YOKOHAMA MATHEMATICAL
JOURNAL VOL. 43, 1995

# $\lambda$-AUTOMORPHISMS OF A RIEMANNIAN FOLIATION ON A COMPLETE MANIFOLD

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(Received July 10, 1995)

**Abstract.** In this paper, we discuss characterizations of transversal Killing and conformal fields on a complete ambient manifold with Riemannian foliation. Our results extend both those of [2], [7] (for the case of harmonic foliation) and those of [9] (for the case of closed manifold).

## 1. Introduction

Let $(M,g,\mathscr{F})$ be a $m$-dimensional oriented, connected Riemannian manifold with transversally oriented Riemannian foliation $\mathscr{F}$ of codimension $q:=m-p$ and a bundle-like metric $g$. It is given by an exact sequence of vector bundles

$$0\to\mathscr{V}\to TM\xrightarrow{\pi}Q\to 0, \tag{1.1}$$

where $\mathscr{V}$ is the tangent bundle and $Q$ the normal bundle of $\mathscr{F}$. The metric $g$ determines an orthogonal decomposition $TM=\mathscr{V}\oplus\mathscr{H}$. We often identify $\mathscr{H}$ with $Q$ by an isometric splitting

$$\sigma:(Q,g_Q:=\sigma^{*}g_{\mathscr{H}})\to(\mathscr{H},g_{\mathscr{H}}). \tag{1.2}$$

We have an associated exact sequence of Lie algebras

$$0\to\Gamma(\mathscr{V})\to V(\mathscr{F})\xrightarrow{\pi}\overline{V}(\mathscr{F})\to 0, \tag{1.3}$$

where $V(\mathscr{F}):=\{Y\in\Gamma(TM)\,|\,[V,Y]\in\Gamma(\mathscr{V})$ for all $V\in\Gamma(\mathscr{V})\}$ and $\overline{V}(\mathscr{F}):=\{s\in\Gamma(Q)|$ $s=\pi(Y),Y\in V(\mathscr{F})\}$, called the space of all transversal infinitesimal automorphisms of $\mathscr{F}$. Here and hereafter, we denote by $\Gamma(\cdot)$ the space of all smooth sections of a vector bundle $(\cdot)$. The transversal Levi-Civita connection $D$ on $Q$ is a torsion free and metric connection with respect to $g_Q$ ([2], [4], [12]).

Throughout this paper, we use the following notations:

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1991 Mathematics Subject Classification: 53C20, 57R30.
Key words and phrases: Riemannian foliation, $\lambda$-automorphism, complete manifold.

* The present Studies were supported (in part) by the Basic Science Research Institute Program, Ministry of Education, 1996, Project No. BSRI-96-1404

** Supported by the Korean Science and Engineering Foundation, 1995

$\tau$: the tension field of $\mathscr{F}$,
$div_D s$: the transversal divergence of $s \in \Gamma(Q)$,
$grad_D f$: the transversal gradient of a function $f \in C^\infty(M)$,
$R_D$: the transversal curvature tensor of $D$,
$\rho_D$: the transversal Ricci operator,
$\Delta := d_D^* d_D$: the Laplacian acting on $\Gamma(Q)$,
$\theta_Y$: the transversal Lie derivative operator for $Y \in V(\mathscr{F})$,
$A_D(Y) := \theta_Y - D_Y$ for $Y \in V(\mathscr{F})$.

It is noted that $A_D(s)$ is well-defined, since for a given $s \in \overline{V}(\mathscr{F})$, $A_D(Y)$ is independent of the choice of $Y$ with $\pi(Y) = s$. Now we introduce a symmetric operator $B_D^\lambda(s) : \Gamma(Q) \to \Gamma(Q)$ defined by

(1.4) $$B_D^\lambda(s) := A_D(s) + {}^t A_D(s) + \lambda(div_D s)\mathrm{Id}\,,$$

for a given $s = \pi(Y) \in \overline{V}(\mathscr{F})$, where Id denotes the identity map of $\Gamma(Q)$.

The study of geometric transversal infinitesimal automorphisms – for example, transversal Killing, affine, conformal, projective fields – of a Riemannian foliation has been attacked by many differential geometers. For the point foliation, such transversal infinitesimal automorphisms on a foliated Riemannian manifold reduce to usual infinitesimal automorphisms on a Riemannian manifold.

In this paper, we are particularly interested in $\lambda$-automorphisms of a Riemannian foliation. This notion was first introduced in [13] for the case where the foliation is harmonic and was recently extended to the general Riemannian foliation in [9]. Indeed transversal Killing, affine, conformal, projective fields are all examples of $\lambda$-automorphisms.

The main interest of this paper is to characterize transversal Killing and conformal fields of a Riemannian foliation on a complete Riemannian manifold. In order to do this, we have to consider $L^2$ transversal infinitesimal automorphisms, that is, transversal infinitesimal automorphisms with finite global norm.

For the closed case, we have obtained

**Theorem ([9]).** *Let $(M, g, \mathscr{F})$ be a m-dimensional oriented, connected, closed Riemannian manifold with transversally oriented Riemannian foliation of codimension $q := m - p \geq 2$ and a bundle-like metric $g$. Let $s \in \overline{V}(\mathscr{F})$ and $\omega$ the $g_Q$-dual of $s$.*

(A) a $\lambda$-automorphism s is a transversal Killing field if and only if

$$d(div_D s) = 0 \text{ and } \langle B_D^0(s)s, \tau\rangle \geq 0.$$

(B) a $\lambda$-automorphism s is a transversal conformal field if and only if

$$\lambda = 1 - \tfrac{2}{q} \text{ and } \langle B_D^{2/q}(s)s, \tau\rangle \geq 0.$$

The above results $(A)$ and $(B)$ extend those of [8] and [11] respectively. When $\mathscr{F}$ is harmonic, these results correspond to those of [5], [10] respectively.

Our main result is then the following:

**Main Theorem.** *Let $(M, g, \mathscr{F})$ be a m-dimensional oriented, connected, complete Riemannian manifold with transversally oriented Riemannian foliation of codimension $q := m - p \geq 2$ and a bundle-like metric $g$. Let $s \in \overline{V}(\mathscr{F})$ and $\omega$ the $g_Q$-dual of $s$.*

(C) *a $L^2\lambda$-automorphism s is a transversal Killing field if and only if*

$$div_D s = 0 \text{ and } \langle B_D^0(s)s, \tau\rangle \geq 0.$$

(D) *a $L^2\lambda$-automorphism s is a transversal conformal field if and only if*

$$\lambda = 1 - \frac{2}{q} \text{ and } \langle B_D^{2/q}(s)s, \tau\rangle \geq 0.$$

When $\mathscr{F}$ is harmonic, these results were obtained in [2], [7].

The authors would like to thank the referee for his kind suggestions.

## 2. A review of $\lambda$-automorphisms.

Let $(M, g, \mathscr{F})$ be a $m$-dimensional oriented, connected, complete Riemannian manifold with transversally oriented Riemannian foliation of codimension $q := m - p \geq 2$ and a bundle-like metric $g$. The basic complex $(\Omega_B, d_B := d|\Omega_B)$ is a subcomplex of the de Rham complex $(\Omega(M), d)$, where

$$\Omega_B := \{\omega \in \Omega(M) | i_V\omega = \theta_V\omega = 0 \text{ for all } V \in \Gamma(\mathscr{V})\}. \tag{2.1}$$

We consider a codifferential operator $\delta_T : \Omega_B^r \to \Omega_B^{r-1}$ defined by

$$\delta_T := (-1)^{r(q+1)+1}\,\overline{*}\,d_B\,\overline{*}, \tag{2.2}$$

where $\overline{*}$ is the star operator associated to the holonomy-invariant metric $g_Q$ on $Q$. It should be noted that in general $d_B$ and $\delta_T$ are not formal adjoint on $\Omega_B$ unless $\mathscr{F}$ is harmonic.

In order to discuss $\lambda$-automorphisms, it is useful to introduce the following operators $\delta, \delta^*$ appeared in [5]. Throughout this paper, We denote by $\{E_A\} = \{E_i, E_\alpha\}$, $E_i \in \Gamma(\mathscr{V}), E_\alpha \in V(\mathscr{F})$ the special local orthonormal frame field about $x \in M$ with $(E_A)_x = e_A$ introduced in [5]. $\delta : \Gamma(S^2Q^*) \to \Gamma(Q^*)$, $S^2Q^*$

being the symmetric tensor product of $Q^*$ of order 2, is given by the local formula

(2.3) $$\delta\beta := -\sum_\alpha (D_{E_\alpha}\beta)(E_\alpha, \bullet), \text{ for } \beta \in \Gamma(S^2Q^*)$$

and $\delta^* : \Gamma(Q^*) \to \Gamma(S^2Q^*)$ by

(2.4) $$(\delta^*\omega)(s,t) := \frac{1}{2}\{(D_{\sigma(s)}\omega)(t) + (D_{\sigma(t)}\omega)(s)\}, \quad \text{for } \omega \in \Gamma(Q^*),\ s,t \in \Gamma(Q).$$

The basic 1-forms (resp. basic symmetric 2-forms) may be identified with a subspace of $\Gamma(Q^*)$ (resp. $\Gamma(S^2Q^*)$). We denote by $(\,,\,)_x$ the local scalar product on $\Gamma(Q)$ or $\Gamma(Q^*)$ at a point $x \in M$ and $|\cdot|_x^2 := (\cdot,\cdot)_x$. The local scalar product may be extended on $\Gamma(\otimes^{r_1} Q \otimes^{r_2} Q^*)$. Let $\Gamma_c(Q)$ (resp. $\Gamma_c(Q^*)$) be the space of all sections of $Q$ (resp. $Q^*$) with compact supports. Let $\langle\,,\,\rangle$ be the global scalar product on $\Gamma_c(Q)$ or $\Gamma_c(Q^*)$ and $\|\cdot\|^2 := (\cdot,\cdot)$. The global scalar product may be also extended on $\Gamma(\otimes^{r_1} Q \otimes^{r_2} Q^*)$. Let $L^2(Q)$ (resp. $L^2(Q^*)$) be the completion of $\Gamma_c(Q)$ (resp. $\Gamma_c(Q^*)$) with respect to $\langle\,,\,\rangle$.

**Definition.** *We say that an element $s \in L^2(Q) \cap \overline{V}(\mathcal{F})$ is a $L^2$ transversal infinitesimal automorphism of $\mathcal{F}$.*

We can verify that (we refer to [5], [9]).

**Proposition 2.1.** *Let $(M, g, \mathcal{F})$ be a m-dimensional oriented, connected, complete Riemannian manifold with transversally oriented Riemannian foliation of codimension $q := m - p \geq 2$ and a bundle-like metric $g$. For $s \in \overline{V}(\mathcal{F})$ and $\omega$ the $g_Q$-dual of $s$*

(2.5) $$2\delta\delta^*\omega = -trD^2\omega - \rho_D(\omega) + d_B\delta_T\omega,$$

(2.6) $$(div_D s)_x = -(\delta_T\omega)_x = (\delta^*\omega, g_Q)_x,$$

(2.7) $$|\delta^*\omega + \frac{1}{q}(\delta_T\omega)g_Q|_x^2 = |\delta^*\omega|_x^2 - \frac{1}{q}(\delta_T\omega)_x^2$$

**Definition.** Given $Y \in V(\mathcal{F})$, $s = \pi(Y)$ is called a $\lambda$-automorphism for $\lambda \in \boldsymbol{R}$ if it satisfies

(2.8) $$\Delta s - D_{\sigma(\tau)}s - \rho_D(s) - \lambda grad_D div_D s = 0,$$

or equivalently the $g_Q$-dual $\omega$ satisfies

$$-trD^2\omega - \rho_D(\omega) + \lambda d_B\delta_T\omega = 0. \tag{2.9}$$

**Proposition 2.2 ([10]).** Let $s = \pi(Y) \in \overline{V}(\mathscr{F})$.
(i) If s is a transversal Killing field, i.e., $\theta_Y g_Q = 0$ then

$$div_D s = 0, \quad \Delta s = D_{\sigma(r)} s - \rho_D(s).$$

(ii) If s is a transversal conformal field, i.e., $\theta_Y g_Q = 2f_Y g_Q$ where $f_Y$ is a function on $M$, then

$$div_D s = q f_Y, \quad \Delta s = D_{\sigma(r)} s + \rho_D(s) + (1 - \frac{2}{q}) grad_D div_D s.$$

Proposition 2.2 says that a transversal Killing field is a $\lambda$-automorphism for all $\lambda$ and a transversal conformal field is a $(1 - \frac{2}{q})$-automorphism.

## 3. The proof of Main Theorem

The method of proof for the extension of the results from the closed to the complete case follows from the construction of a family of cut-off functions whose supports exhaust $M$, which allows to carry out the integration by parts arguments familiar in the closed case.
Let $w \in C^\infty(\boldsymbol{R})$ such that $0 \le w(t) \le 1$, $w(t) = 1$ for $t \le 1$, $w(t) = 0$ for $t \le 2$. Let $d(x) = dist(x_0, x)$ be the geodesic distance from a fixed point $x_0$ of $M$. For every $k > 0$ let $w_k(x) := w(\frac{d(x)}{k})$, which is Lipschitz continuous, and hence differentiable almost everywhere on $M$. Moreover

$$0 \le w_k(x) \le 1,$$

supp $w_k \subset B(2k)$ (geodesic ball of radius $2k$ centered at $x_0$),

$$w_k(x) \le 1 \text{ on } B(k), \tag{3.1}$$

$$\lim_{k \to \infty} w_k = 1,$$

$$|dw_k(x)| \le C/k \text{ almost everywhere on } M,$$

where $C$ is a positive constant independent of $k$ ([1], [2], [3], [6], [7]).

The following divergence formula is fundamental at a point $x \in M$

$$\begin{aligned} (div_{\nabla^M} Z)_x &:= \sum_A g(\nabla^M_{e_A} Z, e_A) \\ &= (div_D \pi(Z))_x - (\pi(Z), \tau)_x, \end{aligned} \tag{3.2}$$

for $Z \in \Gamma(\mathscr{H})$, where $\nabla^M$ is the Levi-Civita connection of $g$.

Now we are in a position to prove Main Theorem. By virtue of Proposition

2.2, it suffices to verify the converse. Let $s$ be a $L^2$ $\lambda$-automorphism and $\omega$ the $g_Q$-dual of $s$. Then we find from (2.5) and (2.9) that $2\delta\delta^*\omega+(\lambda-1)d_B\delta_T\omega=0$. So we need to compute the global scalar products on $B(2k)$.

$$\langle 2\delta\delta^*\omega+(\lambda-1)d_B\delta_T\omega, w_k^2\omega\rangle_{B_{2k}}.$$

First we derive the local scalar product by a similar argument as in [7]

$$(3.3)\qquad (\delta\delta^*\omega, w_k^2\omega)_x=\sum_\alpha(\delta\delta^*\omega)(e_\alpha)(w_k^2\omega)(e_\alpha)$$
$$=-(div_D u)_x+(w_x\delta^*\omega, dw_k\otimes\omega+\omega\otimes dw_k)_x$$
$$+(w_k\delta^*\omega, w_k\delta^*\omega)_x,$$

where $u\in\Gamma(Q)$ is the $g_Q$-dual of the 1-form

$$(3.4)\qquad \zeta(\cdot):=\sum_\alpha(\delta^*\omega)(\cdot, E_\alpha)(w_k^2 w)(E_\alpha)\in\Gamma(Q^*).$$

Hence we get by using (3.2)

$$(3.5)\qquad \langle\delta\delta^*\omega, w_k^2\omega\rangle B_{2k}=-\langle u,\tau\rangle_{B(2k)}+\|w_k\delta^*\omega\|^2_{B(2k)}$$
$$+\langle w_k\delta^*\omega, dw_k\otimes\omega+\omega\otimes dw_k\rangle_{B(2k)},$$

where $\|\cdot\|_{B(2k)}:=\langle\cdot,\cdot\rangle_{B(2k)}$. Moreover,

$$(3.6)\qquad \langle u,\tau\rangle_{B(2k)}=-\frac{1}{2}\langle w_k^2B_D^0(s)s,\tau\rangle_{B(2k)}.$$

Next we derive

$$(3.7)\qquad (d_B\delta_T\omega, w_k^2\omega)_x=\sum_\alpha(d_B\delta_T\omega)(e_\alpha)(w_k^2\omega)(e_\alpha)$$
$$=-(div_D v)_x-2(w_k\delta_T\omega)_x(dw_k,\omega)+(w_k\delta_T\omega, w_k\delta_T\omega)_x,$$

where $v\in\Gamma(Q)$ is the $g_Q$-dual of the 1-form

$$(3.8)\qquad \eta(\cdot):=(w_k^2\delta_T\omega)\omega(\cdot)\in\Gamma(Q^*)$$

Hence applying (3.2) imlies

$$(3.9)\qquad \langle d_B\delta_T\omega, w_k^2\omega\rangle_{B(2k)}=\langle v,\tau\rangle_{B(2k)}+\|w_k\delta_T\omega\|^2{}_{B(2k)}-2\langle w_k(\delta_T\omega),(dw_k,\omega)\rangle_{B(2k)},$$

and

$$(3.10)\qquad \langle v,\tau\rangle_{B(2k)}=-\langle w_k^2(div_D s)s,\tau\rangle_{B(2k)}.$$

It follows that

$$0=\langle 2\delta\delta^*\omega+(\lambda-1)d_B\delta_T\omega, w_k^2\omega\rangle_{B(2k)}$$

$$(3.11) \qquad = 2\|w_k\delta^*\omega\|^2_{B(2k)} + (\lambda-1)\|w_k\delta_T\omega\|^2_{B(2k)} + \langle w_k^2 B_D^{1-\lambda}(s)s,\tau\rangle_{B(2k)}$$
$$+2\langle w_k\delta^*\omega, dw_k\otimes\omega+\omega\otimes dw_k\rangle_{B(2k)} - 2(\lambda-1)\langle w_k(\delta_T\omega),(dw_k,\omega)\rangle_{B(2k)}.$$

On the other hand,

$$(3.12) \qquad \langle w_k\delta^*\omega, dw_k\otimes\omega+\omega\otimes dw_k\rangle_{B(2k)} \geq -2\|w_k\delta^*\omega\|_{B(2k)}\|dw_k\otimes\omega\|_{B(2k)}$$
$$\geq -\frac{1}{2}\|w_k\delta^*\omega\|^2_{B(2k)} - 2\|dw_k\otimes\omega\|^2_{B(2k)}$$
$$\geq -\frac{1}{2}\|w_k\delta^*\omega\|^2_{B(2k)} - \frac{2C^2}{k^2}\|\omega\|^2_{B(2k)},$$

and similarly

$$(3.13) \qquad \langle w_k(\delta_T\omega),(dw_k,\omega)\rangle_{B(2k)} \geq -\frac{1}{2}\|w_k\delta^*\omega\|^2_{B(2k)} - \frac{2C^2}{k^2}\|\omega\|^2_{B(2k)}.$$

Therefore we conclude that for $\omega$ the $g_Q$-dual of a $L^2$ $\lambda$-automorphism $s$

$$(3.14) \qquad (\lambda+1)\frac{2C^2}{k^2}\|\omega\|^2_{B(2k)} \geq \|w_k\delta^*\omega\|^2_{B(2k)} + \frac{1}{2}(\lambda-1)\|w_k\delta^*\omega\|^2_{B(2k)}$$
$$+\langle w_k^2 B_D^{1-\lambda}(s)s,\tau\rangle_{B(2k)}.$$

An important feature of $B_D^\lambda(s)$ is the following

**Proposition 3.1 ([9]).** Let $s\in\overline{V}(\mathscr{F})$ and $\omega$ be the $g_Q$-dual of s.

(i) $s$ is transversal Killing if and only if $B_D^0(s)=0$, or equivalently $\delta^*\omega=0$,

(ii) $s$ is transversal conformal if and only if $B_D^{2/q}(s)=0$, or equivalently $\delta^*\omega = -\frac{1}{q}(\delta_T\omega)g_Q$.

(Case $C$) We assume that $s$ satisfies

$$div_D s = 0 \text{ and } \langle B_D^0(s)s,\tau\rangle \geq 0.$$

Then by letting $k\to\infty$, (3.14) becomes

$$0\geq \|\delta^*\omega\|^2 + \langle B_D^0(s)s,\tau\rangle,$$

which yields that $\delta^*\omega=0$. Thus by Proposition 3.1 $s$ is a transversal Killing field.

(Case $D$) We assume that $s$ satisfies

$$\lambda = 1-\frac{2}{q} \text{ and } \langle B_D^{2/q}(s)s,\tau\rangle \geq 0.$$

Then (3.14) reduces to

$$(1-\frac{2}{q})\frac{2C^2}{k^2}\|\omega\|^2{}_{B(2k)} \geq \|w_k\delta^*\omega\|^2{}_{B(2k)} - \frac{1}{q}\|w_k\delta_T\omega\|^2{}_{B(2k)} + \langle w_k^2 B_D^{2/q}(s)s,\tau\rangle_{B(2k)}$$

$$= \|w_k^2(\delta^*\omega + \frac{1}{q}(\delta_T\omega)g_Q\|^2{}_{B(2k)} + \langle w_k^2 B_D^{2/q}(s)s,\tau\rangle_{B(2k)}.$$

Letting $k \to \infty$ implies

$$\delta^*\omega + \frac{1}{q}(\delta_T\omega)g_Q = 0\,.$$

Thus by Proposition 3.1 $s$ is a transversal conformal field.

**Remarks.**

**(i)** The $L^2$ hypothesis of a $\lambda$-automorphism in Main Theorem is necessary. For example, we consider a product manifold $M := \boldsymbol{R}^1 \times \boldsymbol{R}^3$ with a metric

$$g := f^2(dx_1)^2 + \sum_{\alpha=2}^{4}(dx_\alpha)^2,$$

where $(x_1, x_2, x_3, x_4)$ is a coordinate system of $M$, $(x_1)$ and $(x_2, x_3, x_4)$ coordinate systems of $\boldsymbol{R}^1$ and $\boldsymbol{R}^3$ respectively, and

$$f = f(x_2, x_3, x_4) = \exp(x_3 - x_4).$$

The family $\{\boldsymbol{R}^1 \times \{y\}\}_{y\in \boldsymbol{R}^3}$ defines a Riemannian foliation $\mathscr{F}$ on $M$ with a bundle-like metric $g$ of codimension 3. We consider a vector field $Y$ on $M$ defined by

$$Y := x_2\partial/\partial x_2 + \partial/\partial x_3 + \partial/\partial x_4.$$

Then $Y \in V(\mathscr{F})$, so that $Y$ indices a transversal infinitesimal automorphism $s := \pi(Y)$ of $\mathscr{F}$ . In this case the tension field $\tau$ is given by

$$\tau = -\pi(\partial/\partial x_3) + \pi(\partial/\partial x_4).$$

Then we have

$$\begin{aligned}
&D_{\sigma(\tau)}s = 0,\\
&D_{\sigma(\tau)}s = x_2\pi(\partial/\partial x_2),\\
&g_Q(s,\tau) = 0,\\
&div_D s = 1,
\end{aligned}$$

and hence

$$g_Q(B_D^{2/q}(s)s,\tau) = 0\,.$$

Furthermore, we find

$$\rho_D(s)=0, \qquad D_{\sigma(\tau)}s+\rho_D(s)+(1-\frac{2}{3})grad_D div_D s=0, \qquad \Delta s=0.$$

Therefore we see that $s$ is a $(1-\frac{2}{3})$-automorphism. However, we deduce

$$(\theta_Y g_Q)(\pi(\partial/\partial x_2),\pi(\partial/\partial x_2))=2,$$

$$(\theta_Y g_Q)(\pi(\partial/\partial x_3),\pi(\partial/\partial x_3))=0,$$

which means that $s$ is not a transversal conformal field of $\mathscr{F}$. Of course, $s$ is not a $L^2$ transversal infinitesimal automorphism.

(ii) We finally consider a $L^2$ $\lambda$-automorphism $s$ satisfying

$$d(div_D s)=0.$$

In this case, we see that

$$(div_D s)vol(B(2k))=\langle s,\tau\rangle_{B(2k)},$$

which follows that

$$\|\delta_T\omega\|^2{}_{B(2k)}=\langle (div_D s)s,\tau\rangle_{B(2k)}.$$

By observing that a similar computation yielding (3.14) gives rise to a general formula

$$(3.15) \qquad (2l_1+(\lambda-1)l_2)\frac{C^2}{k^2}\|\omega\|^2_{B(2k)}\geq(2-\frac{2}{l_1})\|w_k\delta^*\omega\|^2{}_{B(2k)}$$
$$+(\lambda-1)(1-\frac{1}{l_2})\|w_k\delta_T\omega\|^2_{B(2k)}+\langle w_k^2B_D^{1-\lambda}(s)s,\tau\rangle_{B(2k)},$$

for $l_1,l_2>0$, we deduce that

$$(2l_1+(\lambda-1)l_2)\frac{C^2}{k^2}\|\omega\|^2_{B(2k)}\geq(2-\frac{2}{l_1})\|w_k\delta^*\omega\|^2{}_{B(2k)}+\langle w_k^2B_D^{(1-\lambda)/l_2}(s)s,\tau\rangle_{B(2k)}.$$

Therefore we conclude a variant of $(C)$:

**(E).** *Under the same situation of Main Theorem, a $L^2$ $\lambda$-automorphism $s$ is a transversal Killing field if and only if*

$$d(div_D s)=0 \text{ and } \langle B_D^{\tilde\lambda}(s)s,\tau\rangle\geq 0$$

*for some $\tilde\lambda\in\boldsymbol{R}$, which is a constant linearly depending only on $\lambda$.*

As a corollary, $(E)$ means that a $L^2$ transversal affine field is a transversal Killing field if and only if $\langle B_D^{\tilde\lambda}(s)s,\tau\rangle\geq 0$ for some $\tilde\lambda\in\boldsymbol{R}$.

It should be noted that $s$ appeared in the previous example is a $\lambda$-automorphism satisfying

$$d(div_D s) = 0 \text{ and } \langle B_D^{(1-\lambda)/l}(s)s, \tau\rangle = 0$$

for all $l > 0$.

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