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A REMARKABLE CLASS OF NONSYMMETRIC DIPOLARIZATIONS IN LIE ALGEBRAS

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Abstract. A dipolarization in a Lie algebra $\mathfrak{g}$ is a pair of polarizations $(\mathfrak{g}^{+}, f)$ and $(\mathfrak{g}^{-}, f)$ satisfying the conditions: the two subalgebras $\mathfrak{g}^{\pm}$ span $\mathfrak{g}$, and the intersection $\mathfrak{g}^{+}\cap\mathfrak{g}^{-}$ is the isotropy subalgebra at the linear form f with respect to the coadjoint representation of $\mathfrak{g}$. We construct here a class of dipolarizations in certain solvable Lie algebras for which the two subalgebras of dipolarization are not isomorphic.

Introduction

Let $\mathfrak{g}$ be a real Lie algebra. In [1] we define a *dipolarization* of $\mathfrak{g}$ as a triple $\{\mathfrak{g}^{+}, \mathfrak{g}^{-}, f\}$, where $\mathfrak{g}^{\pm}$ are subalgebras of $\mathfrak{g}$, f is a linear function on $\mathfrak{g}$, and the following conditions are satisfied:

(D1) $$\mathfrak{g} = \mathfrak{g}^{+} + \mathfrak{g}^{-},$$

(D2) $$f([X, \mathfrak{g}]) = 0 \text{ if and only if } X \in \mathfrak{g}^{+} \cap \mathfrak{g}^{-},$$

(D3) $$f([\mathfrak{g}^{+}, \mathfrak{g}^{+}]) = f([\mathfrak{g}^{-}, \mathfrak{g}^{-}]) = 0.$$

A dipolarization is called *symmetric* if the two subalgebras $\mathfrak{g}^{+}$ and $\mathfrak{g}^{-}$ are isomorphic to each other. Otherwise it is called *nonsymmetric*.

The background of this definition is the geometry of homogeneous parakaehler manifolds. In [2] Kaneyuki obtained a remarkable class of symmetric dipolarizations in real semisimple Lie algebras by using gradations. In [3] the authors gave an example of nonsymmetric dipolarization in the Lie algebra of upper triangular matrices, which is the first known nonsymmetric dipolarization. In this note we shall construct a large class of nonsymmetric dipolarizations in subalgebras of some real forms of complex semisimple Lie algebras, which can be viewed as a generalization of the example of [3].

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Let $\tilde{\mathfrak{g}}^c$ be a complex semisimple Lie algebra, $\tilde{\mathfrak{h}}^c$ be a Cartan subalgebra of $\tilde{\mathfrak{g}}^c$, and Δ be the root system of $\tilde{\mathfrak{g}}^c$ with respect to $\tilde{\mathfrak{h}}^c$. Select a basis of $\tilde{\mathfrak{g}}^c$ mod $\tilde{\mathfrak{h}}^c$ $\{X_\alpha \mid X_\alpha \in (\tilde{\mathfrak{g}}^c)_\alpha, \alpha \in \Delta\}$ such that:

$$[X_\alpha, X_\beta] = c_{\alpha\beta} X_{\alpha+\beta},$$

where $c_{\alpha\beta}$ are real numbers (such a basis must exist, for example, a Chevalley basis will do). Let $H_\alpha = [X_\alpha, X_{-\alpha}]\,(\alpha \in \Delta)$, denote the space $\sum_{\alpha\in\Delta} \boldsymbol{R}H_\alpha$ by $\mathfrak{h}$; set

$$\tilde{\mathfrak{g}} = \mathfrak{h} + \sum_{\alpha\in\Delta} \boldsymbol{R}X_\alpha .$$

Then $\tilde{\mathfrak{g}}$ is a real form of $\tilde{\mathfrak{g}}^c$. Select a positive root system Δ^+ of Δ, and let Π be the set of simple roots of Δ^+. We set

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta^+} \boldsymbol{R}X_\alpha .$$

Then $\mathfrak{g}$ is a real Lie algebra. We define the subalgebras of $\mathfrak{g}$:

$$\mathfrak{g}^+ = \sum_{\alpha\in\Delta^+} \boldsymbol{R}X_\alpha ,$$

$$\mathfrak{g}^- = \mathfrak{h} + \sum_{\alpha\in\Delta^+-\Pi} \boldsymbol{R}X_\alpha .$$

Let $\langle , \rangle$ be an inner product in $\mathfrak{g}$ such that $\{H_{\alpha_i}, X_\alpha \mid \alpha_i \in \Pi, \alpha \in \Delta^+\}$ is an orthonormal basis of $\mathfrak{g}$. We define a linear form f on $\mathfrak{g}$ by

$$f(X) = \sum_{\alpha\in\Pi} \langle X, H_\alpha \rangle + \sum_{\alpha\in\Pi} \langle X, X_\alpha \rangle .$$

Theorem. *Let $\mathfrak{g}$, $\mathfrak{g}^+, \mathfrak{g}^-$ and f be defined as above, then $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is a dipolarization of $\mathfrak{g}$. Furthermore if $\tilde{\mathfrak{g}}^c$ contains a simple ideal which is not isomorphic to $\mathfrak{sl}(2,\boldsymbol{C})$, then the dipolarization is nonsymmetric.*

Proof. We first prove that $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is a dipolarization of $\mathfrak{g}$. It is obvious that $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$, so (D1) is satisfied. If $Y \in \mathfrak{g}^+ \cap \mathfrak{g}^- = \sum_{\alpha\in\Delta^+-\Pi} \boldsymbol{R}X_\alpha$, then $[Y, \mathfrak{g}] \subset \sum_{\alpha\in\Delta^+-\Pi} \boldsymbol{R}X_\alpha$. By the definition of f we see that $f([Y, \mathfrak{g}]) = 0$. Conversely, suppose that $Y \in \mathfrak{g}$ and that $f([Y, \mathfrak{g}]) = 0$. We shall prove $Y \in \mathfrak{g}^+ \cap \mathfrak{g}^-$. We write $Y = H + \sum_{\alpha\in\Delta^+} y_\alpha X_\alpha$, where $H \in \mathfrak{h}$ and $y_\alpha \in \boldsymbol{R}$. First we prove H=0. If $H \neq 0$, then one can select a $\beta \in \Pi$ such that $\beta(H) \neq 0$. Then

$$[Y, X_\beta] = \beta(H) X_\beta + Y_1 ,$$

where $Y_1 \in \sum_{\alpha\in\Delta^+-\Pi} \boldsymbol{R}X_\alpha$. Hence

$$f([Y, X_\beta]) = f(\beta(H) X_\beta + Y_1) = \beta(H) \langle X_\beta, X_\beta \rangle \neq 0.$$

This is a contradiction. So $H = 0$. Now we will prove $y_\alpha = 0, \forall \alpha \in \Pi$. Otherwise there exists a $\gamma \in \Pi$ such that $y_\gamma \neq 0$. Select an $H_0 \in \mathfrak{h}$ such that $\gamma(H_0) \neq 0$ and $\alpha(H_0) = 0$, $\forall \alpha \in \Pi - \{\gamma\}$, then

$$[Y, H_0] = -\gamma(H_0) y_\gamma X_\gamma + Y_2,$$

where $Y_2 \in \sum_{\alpha \in \Delta - \Pi} \boldsymbol{R} X_\alpha$; thus,

$$f([Y, H_0]) = -f(\gamma(H_0) y_\gamma X_\gamma) = -\gamma(H_0) y_\gamma \langle X_\gamma, X_\gamma \rangle \neq 0.$$

This also contradicts $f([Y,\mathfrak{g}])=0$. Thus $y_\alpha = 0, \forall \alpha \in \Pi$, that is, $Y \in \mathfrak{g}^+ \cap \mathfrak{g}^-$ so (D2) is also satisfied. Next we prove (D3). Since $[\mathfrak{g}^+, \mathfrak{g}^+] \subset \sum_{\alpha \in \Delta^+ - \Pi} \boldsymbol{R} X_\alpha$, $[\mathfrak{g}^-, \mathfrak{g}^-] \subset \sum_{\alpha \in \Delta^+ - \Pi} \boldsymbol{R} X_\alpha$ by the definition, we have $f([\mathfrak{g}^+, \mathfrak{g}^+]) = f([\mathfrak{g}^-, \mathfrak{g}^-]) = 0$, thus (D3) is satisfied and $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is a dipolarization of $\mathfrak{g}$.

Now suppose that $\tilde{\mathfrak{g}}^c$ contains a simple ideal that is not isomorphic to $\mathfrak{sl}(2, \boldsymbol{C})$, Then $\Delta^+ - \Pi$ is not empty. Thus

$$[\mathfrak{g}^-, \mathfrak{g}^-] = \sum_{\alpha \in \Delta^+ - \Pi} \boldsymbol{R} X_\alpha.$$

Using this iteratively we see that $\mathfrak{g}^-$ is not a nilpotent Lie algebra. But it is obvious that $\mathfrak{g}^+$ is nilpotent, thus $\mathfrak{g}^+$ is not isomorphic to $\mathfrak{g}^-$ and the dipolarization is nonsymmetric. Q.E.D.

Next we shall give the matrix realizations of $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ in the case $\tilde{\mathfrak{g}}^c = A_l\,(l \geq 2)$, $B_l\,(l \geq 2)$, $C_l\,(l \geq 3)$ or $D_l\,(l \geq 4)$.

In the following, if A denotes a matrix, we always use a_{ij} to denote the (i,j) element of A.

Case 1. $\tilde{\mathfrak{g}}^c = A_l = \mathfrak{sl}(l+1, \boldsymbol{C})\,(l \geq 2)$

In this case we select

$$\tilde{\mathfrak{h}}^c = \left\{ \mathrm{diag}(h_1, h_2, \cdots, h_{l+1}) \,\middle|\, \sum_{i=1}^{l+1} h_i = 0 \right\}.$$

Then

$$\Delta = \left\{ \pm(\lambda_i - \lambda_j) \mid i < j \right\},$$

where

$$\lambda_i(\,\mathrm{diag}(h_1, h_2, \cdots, h_{l+1})) = h_i.$$

Select

$$\Delta^+ = \left\{ \lambda_i - \lambda_j \mid i < j \right\};$$

then

$$\Pi = \{\lambda_i - \lambda_{i+1} \mid 1 \le i \le l\}.$$

By direct computation we obtain:

$$\mathfrak{g} = \{X \in \boldsymbol{R}^{(l+1)\times(l+1)} \mid x_{ij} = 0 \text{ for } i > j \text{ and } \mathrm{Tr}X = 0\},$$

$$\mathfrak{g}^+ = \{X \in \mathfrak{g} \mid x_{ii} = 0,\ i = 1, 2, \cdots, l+1\},$$

$$\mathfrak{g}^- = \{X \in \mathfrak{g} \mid x_{i,i+1} = 0,\ i = 1, 2, \cdots, l\},$$

$$f(X) = \sum_{i=1}^{l} x_{i,i+1} + 2(l+1)\sum_{i=1}^{l}(l+1-i)x_{ii} \text{ for } X \in \mathfrak{g}.$$

Remark. Let $\mathfrak{g}' = \mathfrak{g} \oplus \boldsymbol{R}I_{l+1}$, where I_{l+1} is the identity matrix, and let

$$(\mathfrak{g}')^+ = \mathfrak{g}^+ \oplus \boldsymbol{R}I_{l+1},\quad (\mathfrak{g}')^- = \mathfrak{g}^- \oplus \boldsymbol{R}I_{l+1},$$

$$f'(X + rI_{l+1}) = r + f(X) - 2(l+1)\ \textstyle\sum_{i=1}^{l}(l+1-i)x_{ii}.$$

Then $\{(\mathfrak{g}')^+, (\mathfrak{g}')^-, f'\}$ is the dipolarization of [3].

Case 2. $\tilde{\mathfrak{g}}^c = B_l\ (l \ge 2)$

In this case we use the isomorphism:

$$\tilde{\mathfrak{g}}^c \cong \{X \in \boldsymbol{C}^{(2l+1)\times(2l+1)} \mid SX + X'S = 0\},$$

where $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$. Select

$$\tilde{\mathfrak{h}}^c = \{\mathrm{diag}(0, x_1, x_2, \cdots, x_l, -x_1, -x_2, \cdots, -x_l) \mid x_i \in \boldsymbol{C}\};$$

then the root system is

$$\Delta = \{\pm\lambda_k, \pm(\lambda_i - \lambda_j), \pm(\lambda_i + \lambda_j) \mid 1 \le k \le l, 1 \le i < j \le l\},$$

where

$$\lambda_i(\mathrm{diag}(0, x_1, x_2, \cdots, x_l, -x_1, -x_2, \cdots, -x_l)) = x_i.$$

We can select

$$\Delta^+ = \{\lambda_k, \lambda_i - \lambda_j, \lambda_i + \lambda_j \mid 1 \le k \le l, 1 \le i < j \le l\};$$

then

$$\Pi = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{l-1} - \lambda_l, \lambda_l\}.$$

By direct computation we obtain:

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & 0 & c \\ -c & A & B \\ 0 & 0 & -A' \end{pmatrix} \middle| A, B \in \boldsymbol{R}^{l\times l}, a_{ij} = 0 \text{ for } i > j, B' = -B, c = (c_i) \in \boldsymbol{R}^l \right\},$$

$$\mathfrak{g}^+ = \left\{ \begin{pmatrix} 0 & 0 & c \\ -c' & A & B \\ 0 & 0 & -A' \end{pmatrix} \in \mathfrak{g} \middle| a_{ii} = 0, i = 1, 2, \cdots, l \right\},$$

$$\mathfrak{g}^- = \left\{ \begin{pmatrix} 0 & 0 & c \\ -c' & A & B \\ 0 & 0 & -A' \end{pmatrix} \in \mathfrak{g} \middle| c_l = 0, a_{i,i+1} = 0, i = 1, 2, \cdots, l-1 \right\},$$

$$f(X) = \sum_{i=1}^{l-1} a_{i,i+1} + c_l + 2(2l-1)\sum_{i=1}^{l}(l+1-i)a_{ii},$$

where $X = \begin{pmatrix} 0 & 0 & c \\ -c' & A & B \\ 0 & 0 & -A' \end{pmatrix} \in \mathfrak{g}$.

Case 3. $\tilde{\mathfrak{g}}^c = C_l \ (l \geq 3)$

In this case we use the isomorphism:

$$\tilde{\mathfrak{g}}^c \cong \{X \in \boldsymbol{C}^{2l\times 2l} \mid SX + X'S = 0\},$$

where $S = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$. Select

$$\tilde{\mathfrak{h}}^c = \{\mathrm{diag}(x_1, x_2, \cdots, x_l, -x_1, -x_2, \cdots, -x_l) \mid x_i \in \boldsymbol{C}\};$$

then the root system is

$$\Delta = \{\pm(\lambda_i - \lambda_j), \pm(\lambda_i + \lambda_j), \pm 2\lambda_k \mid 1 \leq i < j \leq l, 1 \leq k \leq l\},$$

where

$$\lambda_i(\mathrm{diag}(x_1, x_2, \cdots, x_l, -x_1, -x_2, \cdots, -x_l)) = x_i.$$

Select

$$\Delta^+ = \{\lambda_i - \lambda_j, \lambda_i + \lambda_j, 2\lambda_k \mid 1 \le i < j \le l, 1 \le k \le l\};$$

then

$$\Pi = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{l-1} - \lambda_l, 2\lambda_l\}.$$

By direct computation we obtain:

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & -A' \end{pmatrix} \mid A, B \in \boldsymbol{R}^{l \times l}, a_{ij} = 0 \text{ for } i > j, B' = B \right\},$$

$$\mathfrak{g}^+ = \left\{ \begin{pmatrix} A & B \\ 0 & -A' \end{pmatrix} \in \mathfrak{g} \mid a_{ii} = 0, \ i = 1, 2, \cdots, l \right\},$$

$$\mathfrak{g}^- = \left\{ \begin{pmatrix} A & B \\ 0 & -A' \end{pmatrix} \in \mathfrak{g} \mid b_{ll} = 0, a_{i,i+1} = 0, \ i = 1, 2, \cdots, l-1 \right\},$$

$$f(X) = \sum_{i=1}^{l-1} a_{i,i+1} + 2b_{ll} + 4(l+1)\sum_{i=1}^{l}(l - i + \frac{1}{2})a_{ii},$$

where $X = \begin{pmatrix} A & B \\ 0 & -A' \end{pmatrix} \in \mathfrak{g}$.

Case 4. $\tilde{\mathfrak{g}}^c = D_l \ (l \ge 4)$

In this case we use the isomorphism:

$$\tilde{\mathfrak{g}}^c \cong \{X \in \boldsymbol{C}^{2l \times 2l} \mid X'S + SX = 0\},$$

where $S = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$. Select

$$\tilde{\mathfrak{h}}^c = \{\mathrm{diag}(x_1, x_2, \cdots, x_l, -x_1, -x_2, \cdots, -x_l) \mid x_i \in \boldsymbol{C}\};$$

then

$$\Delta = \{\pm(\lambda_i - \lambda_j), \pm(\lambda_i + \lambda_j) \mid 1 \le i < j \le l\};$$

where

$$\lambda_i(\mathrm{diag}(x_1, x_2, \cdots, x_l, -x_1, -x_2, \cdots, -x_l)) = x_i.$$

Select

$$\Delta^+ = \{\lambda_i - \lambda_j, \lambda_i + \lambda_j \mid 1 \le i < j \le l\};$$

then

$$\Pi = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{l-1} - \lambda_l, \lambda_{l-1} + \lambda_l\}.$$

By direct computation we obtain:

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} \mid A, B \in \boldsymbol{R}^{l\times l}, a_{ij} = 0 \text{ for } i > j, B^t = -B \right\},$$

$$\mathfrak{g}^+ = \left\{ \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} \in \mathfrak{g} \mid a_{ii} = 0, i = 1, 2, \cdots, l \right\},$$

$$\mathfrak{g}^- = \left\{ \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} \in \mathfrak{g} \mid b_{l-1,l} = 0, a_{i,i+1} = 0, i = 1, 2, \cdots, l-1 \right\},$$

$$f(X) = \sum_{i=1}^{l-1} a_{i,i+1} + b_{l-1,l} + 4(l-1)\sum_{i=1}^{l}(l-i)a_{ii},$$

where $X = \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} \in \mathfrak{g}$.

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