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# ON THE NEIGHBOURHOOD OF PASCU CLASS OF $\alpha$-CONVEX FUNCTIONS

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Let $E$ be the open unit disc in $\boldsymbol{C}$ and $H(E)$ be the class of all functions holomorphic in $E$. A be the class of all $f\in H(E)$ with the normalization $f(0)=0=f'(0)-1$.

**Definition 1.** Let $f\in A$ and $\dfrac{\alpha zf'(z)+(1-\alpha)f(z)}{z}\neq 0$ in $E$ for $0\leq\alpha\leq 1$. Then $f$ is said to be in $A_{\alpha}^{*}$- the class of $\alpha$-convex functions in the sense of Pascu [2] if

$$\mathrm{Re}\left\{\frac{\alpha z(zf'(z))'(z)+(1-\alpha)zf'(z)}{\alpha zf'(z)+(1-\alpha)f(z)}\right\}>0 \text{ in } E$$

or equivalently

$$\frac{\alpha z(zf'(z))'(z)+(1-\alpha)zf'(z)}{\alpha zf'(z)+(1-\alpha)f(z)}<\frac{1+z}{1-z}$$

where < denotes subordination; in otherwords $\alpha zf'(z)+(1-\alpha)f(z)$ is in $S^{*}$ the class of starlike univalent functions.

$A_{1}^{*}=K$- the class of convex univalent functions and $A_{0}^{*}=S^{*}$. In this note we will investigate $f\in A_{\alpha}^{*}$.

The notion of $\delta$-neighbourhood was first introduced by St. Ruscheweyh [3].

**Definition 2.** For $\delta\geq 0$, the $\delta$-neighbourhood of $f(z)=z+\sum_{k=2}^{\infty}a_{k}z^{k}\in A$ is defined by

$$N_{\delta}(f)=\{g(z)=z+\sum_{k=2}^{\infty}b_{k}z^{k}:\sum_{k=2}^{\infty}k|a_{k}-b_{k}|\leq\delta\}.$$

The convolution (Hadamard product) of two functions $f(z)=z+\sum_{k=2}^{\infty}a_{k}z^{k}$ and

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$g(z)=z+\sum_{k=2}^{\infty}b_k z^k$ is given by $(f*g)(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k$.

**Theorem 1.** *Let* $f(z)=z+\sum_{k=2}^{\infty}a_k z^k \in A_\alpha^*$ *and* $0<\alpha<1$. *Then* $N_\delta(f)\subset S^*$ *for* $\delta=\dfrac{G(\frac{1}{\alpha},3,(2+\frac{1}{\alpha});-1)}{1+\frac{1}{\alpha}}$ *where G(a,b,c;z) is the hypergeometric function. The value of* $\delta$ *is best possible.*

**Proof.** It is well known that a function $f\in A$ belongs to the class $S^*$ if and only if for all $\theta\in[0,2\pi]$ we have $\dfrac{zf'(z)}{f(z)}\neq\dfrac{1+e^{i\theta}}{1-e^{i\theta}}$; that is $\dfrac{(h_\theta * f)(z)}{z}\neq 0$ where

$$\begin{aligned} h_\theta(z) &= -\frac{1}{2e^{i\theta}}\left\{\frac{z(1-e^{i\theta})}{(1-z)^2}-\frac{z}{(1-z)}(1+e^{i\theta})\right\} \\ &= -\frac{(1-e^{i\theta})}{2e^{i\theta}}\left\{\frac{-2e^{i\theta}}{1-e^{i\theta}}z+(2-\frac{1+e^{i\theta}}{1-e^{i\theta}})z^2+\cdots\right\} \\ &= z+\sum_{n=2}^{\infty}\frac{(n+1)-(n-1)e^{-i\theta}}{2}z^n = z+\sum_{n=2}^{\infty}h_n z^n \end{aligned}$$

such that $|h_n|\le|\dfrac{(n+1)-(n-1)e^{-i\theta}}{2}|\le\dfrac{(n+1)+(n-1)}{2}=n$. Then, a sufficient condition for $N_\delta(f)\subset S^*$ to hold for some $f\in A$ is $|\dfrac{h_\theta(z)*f(z)}{z}|\ge\delta$, $z\in E$, $\theta\in[0,2\pi]$. For $\theta$=0, $h_\theta(z)*f(z)=f(z)$ and hence we can assume that $0<\theta<2\pi$. Let

$$H_\theta(z)=h_\theta(z)*f(z)=-\frac{(1-e^{i\theta})}{2e^{i\theta}}\left\{zf'(z)-\frac{1+e^{i\theta}}{1-e^{i\theta}}f(z)\right\}.$$

Then

$$zH_\theta'(z)=z(h_\theta(z)*f(z))'=-\frac{(1-e^{i\theta})}{2e^{i\theta}}\{z(zf'(z))'(z)-\frac{1+e^{i\theta}}{1-e^{i\theta}}zf'(z)\}$$

and

$$\begin{aligned} &\alpha zH_\theta'(z)+(1-\alpha)H_\theta(z) \\ &= -\frac{(1-e^{i\theta})}{2e^{i\theta}}\{\alpha z(zf'(z))'(z)+(1-\alpha)zf'(z)-\frac{(1+e^{i\theta})}{(1-e^{i\theta})}(\alpha zf'(z)+(1-\alpha)f(z))\}. \end{aligned}$$

Hence

$$\frac{\alpha zH_\theta'(z)+(1-\alpha)H_\theta(z)}{\alpha zf(z)+(1-\alpha)f(z)}=-\frac{(1-e^{i\theta})}{2e^{i\theta}}\left\{\frac{\alpha z(zf'(z))'(z)+(1-\alpha)zf'(z)}{\alpha zf'(z)+(1-\alpha)f(z)}-\frac{(1+e^{i\theta})}{(1-e^{i\theta})}\right\};$$

$$\frac{-2e^{i\theta}}{1-e^{i\theta}}\frac{\alpha zH_\theta'(z)+(1-\alpha)H_\theta(z)}{\alpha z f_\theta'(z)+(1-\alpha)f(z)}+\frac{1+e^{i\theta}}{1-e^{i\theta}}=\frac{\alpha z(zf'(z))'+(1-\alpha)zf'(z)}{\alpha zf'(z)+(1-\alpha)f(z)}<\frac{1+z}{1-z}$$

for $z\in E$, since $f\in A_\alpha^*$. In otherwords

$$\frac{\alpha zH_\theta'(z)+(1-\alpha)H_\theta(z)}{\alpha zf(z)+(1-\alpha)f(z)}<+\frac{(1-ze^{-i0})}{1-z}, \qquad z\in E,$$

that is $\operatorname{Re}\left\{(1+e^{i\theta})\frac{\alpha zH_\theta'(z)+(1-\alpha)H_\theta(z)}{\alpha zf'(z)+(1-\alpha)f(z)}\right\}>0$ in E.

Therefore we can choose a $\lambda\in\boldsymbol{R}$, $|\lambda|<\frac{\pi}{2}$ so that

$$\operatorname{Re}\left\{e^{i\lambda}\frac{\alpha zH_\theta'(z)+(1-\alpha)H_\theta(z)}{\alpha zf'(z)+(1-\alpha)f(z)}\right\}>0 \text{ in E}.$$

Set $\frac{\alpha zH_\theta'(z)+(1-\alpha)H_\theta(z)}{\alpha zf'(z)+(1-\alpha)f(z)}=p(z)$. Then $\operatorname{Re}(e^{i\lambda}p(z))>0$ and $\alpha zH_\theta'(z)+(1-\alpha)H_\theta(z)=p(z)F(z)$ where $F(z)=\alpha zf'(z)+(1-\alpha)f(z)\in S^*$ since $f\in A_\alpha^*$. Let $p(z)F(z) = zH(z)$. Then $H(z)=p(z)(\frac{g(z)}{z})^{1/\alpha}$ where $g(z)\in S^*$ for $0<\alpha<1$. Now $H(z)\in K(1,\frac{2}{\alpha}+1)$ and $\frac{H(z)}{\alpha}=H_\theta'(z)+(\frac{1}{\alpha}-1)\frac{H_\theta(z)}{z}$. Let $K_{\frac{1}{\alpha}}(z)=\sum_{n=0}^{\infty}\frac{1/\alpha}{n+\frac{1}{\alpha}}z^n$. Then

$$K_{\frac{1}{\alpha}}(z)*H(z)=K_{\frac{1}{\alpha}}(z)*\ (\alpha H_\theta'(z)+(1-\alpha)\frac{H_\theta(z)}{z}).$$

Choosing the branch of $(\frac{h(z)}{z})^{1/\alpha}$ to 1 when $z = 0$ and $h(z)=z(K_{\frac{1}{\alpha}}*H)^\alpha(z)$ we have $(\frac{h(z)}{z})^{\frac{1}{\alpha}}=\frac{H_\theta(z)}{z}$. Now *h(z)* is Bazilevic and hence is univalent for $0<\alpha\le 1$ [1]. In fact

$$(\frac{h(z)}{z})^{\frac{1}{\alpha}}=(K_{\frac{1}{\alpha}}*H)(z)=\frac{1}{\alpha z^{\frac{1}{\alpha}}}\int_0^z t^{\frac{1}{\alpha}-1}H(t)dt=\frac{1}{\alpha}\frac{1}{z^{\frac{1}{\alpha}}}\int_0^z\frac{p(t)g^\alpha(t)}{t}dt;$$

or

$$h(z)=\left\{\frac{1}{\alpha}\int_0^z\frac{p(t)g^{\frac{1}{\alpha}}(t)}{t}dt\right\}^\alpha$$

Now $\frac{h'(z)h^{\frac{1}{\alpha}-1}(z)}{z^{\frac{1}{\alpha}-1}}=H(z)=p(z)F(z)$.

Fix $0<r<1$ and choose $z_0, |z_0|=r$ so that $|h(z_0)|=\min_{|z|=r}|h(z)|$. Since *h(z)* is univalent, the pre-image $L$ of the segment from O to $h(z_0)$ is an arc inside $|z| \le r$. The image of L under the mapping $(h(z))^{1/\alpha}$ will in general consists of many line segments emanating from the origin such that for any $z \in L$.

$$\begin{aligned}|\frac{h(z)}{z}|^{\frac{1}{\alpha}} &\ge \frac{|h(z_0)|^{1/\alpha}}{r^{1/\alpha}} = \frac{1}{r^{1/\alpha}}\int_L |\frac{dh^{1/\alpha}(\zeta)}{d\zeta}||d\zeta| \\ &= \frac{1}{\alpha r^{1/\alpha}}\int_L |h^{\frac{1}{\alpha}-1}(\zeta)h'(\zeta)||d\zeta| = \frac{1}{\alpha r^{1/\alpha}}\int_L |\zeta^{\frac{1}{\alpha}-1}H(\zeta)||d\zeta| \\ &= \frac{1}{\alpha r^{1/\alpha}}\int_L |\zeta|^{\frac{1}{\alpha}-2}\alpha|\zeta H_\theta'(\zeta) + (\frac{1}{\alpha}-1)\mathrm{H}_\theta(\zeta)||d\zeta| \\ &= \frac{1}{\alpha r^{1/\alpha}}\int_L |\zeta|^{\frac{1}{\alpha}-2}|\alpha\zeta H_\theta'(\zeta) + (1-\alpha)\mathrm{H}_\theta(\zeta)||d\zeta| \\ &= \frac{1}{\alpha r^{1/\alpha}}\int_L |\zeta|^{\frac{1}{\alpha}-2}|p(\zeta)|\, F(\zeta)||d\zeta| \ge \frac{1}{\alpha r^{1/\alpha}}\int_0^r |\zeta|^{\frac{1}{\alpha}-1}\frac{1-|\zeta|}{(1+|\zeta|)^3}|d\zeta| \\ &= \frac{1}{\alpha r^{1/\alpha}}\int_0^r t^{\frac{1}{\alpha}-1}\frac{(1-t)}{(1+t)^3}dt.\end{aligned}$$

Hence

$$|\frac{h(z)}{z}|^{\frac{1}{\alpha}} \ge \frac{1}{\alpha r^{1/\alpha}}\int_0^r \frac{t^{\frac{1}{\alpha}-1}(1-t)}{(1+t)^3}dt.$$

By substituting $t = ru$ we obtain

$$\begin{aligned}|\frac{h(z)}{z}|^{\frac{1}{\alpha}} &\ge \frac{1}{\alpha r^{1/\alpha}}\int_0^1 r^{\frac{1}{\alpha}}u^{\frac{1}{\alpha}-1}\frac{(1-ur)}{(1+ur)^3}r\,du \\ &= \frac{1}{\alpha}\int_0^1 u^{\frac{1}{\alpha}-1}\frac{(1-ur)}{(1+ur)^3}du \ge \frac{1}{\alpha}\int_0^1 u^{\frac{1}{\alpha}-1}\frac{(1-u)}{(1+u)^3}du \\ &= \frac{\frac{1}{\alpha}G(\frac{1}{\alpha},3,\frac{1}{\alpha}+2,-1)}{\frac{1}{\alpha}(\frac{1}{\alpha}+1)} = \frac{G(\frac{1}{\alpha},3,\frac{1}{\alpha}+2,-1)}{\frac{1}{\alpha}+1}\end{aligned}$$

where $G(a, b, c; z)$ is the hypergeometric function.

Now $|\frac{H_\theta(z)}{z}| = |(\frac{h(z)}{z})^{\frac{1}{\alpha}}| \ge \frac{G(\frac{1}{\alpha},3,\frac{1}{\alpha}+2,-1)}{\frac{1}{\alpha}+1} = \delta.$

Let $f(z) = \frac{1}{\alpha z^{\frac{1}{\alpha}-1}}\int_0^z \frac{t^{\frac{1}{\alpha}-1}}{(1-t)^2}dt = \frac{1}{\alpha}\int_0^1 \frac{u^{\frac{1}{\alpha}-1}}{(1-uz)^2}du.$

Then

$$f'(z) = \frac{1}{\alpha}\int_0^1 \frac{u^{\frac{1}{\alpha}-1}}{(1-uz)^2}du + \frac{2}{\alpha}\int_0^1 \frac{u^{\frac{1}{\alpha}-1}zu}{(1-uz)^3}du,$$

$$f'(-1)=\frac{1}{\alpha}\int_0^1\frac{u^{\frac{1}{\alpha}-1}}{(1+u)^2}du-\frac{2}{\alpha}\int_0^1\frac{u^{\frac{1}{\alpha}}}{(1+u)^3}du$$
$$=\frac{1}{\alpha}\int_0^1 u^{\frac{1}{\alpha}-1}\frac{(1-u)}{(1+u)^3}du=\frac{G(\frac{1}{\alpha},3,\frac{1}{\alpha}+2,-1)}{(\frac{1}{\alpha}+1)}=\delta.$$

Let $g(z)=f(z)+\frac{\delta}{2}z^2$ where $f(z)=\frac{1}{\alpha z^{\frac{1}{\alpha}-1}}\int_0^z\frac{t^{\frac{1}{\alpha}-1}}{(1-t)^2}dt$. Then $f(z)\in A_\alpha^*$ and $g'(z)=f'(z)+\delta z, g'(-1)=f'(-1)-\delta=0$. Thus the value of $\delta$ is best possible.

When $\alpha=1$, we get $\delta=\frac{1}{4}$ thus getting $N_{\frac{1}{4}}(f)\subset S^*$ for $f\,\varepsilon\,K$ a result due to St. Ruscheweyh [3].

## References


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