtain Figure 7. ∎

We have two irreducible bipartite quadrangulations of the Klein bottle obtained from H by adding (a) and (b) in Figure 7 to F in Case (1) and denote them by Q_K^2 and Q_K^3, respectively. Note that Q_K^2 is isomorphic to $K_{3,6}$.

In Cases (2) and (3), we have to quadrangulate the remaining hexagonal regions $v_1v_2v_4v_1v_3v_4$ and $v_1v_2v_5v_1v_6v_3$, respectively. In Case (2), we add a vertex u of degree 3 by Lemma 5, since the addition of any edge yields multiple edges. In the resulting quadrangulation, denoted by Q_K^7, the vertex u may be assumed to be adjacent to v_1, v_2 and v_4, up to symmetry. In Case (3), there are two ways to add an edge, v_3v_5 or v_2v_6, to the hexagonal region to obtain Q_K^8 or Q_K^9, respectively.

Now, we consider the case when none of l_i is a meridian. In this case, l_1, l_2

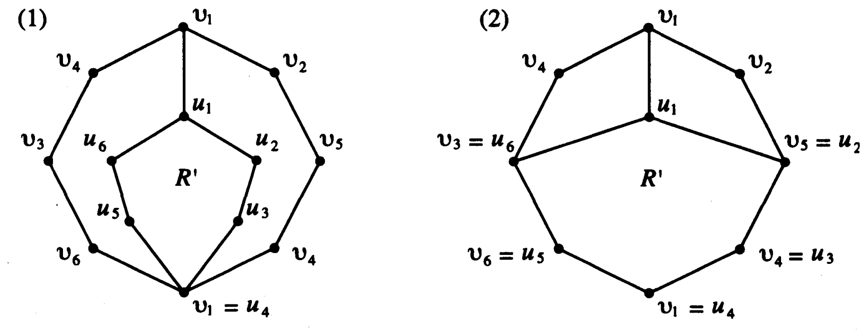

Figure 8

and l_3 are all longitude and homotopic to each other. Since the length of two homotopic cycles have the same parity, $v_1, \dots, v_6$ have to be all distinct. Thus, the embedding of H of this type is essentially unique, up to homeomorphism, as given in Figure 9. (Identify the top and bottom in parallel and the pair of vertical sides in anti-parallel.) This embedding of H into the Klein bottle is not a 2-cell embedding since the face with the boundary cycle $v_2 v_3 v_6 v_5$ is homeomorphic to a Möbius band. Then a quadrangulation of the projective plane will be obtained from the quadrangulated Möbius band by capping off its boundary with an extra quadrilateral face. It is not difficult to see that each of the faces of the projective plane, except the additional face, is not contractible.

Claim 3. *There exists no quadrangulation of the projective plane with precisely one contractible face.*

Proof. Let G be a quadrangulation of the projective plane with precisely one contractible face f. We denote the quadrangulation obtained from G by contracting f by G/f. Since a non-contractible face of G is also non-contractible in G/f, G/f is irreducible and hence G can be obtained from either Q_P^1 or Q_P^2 by *splitting* one vertex v, that is, the inverse of a face contraction. If $\deg(v) \leq 3$, then G would have a vertex of degree 2 and there would be two contractible faces of G incident to the vertex, contrary to our assumption. Thus, we have that $\deg(v) \geq 4$ and hence G/f is not isomorphic to Q_P^2. However, any splitting of each vertex of degree 4 in Q_P^1 yields another contractible face other than f. Therefore, the claim follows. ■

By Claim 3, the irreducible quadrangulations of the Klein bottle in question must be those obtained from two irreudcible quadrangulations P_1 and P_2 of the projective plane by pasting them together along one face. Therefore, we can obtain three irreducible quadrangulations, namely Q_P^4, Q_P^5 and Q_P^{10}, since $P_i (i = 1, 2)$ is isomorphic to either of Q_P^1 or Q_P^2 shown in Figure 2.

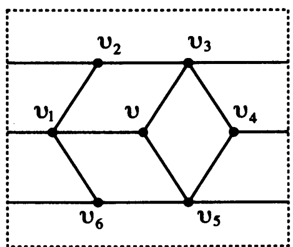

Figure 9 Case that none of l_1, l_2 and l_3 is a meridian

Summarizing all, we have precisely eight irreducible quadrangulations; Q_K^2, $Q_K^3, Q_K^4, Q_K^5, Q_K^6, Q_K^7, Q_K^8, Q_K^9, Q_K^{10}$. The proof of Lemma 7 completes. ■

Nakamoto and Negami [5] have classified 4-regular quadrangulations of the Klein bottle. Their classification plays an important role for our purpose.

Prepare the Cartesian product of $P_{p+1} \times P_{r+1}$ of two paths which quadrangulates a rectangle of size $p \times r$ naturally, denoted by $R_{p,r}$, where P_n stands for a path with n vertices. Identify the pair of horizontal sides of length r in parallel and the pair of vertical sides of length p in anti-parallel to get the Klein bottle. The resulting 4-regular quadrangulation of the Klein bottle is called the *grid type* and is denoted by $Q_g(p,r)$. The simple closed curve which comes from the vertical side of $R_{p,r}$ is a meridian. On the other hand, the horizontal sides of $R_{p,r}$ form a longitude.

Now prepare $R_{2p,r}$ and identify the pair of its horizontal sides to get an annulus with cycles C and C' of length $2p$ on its ends. Attach two Möbius bands to C and C' and add p edges to join antipodal pairs of vertices on each of C and C' in these Möbius bands. The resulting 4-regular quadrangulation in the Klein bottle is called the *ladder type* and denoted by $Q_l(2p,r)$. This quadrangulation $Q_l(2p,r)$ has "Möbius ladders" at both ends. Note that $Q_l(2p,r)$ contains p meridians of length $2r+2$ each of which is obtained from two horizontal paths in $R_{2p,r}$ and two edges in the two Möbius ladders.

The third type of 4-regular quadrangulations, called the *mesh type*, is constructed as follows. Consider the rectangle

$$\Omega = \{(x,y) \in R^2 \mid 0 \le x \le r, 0 \le y \le 2p\}$$

and the set of points

$$V = \{(x,y) \in \Omega \cap Z^2 \mid x - y \equiv 1 \pmod 2\}$$

Let $M_{p,r}$ be the graph with vertex set V such that each vertex (x, y) is adjacent to $(x \pm 1, y \pm 1) \in V$. Identify the pair of horizontal sides of Ω in parallel to get a cylinder and next glue its two ends incoherently to unify the vertices of $M_{p,r}$ along them in pair. The resulting graph in the Klein bottle is the mesh type and is denoted by $Q_m(p,r)$. Note that $Q_m(p,r)$ can be determined uniquely, not depending on the way to glue the ends of the cylinder.

Lemma 8. **(A. Nakamoto and S. Negami [5])** *Every 4-regular quadrangulation of the Klein bottle can be uniquely presented as one of $Q_g(p,r)$, $Q_l(2p,r)$ and $Q_m(p,r)$ with suitable parameters p and r.*

Lemma 9. *There exist precisely two irreducible 4-regular quadrangulations of the Klein bottle, up to homeomorphism.*

Proof. Observe that $Q_g(p,r)$ and $Q_l(2p,r)$ are not irreducible if they are simple and that the only irreducible 4-regular quadrangulations of mesh type are $Q_m(2,3)$ and $Q_m(2,4)$, which are denoted by Q_K^6 and Q_K^1, respectively, in our classification. Thus, the lemma follows. ■

Theorem 1 follows from Lemmas 9 and 7.

3. Minor-minimal 2-representative graphs

In [11], it has been shown that there are precisely two minor-minimal 2-representative graphs embedded in the projective-plane. Also, the author has classified the seven minor-minimal 2-representative graphs on the torus [6]. In this section, we shall determine such graphs on the Klein bottle as an application of our classification of irreducible quadrangulations of the Klein bottle.

The following two propositions are the keys to connect these two classifications.

Proposition 10. *Let G be a 2-connected graph 2-cell embedded in a closed surface F^2. Then G is 2-representative if and only if each face of G is bounded by a cycle.*

Let G be a graph 2-cell embedded in a closed surface F^2 with black vertices. Put a white vertex into each face of G, join it to the black vertices of G lying along the boundary walk of the face and delete all the edges of G. The resulting graph is called the *radial graph* $R(G)$ of G [1]. It is obvious that $R(G)$ is bipartite and each face of $R(G)$ is quadrilateral, but $R(G)$ is not always a quadrangulation since it may not be simple. If there is a face of G whose boundary walk is not a cycle, then $R(G)$ has multiple edges.

Proposition 11. *A graph G is embedded in a closed surface F^2 so that each face of G is bounded by a cycle if and only if $R(G)$ is a bipartite quadrangulation of F^2.*

Observe that a face contraction at white vertices in $R(G)$ corresponds to the deletion of an edge in G, and that a face contraction at black vertices in $R(G)$ corresponds to contraction of an edge in G. Thus, a face contraction of $R(G)$ corresponds to one of two operations which produce a minor of G. Therefore, the following proposition holds immediately from Propositions 10 and 11.

Proposition 12. *Let G be a graph embedded in a closed surface F^2 with its* *radial graph $R(G)$. Then, G is minor-minimal 2-representative on F^2 if and only if $R(G)$ is an irreducible bipartite quadrangulation of F^2.*

By Proposition 12, we can translate Theorem 1 into the following theorem, regarding Q_k^i $(i = 1, \cdots, 5)$ as radial graphs.

Theorem 13. *If G is a minor-minimal 2-representative graph on the Klein bottle, then G is isomorphic to one of $B_1, \cdots, B_9$ shown in* Figure 10, *up to homeomorphism.*

Four pairs $(B_1, B_2), (B_3, B_4), (B_5, B_6)$ and (B_8, B_9) are dual pairs of graphs while B_7 is a self-dual embedding. Moreover, B_3, B_4, B_5 and B_6 are transformed into each other by Y-Δ transformations, and so are B_7, B_8 and B_9. However, $\{B_3, B_4, B_5, B_6\}$ and $\{B_7, B_8, B_9\}$ are distinct Y-Δ equivalence classes.

In fact, the two minor-minimal 2-representative graphs on the projective-plane, determined by Vitray [11], are obtained from Q_P^1 in Figure 2 by regarding it as a radial graph. In [6], the minor-minimal 2-representative graphs on the torus were determined from irreducible bipartite quadrangulations of the torus. For the torus, Schrijver [10] has determined the number of Y-Δ equivalence classes of minor-minimal n-representative graphs and given the

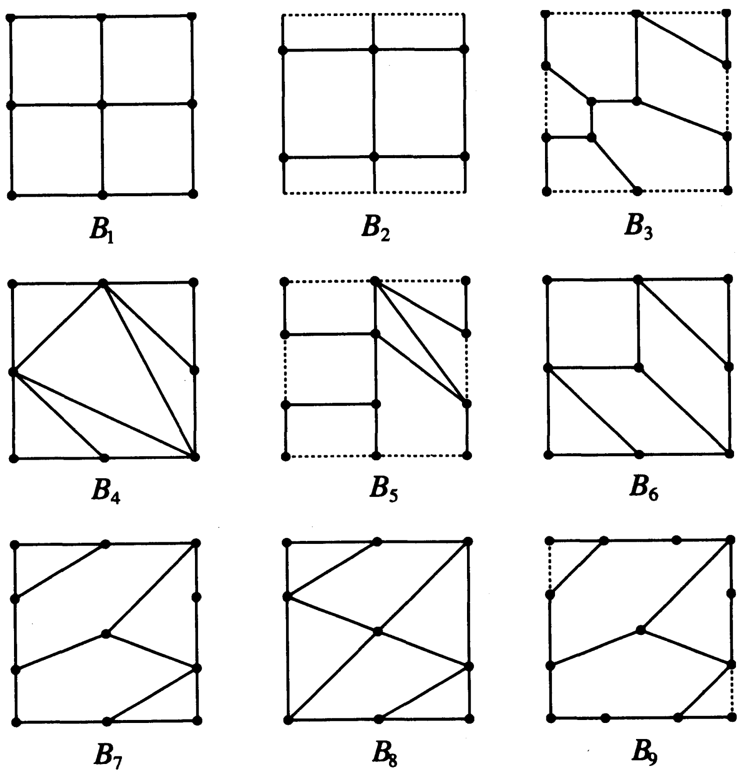

Figure 10 Minor-minimal 2-representative graphs on the Klein bottle.

concrete way to construct them for any n. However, nothing had been known about the Klein bottlal case until we did for $n = 2$.

4. Diagonal transformations in quadrangulations

Investigating the complete list of irreducible quadrangulations of the Klein bottle, we shall determine the precise values of $N(K^2)$ and $N'(K^2)$ in Theorems 2 and 3 for the Klein bottle K^2. The following two lemmas have been shown in [3].

Lemma 14. *Any vertex of degree 2 of a quadrangulation of F^2 can be moved into any face by a sequence of diagonal slides.*

Let Γ_n denote a quadrilateral region which contains n vertices of degree 2 as shown in Figure 11. Let T be a quadrangulation of a closed surface F^2 and $T+\Gamma_n$ a quadrangulation of F^2 obtained from T by adding Γ_n to a face of T. Depending on our choice of a face of T to add Γ_n, $T+\Gamma_n$ denotes various quadrangulations. However, $T+\Gamma_n$ represents an unique equivalence class of quadrangulations, by Lemma 14.

Lemma 15. *Let G and T be two quadrangulations of a closed surface. If G is contractible to T, then $G \approx T+\Gamma_m$ with $m=|V(G)|-|V(T)|$.*

Theorem 16. *Any two bipartite quadrangulations G_1 and G_2 of the Klein bottle K^2 are equivalent to each other if $|V(G_1)|=|V(G_2)|\geq N(K^2)=10$. Furthermore, this lower bound is sharp.*

Proof. Any bipartite quadrangulation G of the Klein bottle with $m\geq 10$ vertices is contractible to one of $Q_K^1,\cdots,Q_K^5$, by Theorem 1, and hence it is equivalent to one of $Q_K^1+\Gamma_{m-8},\cdots,Q_K^5+\Gamma_{m-10}$, by Lemma 15. So it sufficies to show that

$$Q_K^1+\Gamma_{m-8}\approx Q_K^2+\Gamma_{m-9}\approx Q_K^3+\Gamma_{m-9}\approx Q_K^4+\Gamma_{m-10}\approx Q_K^5+\Gamma_{m-10}.$$

Since any vertex of degree 2 can be carried into any face by Lemma 14, the

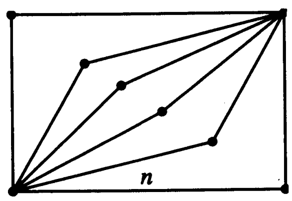

Figure 11 Γ_n

above relation can be transformed into that

$$Q_K^1+\Gamma_2+\Gamma_{m-10}\approx Q_K^2+\Gamma_1+\Gamma_{m-10}\approx Q_K^3+\Gamma_1+\Gamma_{m-10}$$
$$\approx Q_K^4+\Gamma_{m-10}\approx Q_K^5+\Gamma_{m-10}$$

Notice that if two quadrangulations G_1 and G_2 are equivalent to each other, then so are $G_1+\Gamma_n$ and $G_2+\Gamma_n$ for any $n\geq 0$. So, the readers have only to check that

$$Q_K^1+\Gamma_2\approx Q_K^2+\Gamma_1\approx Q_K^3+\Gamma_1\approx Q_K^4\approx Q_K^5.$$

It can be done, and hence the minimum value of $N(K^2)$ does not exceed 10.

Since Q_K^2 is a complete bipartite graph with 9 vertices, we can not apply any diagonal slide and diagonal rotation to Q_K^2. So, it is not equivalent to any other quadrangulation. Thus, Q_K^2 and $Q_K^1+\Gamma_1$ are an inequivalent pair of bipartite quadrangulations with m = 9. This implies that $N(K^2)=10$ is sharp. ■

Observe that Q_K^7 has a meridian cycle of length 4 while Q_K^8 has a meridian cycle of length 3. Since a meridian is unique on the Klein bottle, up to isotopy, the cycle parities of Q_K^7 and Q_K^8 are not congruent. On the other hand, $\{Q_K^6, Q_K^7\}$ and $\{Q_K^8, Q_K^9, Q_K^{10}\}$ form two classes consisting of ones with congruent cycle parities.

Theorem 17. *Any two quadrangulations G_1 and G_2 of the Klein bottle K^2 with $|V(G_1)|=|V(G_2)|\geq N'(K^2)=10$ are equivalent to each other if and only if their cycle parities are congruent.*

Proof. Since the necessity is obvious, we shall show only the sufficiency. Observe that $Q_K^6\approx Q_K^7, Q_K^8+\Gamma_1\approx Q_K^9+\Gamma_1\approx Q_K^{10}, Q_K^8\approx Q_K^9$. Similarly to the proof of Theorem 16, these imply that any two non-bipartite quadrangulations G_1 and G_2 of the Klein bottle K^2 with $|V(G_1)|=|V(G_2)|\geq 7$ are equivalent to each other under diagonal slides and diagonal rotations if $\rho_{G_1}\equiv\rho_{G_2}$, since any face contraction preserves the cycle parity [4]. Unifying Theorem 16 and the above, we obtain the theorem. ■

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