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EXISTENCE AND STRONG RELAXATION THEOREMS FOR NONLINEAR EVOLUTION INCLUSIONS[1]

By

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Abstract. In this paper we study nonlinear evolution inclusions defined on a Gelfand triple of spaces. First we prove an existence and compactness result for the set of solutions of the "convex" problems. Then we look at extremal solutions and show that under reasonable hypotheses such solutions exist. Moreover if the orientor field (multivalued perturbation term) is h-Lipschitz in the state-variable, we show that the set of extremal solutions is dense in the solution set of the convexified problem ("strong relaxation theorem"). We also show that the solution set is compact in $C(T, H)$ if and only if the orientor field is convex-valued. Finally we present two examples of parabolic distributed parameter systems which illustrate the applicability of our abstract results.

1. Introduction

One of the central results in the theory of evolution inclusions (i.e. evolution equations with multivalued terms) is the "relaxation theorem". It says that if the orientor field (i.e. multivalued perturbation term) is h-Lipschitz in the state variable, then the solution set of the original problem is dense in that of the convexified problem (i.e. the system obtained by replacing the orientor field by its closed convex hull). Such a result is important in control theory in connection with the "bang-bang principle". Roughly speaking it tells us that we can produce a control system with essentially the same reachable sets, by economizing in the set of admissible controls. In this paper we prove a strong version of the relaxation theorem, in which the approximating trajectories are "extremal solutions" (i.e. solutions moving through the extreme points of the orientor field). First we prove a general existence result for the convexified problem under very general conditions on the orientor field. Then we establish the existence of extremal trajectories, under the stronger hypothesis that the orientor field is h-continuous in x. Subsequently by strengthening the hypothesis further to h-Lipschitzness, we show that the extremal trajectories are

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2Revised version.

dense in those of the convexified problem. A two-dimensional counterexample due to Plis [11], shows that simple h-continuity in the x-variable is not enough to guarantee the validity of the relaxation theorem. Finally we show that the solution set is compact in $C(T, H)$ if and only if the orientor field is convex-valued. In the last section we present examples of parabolic distributed parameter systems.

2. Preliminaries

Let (Ω,Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X)=\{A\subseteq X:\text{nonempty, closed, (convex)}\}$$
$$\text{and } P_{(\omega)k(c)}=\{A\subseteq X:\text{nonempty, (weakly-) compact, (convex)}\}.$$

A multifunction (set-valued function) $F:\Omega\to P_f(X)$ is said to be measurable if for every $x\in X, \omega\to d(x,F(\omega))=\inf\{\|x-y\|:y\in F(\omega)\}$ is measurable. Now let $\mu(\cdot)$ be a σ-finite measure on (Ω,Σ). By S_F^p, $1\leq p\leq\infty$, we denote the set of selectors of $F(\cdot)$ which belong in the Lebesgue-Bochner space $L^p(\Omega,X)$, i.e. $S_F^p=\{f\in L^p(\Omega,X): f(\omega)\in F(\omega)\ \mu-a.e.\}$. In general this set may be empty. However, if $F:\Omega\to 2^X\setminus\{\phi\}$ satisfies $GrF=\{(\omega,x)\in\Omega\times X: x\in F(\omega)\}\in\Sigma\times B(X)$ with $B(X)$ being the Borel σ-field of X (i.e. $F(\cdot)$ is graph measurable), then $S_F^p\neq\phi$ if and only if $\omega\to\inf\{\|x\|:x\in F(\omega)\}\in L^p(\Omega)$. Note that for $P_f(X)$-valued multifunctions measurability implies graph measurability, while the converse is true if Σ is μ-complete. Also remark that S_F^p is decomposable, i.e. for every $(A,f_1,f_2)\in\Sigma\times S_F^p\times S_F^p$ we have $x_Af_1+x_{A^c}f_2\in S_F^p$, with x_A being the characteristic function of a set $A\in\Sigma$.

On $P_f(X)$ we can define a generalized metric, known in the literature as the "Hausdorff metric" by setting, for $A,C\in P_f(X)$, $h(A,C)=\max\{\sup_{a\in A} d(a,C), \sup_{c\in C} d(c,A)\}$. It is well-known that $(P_f(X),h)$ is complete. A multifunction $F:X\to P_f(X)$ is said to be h-continuous (resp. h-Lipschitz), if it is continuous (resp. Lipschitz) from X into $(P_f(X),h)$.

Let Y, Z be Hausdorff topological spaces and $G:Y\to 2^Z\setminus\{\phi\}$. We say that $G(\cdot)$ is upper-semicontinuous (usc) (resp. closed), if for every $C\subseteq Z$ closed $G^-(C)=\{y\in Y:G(y)\cap C\neq\phi\}$ is closed (resp. $GrG=\{(y,z)\in Y\times Z:z\in G(y)\}$ is closed). For a $P_f(Z)$-valued multifunction upper-semicontinuity implies closedness, while the converse is true if $\overline{G(Y)}$ is compact in Z. We say that $G(\cdot)$ is lower-semicontinuous (lsc), if for every $C\subseteq Z$ closed $G^+(C)=\{y\in Y: G(y)\subseteq C\}$ is closed. For details we refer to DeBlasi-Myjak [2] and Klein-Thompson [8].

Now let H be a separable Hilbert space with norm denoted by $|\cdot|$. Let X be a separable reflexive Banach space which embeds continuously and densely into H. Identifying H with its dual (pivot space), we have $X \to H \to X^*$ with all embeddings being continuous and dense. Such a triple of spaces is known in the literature as "evolution triple" or "Gelfand triple". We will also assume that the embeddings are compact. This is often the case in applications where evolution triples are generated by Sobolev spaces. By $\|\cdot\|$ (resp. $\|\cdot\|_*$) we will denote the norm of X (resp. of X^*). Also by $\langle\cdot,\cdot\rangle$ we will denote the duality brackets for the pair (X, X^*) and by $(\cdot,\cdot)$ the inner-product of H. The two are compatible in the sense that $(\cdot,\cdot) = \langle\cdot,\cdot\rangle|_{X\times H}$. Let $T = [0, b]$ and $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$ and define $W_{pq}(T) = \{x \in L^p(T,X) : \dot{x} \in L^q(T,X^*)\}$ The derivative involved in this definition is understood in the sense of vector-valued distributions. Equipped with the norm

$$\|x\|_{W_{pq}(T)} = (\|x\|^2_{L^p(T,X)} + \|\dot{x}\|_{L^q(T,X^*)})^{\frac{1}{2}},$$

$W_{pq}(T)$ becomes a separable reflexive Banach space. Moreover $W_{pq}(T)$ embeds continuously into $C(T, H)$ and since we have assumed that $X \to H$ compactly, we have that $W_{pq}(T)$ embeds compactly in $L^p(T,H)$. For details we refer to Zeidler [17].

A map $A : X \to X^*$ is said to be hemicontinuous, if for every $x, y, z \in X$ the function $t \to \langle A(x+ty), z\rangle$ is continuous from $[0, 1]$ into $\boldsymbol{R}$. We say that $A(\cdot)$ is monotone if for every $x, y \in X$ $\langle A(x) - A(y), x - y\rangle \geq 0$.

3. Convex existence theorem

Let $T = [0, b]$ and let (X, H, X^*) be an evolution triple of spaces with all embeddings assumed to be compact. We consider the following multivalued Cauchy problem:

$$\left\{\begin{matrix} \dot{x}(t) + A(t, x(t)) \in F(t, x(t)) \;\; a.e. \\ x(0) = x_0 \end{matrix}\right\}$$

By a solution of (1) we understand a function $x \in W_{pq}(T)$ such that $\dot{x}(t) + A(t, x(t)) = f(t)$ *a.e* with $f \in S^q_F(\cdot, x(\cdot))$ and $x(0) = x_0$. Since $W_{pq}(T)$ embeds into $C(T, H)$, $x(0)$ makes sense. We will denote the solution set of (1) by $P_c(x_0)$. In the next theorem we establish the nonemptiness and compactness of $P_c(x_0)$ in $C(T, H)$. For this we need the following hypotheses.

<u>$H(A)$</u>: $A : T \times X \to X^*$ is an operator such that

(i) $t \to A(t,x)$ is measurable, (ii) $x \to A(t,x)$ is hemicontinuous monotone for every t, (iii) $\|A(t,x)\|_* \leq a_1(t)+c_1\|x\|^{p-1}$ $a.e$ for every x with $a_1 \in L^q(T), c_1>0, p\geq 2$ $\frac{1}{p}+\frac{1}{q}=1$, (iv) $\langle A(t,x),x\rangle \geq c[x]^p a.e$ with $c>0$ and $[\cdot]$ a quasinorm on X such that $[x]+\lambda|x| \geq \gamma\|x\|$ for some $\lambda,\gamma>0$.

$\underline{H(F)}: F: T\times H \to P_{fc}(H)$ is a multifunction such that
(i) $t\to F(t,x)$ is measurable, (ii) $GrF(t,.)$ is sequentially closed in $H\times H_\omega$ (here H_ω denotes the Hilbert space H furnished with the weak topology), (iii) $|F(t,x)| = \sup\{|v|: v\in F(t,x)\} \leq a_2(t)+c_2|x|$ *a.e.* with $a_2\in L^2(T), c_2>0\}$.

Theorem 1. *If hypotheses $H(A)$, $H(F)$ hold and $x_0\in H$ then $P_c(x_0)$ $\subseteq C(T,H)$ is nonempty and compact.*

Proof. We start by deriving some a priori bounds for the elements in $P_c(x_0)$. So let $x\in P_c(x_0)$. By definition we have $\dot{x}(t)+A(t,x(t))=f(t)$ *a.e.*, $x(0)=x_0$ with $f\in S^2_{F(\cdot,x(\cdot))}$. Multiply this equation with x and recall that there exists $\beta_1>0$ such that $\|x\|_*\leq \beta_1|x|$, to get:

$$\langle \dot{x}(t),x(t)\rangle + \langle A(t,x(t)),x(t)\rangle = \langle f(t),x(t)\rangle \quad a.e.$$

$\Rightarrow$

$$\frac{1}{2}\frac{1}{dt}|x(t)|^2 + c[x(t)]^p \leq \|f(t)\|_*\|x(t)\| \leq \hat{\beta}_1|f(t)|\cdot[x(t)] + \lambda\hat{\beta}_1|f(t)|\cdot|x(t)| \quad a.e.$$

with $\hat{\beta}_1=\dfrac{\beta_1}{\gamma}$. From Cauchy's inequality with $\varepsilon>0$ we get

$$\hat{\beta}_1|f(t)|[x(t)] \leq \hat{\beta}_1\frac{\varepsilon^p}{p}[x(t)]^p + \hat{\beta}_1\frac{1}{\varepsilon^q q}|f(t)| \quad a.e..$$

In addition from classical Cauchy's inequality we have

$$\lambda\hat{\beta}_1|f(t)||x(t)| \leq \frac{1}{2}\lambda\hat{\beta}_1|f(t)|^2 + \frac{1}{2}\lambda\hat{\beta}_1|x(t)|^2.$$

So after integration over $[0,t]$, we get

$$\frac{1}{2}|x(t)|^2 + c\int_0^t[x(s)]^p ds \leq \frac{1}{2}|x_0|^2 + \hat{\beta}_1\frac{\varepsilon^p}{p}\int_0^t[x(s)]^p ds + \hat{\beta}_1\frac{1}{\varepsilon^q q}\int_0^t|f(s)|^q ds$$

$$+\frac{1}{2}\lambda\hat{\beta}_1\int_0^t|f(s)|^2 ds + \frac{1}{2}\lambda\hat{\beta}_1\int_0^t|x(s)|^2 ds.$$

Let $\varepsilon>0$ be such that $\hat{\beta}_1\dfrac{\varepsilon^p}{p}=\dfrac{c}{2}$. Also note that since $2\leq p$ (hence $1<q\leq 2$), we have

$$|f(s)|^q \le [\max(1, |f(t)|)]^2 \le (1+|f(t)|)^2 \le 2+2|f(t)|^2.$$

So $\int_0^t |f(s)|^q ds \le 2b + 2\int_0^t |f(s)|^2 ds$. Hence we deduce that there exists $\gamma_1 > 0$ for which we have

$$\frac{1}{2}|x(t)|^2 + \frac{c}{2}\int_0^t [x(s)]^p ds \le \frac{1}{2}|x_0|^2 + \gamma_1 \int_0^t |f(s)|^2 ds + \gamma_1 \int_0^t |x(s)|^2 ds.$$

Because of hypothesis $H(F)$ (iii) we have $|f(s)|^2 \le 2a_2(s)^2 + 2c_2|x(s)|^2$ *a.e.*. So we can find some $\gamma_2 > 0$ such that

$$\frac{1}{2}|x(t)|^2 + \frac{c}{2}\int_0^t [x(s)]^p ds \le \gamma_2 + \gamma_2 \int_0^t |x(s)|^2 ds. \tag{2}$$

Through Gronwall's lemma, we see that there exists $M_1 > 0$ such that

$$|x(t)| \le M_1 \text{ for every } t \in T \text{ and every } x \in P_c(x_0). \tag{3}$$

From (2) and (3) above we get the existence of $M_2 > 0$ such that

$$\int_0^b [x(s)]^p ds \le M_2 \text{ for every } x \in P_c(x_0). \tag{4}$$

From (3) and (4) above and since $\gamma\|x\| \le [x] + \lambda|x|$ and from $H(A)$ (iii) we get an $M_3 > 0$ such that

$$\|x\|_{L^p(T,X)} \le M_3 \text{ and } \|x\|_{L^q(T,X^*)} \le M_3 \text{ for every } x \in P_c(x_0). \tag{5}$$

Thus without any loss of generality we may assume that $|F(t,x)| \le \alpha_2(t) + c_2 M_1 = \psi(t)$ *a.e.* with $\psi \in L^2(T)$. Otherwise replace $F(t,x)$ by $F(t, r_{M_1}(x))$ where $r_{M_1}(\cdot)$ is the M_1-radial retraction on H. Let $K = \{g \in L^2(T,H) : |g(t)| \le \psi(t)\ a.e.\}$. Furnished with the relative weak-$L^2(T,H)$ topology K becomes a compact metrizable space. In what follows this is the topology considered on K. Let $R: K \to P_{fc}(K)$ be the multifunction defined by $R(g) = S^2_{F(\cdot,p(g)(\cdot))}$, where $p: L^2(T,H) \to C(T,H)$ is the map which to each $g \in L^2(T,H)$ assigns the unique solution of $\dot{x}(t) + A(t,x(t)) = g(t)$ *a.e.*, $x(0) = x_0$. We claim that GrR is sequentially closed in $K \times K$. To this end let $[g_n, f_n] \in GrR$ $n \ge 1$ and assume that $[g_n, f_n] \to [g, f]$ in $K \times K$. From the same a priori estimation as in the beginning of this proof, we get that $\{p(g_n)\}_{n\ge 1}$ is bounded in $W_{pq}(T)$, thus relative compact in $L^p(T,H)$. So by passing to a subsequence if necessary we may assume that $p(g_n)(t) \to p(g)(t)$ *a.e.* in H. Using theorem 3.1 of [9] and hypothesis H(F)(ii) we get $f(t) \in F(t, p(g)(t))$ *a.e.*. So $[g,f] \in GrR$ and hence GrR is sequentially closed in $K \times K$. Since K (with the relative weak-$L^2(T,H)$ topology) is compact metrizable, we can apply the Kakutani-Ky Fan fixed point theorem (see for example Klein-Thompson [8]) and get $f \in R(f)$. Evidently $p(f) \in P_c(x_0) \ne \varnothing$.

Now we will show that $P_c(x_0)$ is compact in $C(T,H)$. To this end let

$\{x_n\}_{n\geq 1} \subseteq S_c(x_0)$. We know that $\{x_n\}_{n\geq 1}$ is bounded in $W_{pq}(T)$, hence relatively compact in $L^2(T,H)$. So we may assume that $x_n \to x$ in $L^2(T,H)$ and $x_n \xrightarrow{\omega} x$ in $W_{pq}(T)$, Note that $x_n \in W_0 = \{y \in W_{pq}(T): y(0) = x_0\}$ and W_0 is convex closed (hence weakly closed). So $x \in W_0$ and thus $x(0) = x_0$. Also let $f_n \in S^2_{F(\cdot,x_n(\cdot))}$ such that $x_n = p(f_n)$, $n \geq 1$. We may assume that $f_n \xrightarrow{\omega} f$ in $L^2(T,H)$. Let $\hat{A}: L^p(T,X) \to L^q(T,X^*)$ be the Nemitsky operator corresponding to $A(t,x)$, *i.e.* $\hat{A}(x)(\cdot) = A(\cdot,x(\cdot))$. Moreover let $((\cdot,\cdot))$ be the duality brackets for the pair $(L^p(T,X), L^q(T,X^*))$. We have

$$((x_n, x_n - x)) + ((\hat{A}(x_n), x_n - x)) = ((f_n, x_n - x)). \tag{6}$$

From the integration by parts formula for functions in $W_{pq}(T)$ (see Zeidler [17], p. 422) we have

$$((x_n, x_n - x)) = \frac{1}{2}|x_n(b) - x(b)|^2 + ((x, x_n - x)) \tag{7}$$

Replacing (7) in (6) we get that

$$((\hat{A}(x_n), x_n - x)) \leq ((f_n, x_n - x)) - ((x, x_n - x)).$$

Also $((f_n, x_n - x)) = (f_n, x_n - x)_{L^2(T,H)}$ (here by $(\cdot,\cdot)_{L^2(T,H)}$ we denote the inner product in the Hilbert space $L^2(T,H)$). So we get $\overline{\lim}((\hat{A}(x_n), x_n - x)) \leq 0$. Now let $\xi_n(t) = \langle A(t,x_n(t)) - A(t,x(t)), x_n(t) - x(t)\rangle \geq 0$. From Fatou's lemma we have

$$0 \leq \int_0^b \underline{\lim}\,\xi_n(t)dt \leq \overline{\lim}\int_0^b \xi_n(t)dt = \overline{\lim}[((\hat{A}(x_n), x_n - x)) - ((\hat{A}(x), x_n - x))] \leq 0$$

hence $\xi_n \to 0$ in $L^1(T)$ and so we may assume that $\xi_n(t) \to 0$ a.e. . We have

$$\xi_n(t) \geq c[[x_n(t)]^p + [x(t)]^p - \|x(t)\|[a_1(t) + c_1\|x(t)\|^{p-1}] - \|x(t)\|[a_1(t) + c_1\|x_n(t)\|^{p-1}] \geq$$
$$c_3[\|x_n(t)\|^p + \|x(t)\|^p] - c_4 - \|x_n(t)\|[a_1(t) + c_1\|x(t)\|^{p-1}] - \|x(t)\|[a_1(t) + c_1\|x_n(t)\|^{p-1}] \ a.e.$$

for some $c_3, c_4 > 0$. From the above inequality we deduce that $\{x_n\}_{n\geq 1}$ is bounded in $L^\infty(T,H)$. So we can apply corollary 4, p. 85 of Simon [14] and conclude that $\{x_n\}_{n\geq 1}$ is relatively compact in $C(T,H)$. Thus $x_n \to x$ in $C(T,H)$. and $x = p(f)$, $f \in S^2_{F(\cdot,x(\cdot))}$, i.e. $x \in P_c(x_0)$. So $P_c(x_0)$ is compact in $C(T,H)$. Q.E.D.

A careful reading of the above proof reveals that the following result holds:

Corollary 2. *If hypothesis $H(A)$ holds and $x_0 \in H$, then the solution map $p: L^2(T,H) \to C(T,H)$ which to each $g \in L^2(T,H)$ assigns the unique solution of $x(t) + A(t,x(t)) = g(t)$ a.e., $x(0) = x_0$, is compact.*

4. Extremal solutions

In conjunction with (1) we also consider the following problem:

$$\left\{ \begin{array}{c} x(t) + A(t, x(t) \in extF(t, x(t) \ \ a.e \\ x(0) = x_0 \end{array} \right\} \tag{8}$$

Here by $extF(t, x)$ we denote the extreme points of the orientor field $F(t, x)$. By $P_e(x_0)$ we will denote the solution set of (8). In this section we establish the nonemptiness of $P_e(x_0)$.

In what follows let $L^1_\omega(T, H)$ denote the Lebesgue-Bochner space $L^1(T, H)$ equipped with norm $\|x\|_\omega = \sup[\|\int_s^t x(\tau) d\tau \| : 0 \le s \le t \le b]$ ("the weak norm"). We will need the following simple auxiliary result.

Lemma 3. $\{f_n, f\}_{n\geq 1} \subseteq L^2(T, H)$, $f_n \xrightarrow{\|\cdot\|_\omega} f$ and $\sup_{n\geq 1} \|f_n\|_{L^2(T,H)} < \infty$, then $f_n \xrightarrow{w} f$ in $L^2(T, H)$.

Proof. Let $s(t) = \sum_{k=1}^{N} x_{(t_{k-1}, t_k)}(t)_{V_k} \in L^2(T, H)$ be a step function. Then we have $|(f_n - f, s)_{L^2(T,H)}| \le \sum_{k=1}^{N} \|\int_{t_{k-1}}^{t_k} (f_n(s) - f(s)) ds\| \cdot \|\upsilon_k\| \le \|f_n - f\|_\omega \cdot \sum_{k=1}^{N} \|\upsilon_k\| \to 0$ as $n \to \infty$.

Since step functions are dense in $L^2(T, H)$, we conclude that $f_n \xrightarrow{w} f$ in . $L^2(T, H)$. Q.E.D.

To establish the existence of solutions for (8) we will need the following stronger hypothesis on the orientor field $F(t, x)$.

$\underline{H(F)_1}$: $F: T \times H \to P_{\omega kc}(H)$ is a multifunction such that

(i) $t \to F(t,x)$ is measurable, (ii) $x \to F(t,x)$ is h-continuous, (iii) $|F(t,x)| \le a_2(t) + c_2|x|$ *a.e.* with $a_2 \in L^2(T), c_2 > 0$.

Theorem 4. *If hypotheses $H(A)$, $H(F)_1$ hold and $x_0 \in H$, then $P_e(x_0)$ is nonempty.*

Proof. As in the proof of Theorem 1, we may assume without any loss of generality that $|F(t,x)| \le \psi(t)$ *a.e.* with $\psi \in L^2(T)$. Again set $K = \{g \in L^2(T,H) : |g(t)| \le \psi(t) \ \ a.e.\}$ and let $V_0 = \overline{p(K)}^{C(T,H)}$. From corollary 2 we know that V_0 is compact in $C(T, H)$. Hence so is $V = \overline{\text{conv}} V_0$. Let $R: V \to P_{\omega kc}(L^1(T,H))$ be defined by $R(y) = S^1_{F(\cdot, y(\cdot))}$. Invoking theorem 1.1 of Tolstonogov [15] we get $r: V \to L^1_\omega(T,H)$ a continuous map such that

$r(y) \in extR(y) = ext\, S^1_{F(\cdot,y(\cdot))}$ for every $y \in V$. From Benamara [1] we know that $ext\, S^1_{F(\cdot,y(\cdot))} = S^1_{extF(\cdot,y(\cdot))}$. Let $u = p \circ r : V \to V$. Using Lemma 3 and Corollary 2 we see that $u(\cdot)$ is compact. Thus via Schauder's fixed point theorem we get $x=u(x)$. Evidently $x \in P_e(x_0) \neq \varnothing$. Q.E.D.

5. Strong relaxation theorem

In this section we establish the density of $P_e(x_0)$ in $P_c(x_0)$ for the $C(T, H)$-norm and in addition that $P_e(x_0)$ is a G_δ −subset of $P_e(x_0)$. To prove this last property we will have to use the Choquet function associated with the orientor field $F(t, x)$. More specifically let $\{z_k\}_{k\geq 1}$ be a sequence which is dense in the unit sphere of H. Following DeBlasi-Pianigiani [3], [4] we define γ_F: $T \times H \times H \to \mathbf{R} \cup \{+\infty\}$ by

$$\gamma_F(t,x,v) = \begin{cases} \sum_{k\geq 1} 2^{-k}(z_k, v)^2 & \text{if } v \in F(t,x) \\ +\infty & \text{otherwise} \end{cases}$$

Let $Aff(H)$ = {set of all continuous affine functions $a : H \to \mathbf{R}\}$. Let $\hat{\gamma}_F(t,x,v) = \inf\{a(v) : a \in aff(H)$ and $a(z) \geq \gamma_F(t,x,v)$ for every $z \in F(t,x)\}$ (as usual we adopt the convention that $\inf\varnothing = -\infty$). Then the Choquet function $\delta_F : T \times H \times H \to \mathbf{R} \cup \{-\infty\}$ is defined by $\delta_F(t,x,v) = \hat{\gamma}_F(t,x,v) - \gamma_F(t,x,v)$. The next proposition summarizes the properties of $\delta_F(\cdot,\cdot,\cdot)$ (see DeBlasi-Pianigiani [3], [4] and Papageorgiou [10]).

Proposition 5. *If hypothesis $H(F)_1$ holds, then (a) $(t,x,v) \to \delta_F(t,x,v)$ is measurable, (b) $(x,v) \to \delta_F(t,x,v)$ is usc, (c) $v \to \delta_F(t,x,v)$ is concave and strictly concave on $F(t,x,)$, (d) $0 \leq \delta_F(t,x,v) \leq 4a_2(t)^2 + 4c_2^2|x|^2$ for every $v \in F(t,x)$, (e) $\delta_F(t,x,v) = 0$ if and only if $v \in extF(t,x)$.*

As we already mentioned in the introduction, simple h-continuity of $F(\cdot,\cdot)$ does not suffice to have the desired relaxation theorem. So we impose the following stronger conditions on $F(t,x)$:

$\underline{H(F)_2}$: $F : T \times H \to P_{wkc}(H)$ is a multifunction such that

(i) $t \to F(t,x)$ is measurable, (ii) $h(F(t,x),F(t,y)) \leq k(t)|x-y|$ *a.e* for every x with $k \in L^1(T)$, (iii) $|F(t,x)| \leq a_2(t) + c_2|x|$ *a.e* with $a_2 \in L^2(T), c_2 > 0$.

Then our stronger relaxation theorem reads as follows:

Theorem 6. *If hypotheses $H(A), H(F)_2$ hold and $x_0 \in H$, then $P_e(x_0)$ is a dense G_δ-subset of $P_c(x_0) \subseteq C(T,H)$.*

Proof. Let $\lambda > 0$ and define

$$D_\lambda = \left\{ x \in S_c(x_0) : \int_0^b \delta_F(t, x(t), f(t))dt < \lambda, \text{ with } f \in S^2_{F(\cdot,x(\cdot))} \text{ such that } x = p(f) \right\}.$$

Our claim is that D_λ is open in $P_c(x_0)$. For this we will show that $P_c(x_0) \setminus D_\lambda$ is closed in $C(T,H)$. So let $\{x_n\}_{n\geq 1} \subseteq P_c(x_0) \setminus D_\lambda$ and suppose that $x_n \to x$ in $C(T,H)$. We have $x_n = p(f_n)$, $n \geq 1$ with $f_n \in S^2_{F(\cdot,x(\cdot))}$. We may assume that $f_n \xrightarrow{\omega} f$ in $L^2(T,H)$ and $f \in S^2_{F(\cdot,x(\cdot))}$. Moreover from corollary 2 we have $x_n \to x = p(f)$ in $C(T,H)$. Also using proposition 5 and the upper semicontinuity of the concave integral functional $(x,f) \to \int_0^b \delta_F(t,x(t),f(t))\,dt$ on $L^1(T,H) \times L^1(T,H)_\omega$, we get

$$\lambda \leq \overline{\lim} \int_0^b \delta_F(t, x_n(t), f_n(t))\,dt \leq \int_0^b \delta_F(t, x(t), f(t))\,dt \Rightarrow x = p(f) \in P_c(x_0) \setminus D_\lambda,$$

which shows that D_λ is open in $S_c(x_0)$.
Now let $\lambda_n \downarrow 0$ and set $D_n = D_{\lambda_n}, n \geq 1$. We claim that $P_e(x_0) = \bigcap_{n\geq 1} D_n$. Evidently from proposition 5 we have that $P_e(x_0) \subseteq \bigcap_{n\geq 1} D_n$. Then $\int_0^b \delta_F(t,x(t),f(t))\,dt = 0$, $x = p(f)$, $f \in S^2_{F(\cdot,x(\cdot))}$. So $f(t) \in extF(t,x(t))\ a.e.$ and thus $x \in P_e(x_0)$. Therefore finally $P_e(x_0) = \bigcap_{n\geq 1} D_n$ which proves that $P_e(x_0)$ is a G_δ-set in $P_c(x_0)$.

Next we will show that $P_e(x_0)$ is dense in $P_c(x_0)$. To this end let $x \in P_e(x_0)$. By definition $x = p(f)$ for some $f \in S^2_{F(\cdot,x(\cdot))}$. Let $V \in P_{kc}(C(T,H))$ be as in the proof of Theorem 4. Let $\theta = |V| = \sup\{\|\upsilon\|\}_{C(T,H)} : y \in V\}$ and define $R : V \to 2^{L^1(T,H)}$ by

$$R(y) = \{g \in S^1_{F(.,y(\cdot))} : |f(t) - g(t)| < \frac{\varepsilon}{4b\theta} + k(t)|x(t) - y(t)|\ \ a.e\}.$$

A straightforward application of Aumann's selection theorem (see for example Wagner [16], Theorem 5.10) reveals that $R(\cdot)$ has nonempty values which are decomposable. Moreover proposition 2.3 of Fryszkowski [5] tells us that $y \to R(y)$ is lsc.. Hence $y \to \overline{R(y)}$ is lsc with closed and decomposable values. Apply the selection theorem of Fryszkowski [5] to get $r_\varepsilon : V \to L^1(T,H)$ a continuous map such that $r_\varepsilon(y) \in \overline{R(y)}$ for every $y \in V$. In addition theorem 1.1 of Tolstonogov [15] gives us $\upsilon_\varepsilon : V \to L^1_\omega(T,H)$ continuous such that $\upsilon_\varepsilon(y) \in extR(y) = S^1_{extF(\cdot,y(\cdot))}$ and $\|r_\varepsilon(y) - \upsilon_\varepsilon(y)\|_\omega < \frac{\varepsilon}{2}$ for every $y \in V$.

Now let $\varepsilon_n = \frac{1}{n}$ and set $r_n = r_{\varepsilon_n}$, $\upsilon_n = \upsilon_{\varepsilon_n}$ $n \geq 1$ as above. Let $y_n \in V$ such that $(p \circ v_n)(y_n) = y_n$ (such an element exists by Schauder's fixed point theorem). By passing to a subsequence if necessary we may assume that $y_n \to y$ in $C(T, H)$. We have

$$\langle y_n(t)-x(t), y_n(t)-x(t)\rangle + \langle A(t,y_n(t))-A(t,x(t)), y_n(t)-x(t)\rangle$$
$$=\langle v_n(y_n)(t)-f(t), y_n(t)-x(t)\rangle \ a.e.$$
$$\Rightarrow \frac{1}{2}\frac{d}{dt}|y_n(t)-x(t)|^2 \le \langle v_n(y_n)(t)-f(t), y_n(t)-x(t)\rangle \ a.e.$$
$$\Rightarrow \frac{1}{2}|y_n(t)-x(t)|^2 \le \int_0^t (v_n(y_n)(s)-f(s), y_n(s)-x(s))ds$$
$$=\int_0^t (v_n(y_n)(s)-r_n(y_n)(s), y_n(s)-x(s))ds \int_0^t (r_n(y_n)(s)-f(s), y_n(s), -x(s))ds$$
$$\le \int_0^t (v_n(y_n)(s)-r_n(y_n)(s), y_n(s)-x(s)ds + \int_0^t k(s)|\, y_n(s)-x(s)\,|^2 ds + \frac{1}{2n}.$$

Note that $\|v_n(y_n)-r_n(y_n)\|_\omega \to 0$ and so by lemma 3 we have $v_n(y_n)-r_n(y_n)$ $\xrightarrow{\omega} 0$ in $L^2(T,H)$. Therefore we have

$$\int_0^t (v_n(y_n)(s)-r_n(y_n)(s), y_n(s)-x(s))ds \to 0, \quad \text{as } n\to\infty.$$

Thus in the limit as $n\to\infty$ we get

$$|y(t)-x(t)|^2 \le 2\int_0^t k(s)|y(s)-x(s)|^2 ds$$
$$\Rightarrow x=y \text{ (Gronwall's lemma).}$$

So $y_n \to x$ in $C(T, H)$ and since $y_n \in P_e(x_0)$ and $P_c(x_0)$ is compact (cf. Theorem 1) we conclude that $P_c(x_0)=\overline{P_e(x_0)}^{C(T,H)}$. Q.E.D.

6. Convexity vs compactness.

In this section we show that the solution set of (1) is compact in $C(T, H)$ if and only if the multivalued perturbation term $F(t, x)$ is convex-valued. This explains why in optimal control theory relaxable systems are the right systems to study.

We will need the following hypotheses:

$\underline{H(F)}_3 : F: T\times H \to P_f(H)$ is a multifunction such that
(i) $t\to F(t,x)$ is measurable, (ii) $h(F(t,x),F(t,y))\le k(t)|x-y|a.e.$ for every x,y with $k\in L^1(T)$, (iii) $|F(t,x)|\le a_2(t)+c_2|x|$ $a.e$ with $a_2\in L^2(T)$, $c_2>0$.

$\underline{H_0}$:For every $t_0\in[0,b)$ and for every $x_0\in H$ there exists $\delta>0$ such that $T_\delta=[t_0,t_0+\delta]\subseteq T$ for which the solution set of $x(t)+A(t,x(t))\in F(t,x(t))$ $a.e.$ on T_δ, $x(t_0)=x_0$ denoted by $P(x_0;T_\delta)$ is nonempty compact in $C(T_\delta,H)$ (the solution set of the evolution inclusion ith $F(t,x)$ replaced by $\overline{\text{conv}}F(t,x)$ will be denoted as before by $P_c(x_0;T_\delta)$.

We will need the following auxiliary result which can be found in Papageorgiou [6], proposition 3.1.

Lemma 7. *If V is a compact metric space, Y is a metric space, Z a Polish space and $F : V \times Y \to P_f(Z)$ is a multifunction such that*

(a) $(v, y) \to F(v, y)$ is measurable,

(b) $y \to F(v, y)$ is l.s.c.,

then for every $\varepsilon > 0$ there exists $V_\varepsilon \subseteq V$ compact with $\lambda(V_\varepsilon^c) < e$ such that $F|_{V_\varepsilon \times Y}$ is l.s.c..

This lemma provides a multivalued version of the well-known Scorza-Dragoni theorem for single-valued maps. Using this lemma we can now prove the following theorem.

Theorem 8. *If hypotheses $H(A)$, $H(F)_3$ and H_0 hold, then for every $[t,x] \in (T \setminus N) \times H$, $\lambda(N) = 0$, $F(t,x) \in P_{\omega kc}(H)$.*

Proof. Suppose not. We will establish the existence of a $(t_0, x_0) \in [0,b) \times H$ such that $P(x_0;T_\delta)$ is closed in $C(T, H)$ and $P(x_0;T_\delta) \neq P_c(x_0;T_\delta)$, which of course contradicts Theorem 6.

So let $T_1 \subseteq T$ be Lebesgue measurable with $\lambda(T_1) > 0$ ($\lambda(\cdot)$ being the Lebesgue measure) such that for every $t \in T_1$ we have there exists $x_t \in H$ for which $F(t, x_t)$ is not convex. From hypotheses $H(F)_3$ (i) and (ii) we have that $(t,x) \to F(t,x)$ is measurable. Via Rzezuchowski's result [13] we can find a multifunction $F_0 : T \times H \to P_f(H)$ such that (a) $F_0(t,x) \subseteq F(t,x)$ for every $(t,x) \in (T \setminus N) \times H$, $\lambda(N) = 0$; (b) If $\Delta \subseteq T$ is Lebesgue measurable and $x, y : \Delta \to H$ are measurable functions, then $y(t) \in F(t, x(t))$ *a.e.* on Δ implies $y(t) \in F_0(t, x(t))$ *a.e.* on Δ ; (c) for every $\varepsilon > 0$ there exists $C_\varepsilon \subseteq T$ closed subset with $\lambda(T \setminus C_\varepsilon) \leq \varepsilon$ such that $F_0 | C_\varepsilon \times H$ has a closed graph. Note that the nonemptiness of the values of F_0 follows from theorem 3.1 of Jarnik-Kurzweil [7]. Invoking Lemma 7 above with $e = \lambda(T_1) > 0$ and $V = C_{\varepsilon/2}$ (with $C_{\varepsilon/2}$ as above) we can produce $T_2 \subseteq C_{\varepsilon/2} \subseteq T$ closed with $\lambda(C_{\varepsilon/2} \setminus T_2) < \frac{\varepsilon}{2}$, hence $\lambda(T_1 \setminus T_2) < \varepsilon$ and $F|_{T_2 \times H}$ *l.s.c.*. Evidently $\lambda(T_2) > b - \lambda(T_1)$ and $\lambda(T_1 \cap T_2) > 0$. Let $t_0 \in T_1 \cap T_2$ $t_0 < b$ be a Lebesgue point (i.e. a point of density) for $T_1 \cap T_2$. Then there exists $x_0 \in H$ such that $F(t_0, x_0)$ is not convex. This means that we can find $y_0 \in \overline{\text{conv}} F(t_0, x_0) \setminus F(t_0, x_0)$. Use Michael's selection theorem (see Klein-Thompson [8]) to find $g : T_2 \times H \to H$ a continuous map such that $g(t,x) \in \overline{\text{conv}} F(t,x)$ for every $(t,x) \in T_2 \times H$ and $g(t_0, x_0) = y_0$. Define $\hat{F} : T \times H \to P_f(H)$ by setting $\hat{F}(t,x) = \overline{\text{conv}} F(t,x)$ if $(t,x) \in T_2^c \times H$ and $\hat{F}(t,x) = \{g(t,x)\}$ if $(t,x) \in T_2 \times H$. Clearly $\hat{F}(\cdot,\cdot)$ is measurable in (t,x) and l.s.c. in x. Then consider the following multivalued Cauchy problem

$$\left\{\begin{matrix} \dot{x}(t)+A(t,x(t))\in \hat{F}(t,x(t)) \ \ a.e. \ \ on \ \ T_0=[t_0,b] \\ x(t_0)=x_0 \end{matrix}\right\} \tag{9}$$

Problem (9) above has a solution $x \notin P(x_0;T_0)$, but $x \notin P(x_0;T_0)$ as we will now show. Suppose the contrary. Then $\dot{x}(t)+A(t,x(t))\in F(t,x(t))$ *a.e.* on T_0, $x(t_0) = x_0$. Then $\dot{x}(t)+A(t,x(t))\in F_0(t,x(t))$ *a.e.* on T_0 and so $g(t,x(t))\in F_0(t,x(t))$ *a.e.* on $T_2 \setminus N, \lambda(N)=0$, For any $\delta>0$, $[t_0,t_0+\delta]\cap T_2\setminus N$ has positive Lebesgue measure. We claim that there exists $\delta_1>0$ such that for $t\in[t_0,t_0+\delta_1]\cap T_2\setminus N$ we have $d(g(t,x(t)),F_0(t,x(t))>0$. If not there exists a sequence $t_n\to t_0$ such that $g(t_n,x(t_n))\in F_0(t_n,x(t_n))$ $n\geq 1$. But recall that $F_0|T_2\times H$ has a closed graph. So $g(t_0,x_0)\in F_0(t_0,x_0)$ and $d(g(t_0,x_0),F_0(t_0,x_0))=0$ a contradiction. Thus $x\notin P(x_0;T_0)$. Now if $\delta_2>0$ is as postulated by hypotheses H_0 and $\delta=\min[\delta_1,\delta_2]$, we have $\overline{P(x_0;T_\delta)}^{C(T,H)}\neq P_c(x_0;T_\delta)$ a contradiction to theorem 6.

Q.E.D.

7. Examples

Let $T=[0,b]$ and Z a bounded domain in $\boldsymbol{R}^N$ with smooth boundary Γ. Let $D_k=\dfrac{\partial}{\partial z_k}$ $k\in\{1,2,\cdots,N\}$ and for any multi-index $\alpha=(\alpha_1,\cdots,\alpha_N)$ of positive integers let $D^\alpha=D_1^{\alpha_1}\cdots D_N^{\alpha_N}$. Also let $|\alpha|$ be the length of the multi-index. Let $N_m=\dfrac{(N+m)!}{N!m!}$. Given $x\in W_0^{m,p}(Z)$ by $\eta(x)$ we denote the N_m-tuple of partial derivatives of $x(\cdot)$ up to order m, i.e. $\eta(x)=\{D^\alpha x:|\alpha|\leq m\}\subseteq L^2(Z)\times\cdots\times L^2(Z)$ (N_m-times). We consider the following problem:

$$\left\{\begin{matrix} \int_Z \xi(z,x(b,z))dz\to\inf=\xi \\ \dfrac{\partial x}{\partial t}+\sum\limits_{|\alpha|\leq m}(-1)^{|\alpha|}A_\alpha(t,z,\eta(x(t,z)))=u(t,z) \ \ a.e \ \ on \ \ T\times Z \\ D^\beta|x|_{T\times\Gamma}=0 \ \ a.e. \ \ for |\beta|\leq m-1, \ \ x(0,z)=x_0(z) \ \ a.e \ \ on \ Z \\ \|u(t,\cdot)\|_2\leq r(t,x(t,\cdot)) \ \ a.e. \ \ on\, T \end{matrix}\right\} \tag{10}$$

Here $A_\alpha(t,z,\eta)$ $|\alpha|\leq m$ are $\boldsymbol{R}$-valued functions defined on $T\times Z\times\boldsymbol{R}^{N_m}$ and $r:T\times L^2(Z)\to\boldsymbol{R}_+$. Problem (10) corresponds to a terminal optimal control of a parabolic distributed parameter systems with a priori feedback (i.e. state-dependent control constraints).

Our hypotheses on the data of (10) are the following:

$\underline{H(A)_1}$: $A_\alpha:T\times Z\to\boldsymbol{R}^{N_m}\to\boldsymbol{R}$ is a function such that

(i) $(t,z)\to A_\alpha(t,z,\eta)$ is measurable,

(ii) $\eta\to A_\alpha(t,z,\eta)$ is continuous,

(iii) $\sum_{|\alpha|\leq m}(A_\alpha(t,z,\eta)-A_\alpha(t,z,\eta'))(\eta_\alpha-\eta'_\alpha)\geq 0$ *a.e.* on Z,

(iv) $|A_\alpha(t,z,\eta)|\leq a_1(t,z)+c_1\|\eta\|^{p-1}$ *a.e.* on $T\times Z$ with $a_1\in L^q(T\times Z)$, $c_1>0$, $2\leq p<\infty, \frac{1}{p}+\frac{1}{q}=1$,

(v) $\sum_{|\alpha|\leq m}A_\alpha(t,z,\eta)\eta_\alpha\geq c\|\eta\|^p$ *a.e.* on Z with $c>0$.

$\underline{H(r)}: r:T\times L^2(Z)\to \boldsymbol{R}_+$ is a function such that

(i) $t\to r(t,x)$ is measurable,

(ii) $|r(t,x)-r(t,y)|\leq k(t)\|x-y\|_2$ *a.e.* with $k\in L^1(T)$,

(iii) $r(t,x)\leq a_2(t)+c_2\|x\|_2$ *a.e.* with $a_2\in L^2(T)$, $c_2>0$.

$\underline{H(\xi)}: \xi: Z\times \boldsymbol{R}\to\boldsymbol{R}$ is a function such that

(i) $z\to\xi(z,r)$ is measurable,

(ii) $r\to\xi(z,r)$ is continuous,

(iii) $|\xi(z,r)|\leq a_3(z)+ c_3|r|$ *a.e.* with $a_2\in L^2(Z)$, $c_3>0$.

In this case the evolution triple is $X=W_0^{m,p}(Z), H=L^2(Z)$ and $X^*=W^{-m,q}$ (Z). From the Sobolev embedding theorem we know that all the embeddings are compact. Let $a:T\times W_0^{m,p}(Z)\times W_0^{m,p}(Z)\to\boldsymbol{R}$ be the time-varying Dirichlet form defined by $a(t,x,y)=\sum_{|\alpha|\leq m}\int_Z A_\alpha(t,z,\eta(x(z)))D^\alpha y(z)dz$. Via Hölder's inequality we can show that

$$|a(t,x,y)|\leq\gamma_1(\|a_1(t,\cdot)\|_q+\|x\|_{m,p}^{p-1})\|y\|_{m,p}$$

for some $\gamma_1>0$ (here $\|\cdot\|_{m,p}$ denotes the norm of $W_0^{m,p}(Z)$) So we can find $A:T\times W_0^{m,p}(Z)\to W^{-m,q}(Z)$ such that $a(t,x,y)=\langle A(t,x),y\rangle$. Using hypothesis $H(A)_1$ we can easily check that $H(A)$ holds. Also let $F:T\times L^2(Z)\to P_{\omega kc}(L^2(Z))$ be defined by $F(t,x)=\{u\in L^2(Z):\|u\|_2\leq r(t,x)\}$. By virtue of $H(r)$ we immediately see that $H(F)_2$ holds. Finally let $\hat{\xi}:L^2(Z)\to\boldsymbol{R}$ be defined by $\hat{\xi}(x)=\int_Z\xi(z,x(z))dz$. Because of $H(\xi)$, $\hat{\xi}(\cdot)$ is continuous. Rewrite (10) in the following equivalent abstract form:

$$\left\{\begin{array}{c}\hat{\xi}(x)\to\inf=m\\ s.t.\ \dot{x}(t)+A(t,x(t))\in F(t,x(t))\ a.e.\\ x(0)=x_0(\cdot)\end{array}\right\}\qquad (10)'$$

Using Theorems 1 and 6 we get:

Theorem 9. *If hypotheses* $H(A_1), H(r), H(\xi)$ *hold and* $x(0,\cdot)\in L^2(Z)$ *then* *problem* (10) *has a solution* $x \in C(T, L^2(Z)) \cap L^2(T, W_0^{m,p}(Z))$ *with* $\frac{\partial x}{\partial t} \in L^2(T, W^{-m,q}(Z))$ *and also given any* $\varepsilon > 0$ *there exists a state* $y_\varepsilon \in C(T, L^2(Z)) \cap L^2(T, W_0^{m,p}(Z))$ *with* $\frac{\partial y_\varepsilon}{\partial t} \in L^2(T, W_0^{-m,p}(Z))$ *generated by a control function* $\upsilon \in extF(t, y_\varepsilon)$ *("bang-bang control") such that* $\hat{\xi}(y_\varepsilon) - \hat{\xi}(x) = \hat{\xi}(y_\varepsilon) - \xi \leq \varepsilon$.

Now consider a problem with a discontinuous nonlinearity $u : Z \times \boldsymbol{R} \to \boldsymbol{R}$. Following Rauch [12] we define $\underline{u}(z,x) = \varliminf_{y \to x} u(x,y)$ and $\bar{u}(z,x) = \varlimsup_{y \to x} u(z,y)$. We know (see Rauch [12]) that $\underline{u}(z,\cdot)$ is lsc and $\bar{u}(z,\cdot)$ is usc. So if we set $F_0(t,z,x) = \{\upsilon \in \boldsymbol{R} : \ \xi(t)\underline{u}(z,x) \leq \upsilon \leq \xi(t)\bar{u}(z,x)\}$, $\xi(t) \geq 0 \ \ t \in T$, we have that $F_0(t,z,\cdot)$ is usc (see Klein-Thompson [8]). Consider the following multivalued p.d.e.

$$\left\{ \begin{array}{c} \frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k(|D_k x|^{p-2} D_k x) \in F_0(t,z,x(t,z))) \ a.e. \ on \ T \times Z \\ \sum_{k=1}^{N} |D_k x|^{p-2} D_k x \cos(n, z_k)_{\boldsymbol{R}^N} = \upsilon(t,z) \ \ on \ T \times \Gamma, \ \ p \geq 2 \\ x(0,z) = x_0(z) \ \ a.e. \ \ on \ Z \end{array} \right\} \tag{11}$$

We need the following hypotheses on the data of (11).

$\underline{H(u)}: u : Z \times \boldsymbol{R} \to \boldsymbol{R}$ is a function such that both $\underline{u}$ and $\bar{u}$ are superpositionally measurable (i.e. for every $x : Z \to \boldsymbol{R}$ .measurable, $z \to \underline{u}(z, x(z))$ and $z \to \bar{u}(z, x(z))$ are both measurable) and $|u(z,x)| \leq a_2(z) + c_2|x|$ *a.e.* with $a_2 \in L^2(z)$, $c_2 > 0$.

$\underline{H(\xi)_1}: \xi \in L^2(T), \xi(t) \geq 0$ *a.e.*

$\underline{H(\upsilon)_1}: \upsilon \in L^2(T, W^{\frac{1}{q},p}(\Gamma))$.

In this case the evolution triple is $X = W^{1,p}(Z), H = L^2(Z)$ and $X^* = [W^{1,p}(Z)]^*$. As before by Sobolev's embedding theorem, all embeddings are compact. Let $A : X \to X^*$ be defined by

$$\langle A(x), y \rangle = \sum_{k=1}^{N} \int_Z |D_k x|^{p-2} D_k x D_k y dz.$$

Using the seminorm $[x] = (\sum_{k=1}^{N} \int_Z |D_k x|^p dz)^{\frac{1}{p}}$ on $W^{1,p}(Z)$, we see that $A(\cdot)$satisfies hypothesis $H(A)$. Let $F(t,x) = S^2_{F_0(t,\cdot,x(\cdot))}$. Hypothesis $H(u)$guarantees that $H(F)$ holds. So we can apply Theorem 1 and get:

Theorem 10. *If hypotheses* $H(u), H(\xi)_1$ *hold and* $x_0(\cdot) \in L^2(Z)$, *then problem* (11) *has a solution* $x \in C(T, L^2(Z)) \cap L^2(T, W^{1,p}(Z))$ *with* $\frac{\partial x}{\partial t} \in L^2(T, [W^{1,p}(Z)]^*)$ *and the solution set is compact in* $C(T, L^2(Z))$.

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