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FINITE RINGS WITH CENTRAL PRIME RADICAL

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Abstract. In this note, certain structure theorems will be given for finite rings whose prime radicals are in the center.

The famous theorem of Wedderburn tells us that a finite division ring is necessarily commutative. In addition to the original proof by Wedderburn, purely algebraic proofs of this theorem have been looking for. Detailed information on this subject are in Nagahara and Tominaga [5]. As a consequence of this research, in [2], Herstein proved that a finite ring R is commutative, if all nilpotent elements are contained in center of R. We shall state related results for a finite ring whose prime radical is in the center. Also we shall prove that a subdirectly irreducible local ring with commuting nilpotent elements is left and right self-injective.

In this note, all rings are finite rings which do not necessarily have identity. For a ring R, $P(R)$ denotes the (prime) radical of R and $Z(R)$ denotes the center of R. It is well known that $P(R)$ coincides with the Jacobson radical for any finite ring R. A finite local ring is a finite ring with identity such that $R/P(R)$ is a finite field.

We begin with the following theorem:

Theorem 1. *Let R be a finite ring. If the radical $P(R)$ of R is in the center $Z(R)$ of R, then R is the direct sum of a finite commutative ring and finitely many matrix rings over finite fields.*

Proof. It is well known that $R/P(R)$ is a finite direct sum of matrix rings over finite fields. In particular, $R/P(R)$ has an element f. Let

$$f = f_1 + f_2 + \cdots + f_n$$

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be the decomposition of f as the sum of orthogonal central primitive idempotents. By [3, Proposition III.8.5], there exist orthogonal idempotents $e_1, e_2, \cdots, e_n$ such that $e_i + P(R) = f_i$ for $i = 1, 2, \cdots, n$. Set

$$e = e_1 + e_2 + \cdots + e_n.$$

Even if R does not have an identity, we shall use the notation $(1-e)a$ (resp. $a(1-e)$) to mean $a-ea$ (resp. $a-ae$) for $a \in R$. Then $(1-e)R$ and $R(1-e)$ are contained in $P(R)$, and hence in $Z(R)$. Therefore we have

$$(1-e)Re = eR(1-e) = 0.$$

If $n > 1$, then, for any $a \in R$. $e_1 a e_2 \in P(R) \subset Z(R)$, and so

$$e_1 a e_2 = e_1^2 a e_2 = e_1 a e_2 e_1 = 0.$$

Similarly we obtain that $e_i R e_j = 0$ whenever $i \neq j$. These mean that $e_1, e_2, \cdots, e_n$ are central orthogonal idempotents of R. Therefore

$$R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R \oplus (1-e)R$$

is a ring decomposition of R with each $e_i R$ a primary ring, that is $e_i R / P(e_i R) = M_{ni}(L_i)$ for some finite field L_i. If $n_i = 1$, then $e_i R / P(e_i R) - \{0\}$ forms a finite cyclic group. In this case, let u be an element of $e_i R$ such that $u + P(e_i R)$ generates this cyclic group. Then the ring $e_i R$ is generated by u and $P(e_i R)$. Since $P(e_i R)$ is contained in the center of $e_i R$, we conclude that $e_i R$ is commutative. Now suppose that $n_i > 1$. By [3, Theorem III.9.1], $e_i R = M_{n_i}(S_i)$ for some finite local ring S_i. By hypothesis, $P(M_{n_i}(S_i)) = M_{n_i}(P(S_i))$ is contained in the center of $M_{n_i}(S_i)$, and hence we conclude that $P(S_i) = 0$, that is, S_i is a finite field. Therefore, if $n_i > 1$, then $e_i R$ is a matrix ring over a finite field. Since $(1-e)R$ is a commutative nilpotent ring, this completes the proof.

As a corollary, we obtain the following generalization of [2, Theorem].

Corollary 1. *Let R be a finite ring. If the radical $P(R)$ of R is in the center $Z(R)$ of R and if any two nilpotent elements of R commute each other, then R is commutative.*

A ring R is subdirectly irreducible if the intersection of all its nonzero ideals is not the zero ideal.

Theorem 2. *Let R be a finite local ring R with commutative radical. If R is subdirectly irreducible, then R is left and right self-injective.*

Proof. If we set $I = Soc({}_R R)$, then either $I = R$ or $I \subset P(R)$. In case $I = R$, R is finite field. So, assume that $I \subset P(R)$. By hypothesis, we have $IP(R) = P(R)I$

$= 0$. Hence I is a $(R/P(R)\ ,R/P(R))$-bimodule. Let $R/P(R) = GF(p^m)$. Then I is a $GF(p^m)\otimes_{GF(p)}GF(p^m)$-module. By [4, Lemma 3.9], $GF(p^m)\otimes_{GF(p)}GF(p^m)$ is isomorphic to the direct sum of m-copies of $GF(p^m)$. Since R is subdirectly irreducible, I is 1-dimensional over $GF(p^m)$. Hence $Soc({}_R R)\cong_R R/P(R)$.Similarly we can show that $Soc(R_R)\cong R/P(R)_R$. Hence R is left and right self-injective by [1, Theorem 31.3],

First we give an example of a self-injective finite local ring with commutative radical.

Example 1. The abelian group $GF(4)\oplus GF(4)$ together with the multiplication

$$(a,b)(c,d)=(ac,a^2d+bc)$$

forms a ring, which we denote by R. This R is a noncommutative local ring $P(R)$ is the unique minimal ideal consisting of elements of the form $(0, b)$. Clearly $P(R)$ is commutative, and hence R is self-injective.

The following example shows that a subdirectly irreducible finite local ring need not be self-injective.

Example 2. Let F be a finite field and consider the subring

$$R=\left\{\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix} \mid a,b,c,d\in F\right\}$$

of $M_3(F)$. Then R is a subdirectly irreducible local ring with unique minimal ideal

$$I=\left\{\begin{pmatrix} 0 & 0 & c\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \mid c\in F\right\}.$$

We can easily see that

$$Soc({}_R R)=\left\{\begin{pmatrix} 0 & b & c\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \mid b,c\in F\right\}$$

and

$$Soc(R_R)=\left\{\begin{pmatrix}0&0&c\\0&0&d\\0&0&0\end{pmatrix} \mid c,d\in F\right\}.$$

Since $Soc({}_RR)\neq Soc(R_R)$, R is not self-injective by [1, Corollary 31.8].

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