YOKOHAMA MATHEMATICAL
JOURNAL VOL. 43, 1995

A CRITERION OF ALMOST SURE CONVERGENCE OF ASYMPTOTIC MARTINGALES IN A BANACH SPACE

By

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(Recived September 7, 1994, Revised July 3, 1995)

Abestract. In this paper we give a necessary and sufficient condition for a L^1-bounded asymptotic martingale (amart) taking values in a Banach space to converge almost surely in norm: such an asymptotic martingale $(X_n, F_n, n \geqq 1)$ converges a.s. iff it is strongly tight, i.e. for every $\varepsilon > 0$ there exists a compact set K_ε such that $P\left(\bigcap_{n=1}^{\infty}[X_n \in K_\varepsilon]\right) > 1-\varepsilon$. Moreover, we show that for realvalued martingales the well known theorem of Doob is, in some sense, the best possible-there exists a martingale $(X_n, n \geqq 1)$ such that $\sup_n E|X_n|^\alpha < \infty$ for every $\alpha \in (0,1)$ and it diverges a.s. (in fact, it does not even converge in law, although it is strongly tight).

1. Introduction.

A classic problem in the theory of martingales is to give conditions which assure their almost sure (a.s.) convergence. The well known Doob's theorem states that every real-valued L^1-bounded submartingale converges a.s. [8]. It is, in general, false in case of L^1-bounded martingales taking values in a separable Banach space. Namely, the following is well known: for every separable Banach space E the fact that every L^1-bounded E-valued martingale converges a.s. in norm to an integrable X-valued random variable (r.v.) is equivalent to the Radon-Nikodym theorem for E-valued measures with a finite variation and absolutely continuous with respect to the probability measure [3], [8]. This theorem gives an exhaustive answer to the question in which spaces every L^1-bounded martingale converges a.s. But it may happen that a L^1-bounded martingale taking values in a space which does not fulfil the Radon-Nikodym condition converges a.s. (e.g. take the space $E=c_0$ of all real sequences converging to zero with a sup norm, $e \in c_0$ such that $\|e\|=1$ and an arbitrary real-valued martingale $(X_n, F_n, n \geqq 1)$; the sequence $(X_n e, F_n, n \geqq 1)$ obviously converges a.s., although an example in [8] shows that not every L^1-bounded martingale taking values in this space converges a.s.). Thus a question origins: can we give a necessary and sufficient condition for a L^1-bounded martingale taking values in a Banach space to converge a.s.? Theorem V.3.9. in [4], which is due to Chatterji [3], answers this question completely in terms of decomposition of vector measures and Radon-Nikodym derivatives. A case of asymptotic martingales is more complicated: if E has the Radon-Nikodym property and a separable dual space, we need an assumption $\sup_{\tau\in T} E\|X_\tau\|<\infty$, where T is the set of bounded stopping times, to assure weak convergence of the sequence $(X_n(\omega),\ n\geqq 1)$ for almost all $\omega\in\Omega$. It is known that the condition $\sup_{\tau\in T}E\|X_\tau\|<\infty$ cannot be replaced by L^1-boundedness of the sequence (X_n) and none of the conditions concerning E cannot be omitted. Moreover, convergence in norm need not hold [6]. In this paper we solve the problem of a.s. convergence of amarts by giving a topological characterization: a L^1 bounded asymptotic martingale X_n taking values in a Banach space E converges a.s. in norm iff it is strongly tight, i.e. for every $\varepsilon>0$ there exists a compact set K_ε such that $P(\bigcap_{n=1}^\infty[X_n\in K_\varepsilon])>1-\varepsilon$. It can seem a little surprising because in an infinitely dimensional Banach space a ball is not compact. The Baire category theorem (see e.g. [5], theorem I.6.9) states that it is not even σ-compact (i.e. is not a sum of a countable family of compact sets) and thus compact sets in an infinitely dimensional Banach spaces can be regarded as "small".

2. Notation and Definitions.

Let N denote a set of natural numbers, i.e. $N=\{1, 2, 3, \cdots\}$. Let (Ω, A, P) be a probability space and let $(F_n,\ n\geqq 1)$ be an increasing sequence of sub-σ-fields of A (i.e. $F_n\subset F_{n+1}\subset A$ for every $n\in N$). A mapping $\tau: \Omega\to N\cup\{\infty\}$ will be called a stopping time with respect to (F_n) iff for every $n\in N$ the event $\{\tau=n]$ belongs to F_n. A stopping time τ will be called bounded iff there exists $M\in N$ such that $P\{\tau\leqq M\}=1$. A set of all bounded stopping times will be denoted by T. Let E be a Banach space with a norm $\|\ \|$. Let E' be its dual and let $\|\ \|_*$ be a norm in E'. We shall say that a function $X: \Omega\to E$ is weakly measurable iff for every $x'\in E'$ a function $x'(X)$ is measurable. A weakly measurable mapping X is said to be Pettis integrable iff for every $B\in A$ there exists $x_B\in E$ such that for every $x'\in E'$ we have $\int_B x'(X)dP=x'(x_B)$. The element x_B is called a Pettis integral of X on the set B and denoted by $\int_B XdP$. Moreover, if X is measurable and $E\|X\|<\infty$ a.s., then X is also Bochner integrable and the Bochner integral EX obviously coincides with the Pettis integral of X on Ω [4]. The set of all Bochner integrable r.v.s. with values in E (more precisely, the set of all their equivalence classes) will be denoted by L_E^1 or simply by L^1, where it does not lead to confusion. Let F be a sub-σ-field of A. Definitions and basic properties of the Bochner integral EX and the conditional expectation $E^F X$ of a r.v. $X \in L_E^1$ can be found e.g. in [8].

Definition 1. A sequence $(X_n, F_n, n \geqq 1)$ will be called a martingale if, for every $n \in N$, the following conditions are satisfied.

(1) X_n is F_n-measurable and $X_n \in L_E^1$,

(2) $E^{F_n} X_{n+1} = X_n$ a.s.

Definition 2. [6] A sequence $(X_n, F_n, n \geqq 1)$ of Pettis integrable r.v.s. is called an asymptotic martingale (amart) iff X_n is F_n-measuracle for every $n \in N$ and if for every $\varepsilon > 0$ there exists $\tau_0 \in T$ such that for every $\tau, \upsilon \in T$, $\tau, \upsilon \geqq \tau_0$ we have

$$(3) \quad \left\| \int X_\tau dP - \int X_\upsilon dP \right\| < \varepsilon .$$

Obviously, every martingale is an asymptotic martingale.

It is well known that every (strongly) measurable r.v. with values in E is essentially separably valued (see [4], theorem 2.1.2). Thus, considering a sequence (indexed by elements of N) of such r.v.s, we can always assume that they take values in a separable subspace of E. For simplicity, we assume that E is itself separable.

Definition 3. We shall say that a sequence $(X_n, n \geqq 1)$ of E-valued r.v.s. is L_E^1 (or simply L^1)-bounded iff $\sup_n E\|X_n\| < \infty$ and that it is strongly tight iff for every $\varepsilon > 0$ there exists a compact subset K_ε of E such that

$$(4) \quad P\left(\bigcap_{n=1}^{\infty} [X_n \in K_\varepsilon] \right) > 1 - \varepsilon .$$

Let us recall that an indexed family $\{\mu_t, t \in T\}$ of probability measures defined on the σ-field $B(E)$ of the Borel subsets of E is called tight iff for every $\varepsilon > 0$ there exists compact set $K \subset E$ such that for every $t \in T$ we have $\mu_t(K) > 1 - \varepsilon$. A classic theorem of Prohorov ([2], p. 37) states that in a Polish (i.e. a complete and separable metric) space a family of probability measures is weakly relatively compact iff it is tight. Obviously if a sequence $(X_n, n \in N)$ is strongly tight, the family of their distributions $\{\mu_{X_n} : n \in N\}$ is tight, but the reverse implication does not hold, e.g. take a sequence of i.i.d. real r.v.s. having a standard normal distribution.

3. Main results.

The following theorem seems to be, in some sense, a counterpart of the mentioned theorem of Prohorov (for almost sure convergence instead of weak convergence) and is crucial to everything what follows.

Theorem 1 [7]. *Let S be a Polish space and let $(X_n, n\geqq 1)$ be a sequence of r.v.s taking values in S.*

If $X_n \xrightarrow{a.s.} X$, $n\to\infty$, for some r.v. X, then the sequence X_n is strongly tight.

Proof. It is easy to see that if $X_n \xrightarrow{a.s.} X$, $n\to\infty$, for some r.v. X, and $F_n=\sigma(X_1, \cdots, X_n)$, then X_n is randomly convergent in law to X, i.e. for every $\tau_0\in T$ such that for every $\tau\geqq\tau_0$ a.s. $d(X_\tau, X)<\varepsilon$, where d denotes the Prokhorov metric. Now we shall show that the family of distributions $\{P_{X_\tau}, \tau\in T\}$ is tight.

Fix $\delta>0$ and a countable dense subset of S. Let

$$B_m(\delta)=\bigcup_{i=1}^{m} K(x_i, \delta),$$

where

$$K(x_i, \delta)=\{x\in S:\ \rho(x_i, x)<\delta\}.$$

Now we shall show that for every $\varepsilon>0$ there exists m such that for every $\tau\in T$

$$P[X_\tau\in B_m(\delta)]>1-\varepsilon.$$

Assume that the last statement is false, i.e. there exists $\varepsilon>0$ such that for every $m\in N$ we can choose $\tau_m\in T$ such that $P[X_{\tau_m}\in B_m(\delta)]\leqq 1-\varepsilon$. For every n there exists a number $m(n)$ such that

$$P\Big(\bigcup_{i=1}^{n} [X_i\notin B_{m(n)}(\delta)]\Big)\leqq\frac{\varepsilon}{2}.$$

Moreover, we can assume that $m(n)>m(n-1)$ and that $m(n)>n$. If we put $\tau'_{m(n)}=\max(\tau_{m(n)}, n+1)$, then it is easy to see that

$$P[X_{\tau'_{m(n)}}\notin B_{m(n)}(\delta)]\geqq\frac{\varepsilon}{2}.$$

Thus, by theorem 2.1 [2], for every n we have

$$P_X(B_{m(n)}(\delta))\leqq\liminf_{k\to\infty} P_{X_{\tau'_{m(k)}}}(B_{m(n)}(\delta))$$

$$\leqq\liminf_{k\to\infty} P_{X_{\tau'_{m(k)}}}(B_{m(k)}(\delta))\leqq 1-\frac{\varepsilon}{2},$$

but, on the basis of the axiom of continuity,

$$\lim_{n\to\infty} P_X(B_{m(n)}(\delta))=1,$$

contradiction.

Thus for an arbitrary $\varepsilon>0$ and $k\geqq 1$ we can choose a number n_k such that

$$P\left[X_\tau\in B_{m(n_k)}\left(\frac{1}{k}\right)\right]>1-\frac{\varepsilon}{2^k}\,.$$

Now let

$$K=\overline{\bigcap_{k=1}^{\infty} B_{m(n_k)}\left(\frac{1}{k}\right)}.$$

It is easy to see that K is compact and $P[X_\tau\in K]>1-\varepsilon$ for every $\tau\in T$. Thus the family $\{P_{X_\tau},\ \tau\in T\}$ is tight.

We are now ready to finish the proof. Assume that theorem 1 is false, i.e. there exists $\varepsilon>0$ such that for any compact set K

$$P\left(\bigcap_{n=1}^{\infty}[X_n\in K]\right)\leqq 1-2\varepsilon\,.$$

By the other hand, there exists a compact set K_ε such that

$$P[X_\tau\in K_\varepsilon]>1-\varepsilon \qquad \text{for every} \quad \tau\in T\,.$$

Let $\tau(\omega)=\inf\{s:\ X_s\notin K_\varepsilon\}$. If $\tau_n=\min(\tau,\ n)$, then $\tau_n\in T$ and

$$P\left(\bigcup_{n=1}^{\infty}[X_n\notin K_\varepsilon]\right)\leqq\lim_{n\to\infty}P[X_{\tau_n}\notin K_\varepsilon]\leqq\varepsilon\,.$$

The proof is complete.

Thus every a.s. convergent sequence of r.v.s. taking values in a Banach space is tight. Now we shall show that strong tightness assures a.s. convergence of a L^1-bounded asymptotic martingale.

Lemma 1. *Let E be a Banach space and let $K\subset E$ be compact. There exists a countable family $x_k'\in E'$, $k\geqq 1$, such that for an arbitrary sequence $\{x_n\}$ of elements of K $x_n\to x_\infty$ for some x_∞ iff for every $k\in N$ the sequence $\{x_k'(x_n),\ n\geqq 1\}$ is convergent.*

Remark. Let us mention that in general if x_n is a sequence of elements of a Banach space E, convergence of all the sequences $x'(x_n)$, $\geqq 1$, $n'\in E'$, does not even imply weak convergence of x_n to some $x_\infty\in E$, for example a sequence $x_n=(\underbrace{1,\ 1,\ \cdots\ 1}_{n},\ 0,\ 0,\ \cdots)$ in the space c_0 of real sequences converging to zero does not converge weakly.

Lemma 2. *Let E be a Banach space and let $(X_n,\ n\geqq 1)$ be a strongly tight sequence of E-valued r.v.s. There exists a countable subset $\{x_k',\ k\in N\}\in E'$ such that $X_n\xrightarrow{a.s.}X$ for some r.v. X iff for every $k\in N$ the sequence $\{x_k'(X_n),\ n\geqq 1\}$ converges a.s.*

Proof. It is obvious that if $X_n\xrightarrow{a.s.}X$, then for every $x'\in E'x'(X_n)\xrightarrow{a.s.}x'(X)$. Conversely, let us, for $p\in N$, take a compact set $K_{1/p}$ fulfilling (4) for $\varepsilon=$

$1/p$. By lemma 1 there exist functionals $\{x_l'^p, l\geqq 1\}$ such that for every sequence $\{x_n\}$ of elements of $K_{1/p} x_n \to x$ for some x iff all the sequences $\{x_l'^p(x_n), n\geqq 1\}$, $l\geqq 1$, converge. Take $\{x_k'\}=\{x_l'^p;\ p, l\in N\}$. Let us suppose that all the sequences $\{x_k'(X_n), n\geqq 1\}$ converge a.s. Let $\Omega_0=\{\omega\in\Omega:$ the sequence $(x_k'(X_n(\omega)), n\geqq 1)$ converges for every $k\in N\}$. We have $P(\Omega_0)=1$. Let $A_p=\bigcup_{n=1}^{\infty}[X_n\in K_{1/p}]$. By (3), $P(A_p)>1-1/p$, so if we put $\Omega_1=\bigcup_{p=1}^{\infty}A_p$, we have $P(\Omega_1)=1$. Let $\omega\in\Omega_0\cap\Omega_1$. There exists $p\in N$ such that $\omega\in A_p$, so $X_n(\omega)\in K_{1/p}$ for all $n\in N$ and, because $\omega\in\Omega_0$, the sequence $x_l'^p(X_n(\omega))$ converges for all $l\in N$. Thus $X_n(\omega)\to X(\omega)$ for some $X(\omega)\in E$, so the sequence X_n converges a.s. and obviously its limit X is measurable. The proof is complete.

Corollary. *A sequence $(X_n, n\geqq 1)$ of r.v.s taking values in a Banach space converges a.s. iff it is strongly tight and for every $x'\in E'$ the sequence $x'(X_n)$ converges a.s.*

Now we are ready to prove our main result.

Theorem 2. *Let $(X_n, F_n, n\geqq 1)$ be a L^1-bounded asymptotic martingale taking values in a Banach space E. $X_n \longrightarrow X$ for some integrable r.v. X if and only if the sequence X_n is strongly tight.*

Proof. Necessity of strong tightness of (X_n) for its a.s. convergence follows from theorem 1 (see also a remark after definition 1). Conversely, assume that (X_n) is strongly tight. For every $x'\in E'$ the sequence $(x'(X_n), F_n, n\geqq 1)$ is a L^1-bounded real asymptotic martingale and thus converges a.s. [1]. Indeed, let $\varepsilon>0$ be arbitrary and let $\tau_0\in T$ be such that for every $\tau, \sigma\in T$, $\tau, \sigma\geqq\tau_0$ a.s., (3) holds. Thus

$$|Ex'(X_\tau)-Ex'(X_\sigma)|=|x'(EX_\tau)-x'(EX_\sigma)|\leqq\|x'\|_*|EX-EX_\tau|\leqq\|x'\|_*\varepsilon,$$

what proves the amart property. L^1-boundedness follows from

$$\sup_n E|x'(X_n)|\leqq\|x'\|_*\sup_n E\|X_n\|<\infty.$$

By the last corollary X_n converges a.s. Integrability of its limit follows easily from L--boundedness of X_n and the Fatou lemma.

We shall also give another proof, which makes use of theorem 5 in [1].

Fix n and let $S=K_{1/n}$ (see definition 3). Let $C=\bigcap_{n=1}^{\infty}[X_n\in S]$, by hypothesis $P(C(>1-1/n$. Lemma 1 says, in the terminology of [5], that there exists a determining set $\mathcal{K}$ for S which consists of linear functionals from E' truncated to S. Let $x'\in\mathcal{K}$. Consider a probability space $(C, C\cup A, P_1)$ and a sequence of real r.v.s $(Y_n, C\cap F_n, n\geqq 1)$, where $C\cap A=\{C\cap D\colon D\in A\}$, similarly $C\cap F_n=\{C\cap D\colon D\in F_n\}$, Y_n is the r.v. $x'(X_n)$ truncated to C, and P_1

is simply the measure P truncated to $C\cap A$ and divided (normalized) by dividing by $P(C)$. $x'(S)$ is compact, hence bounded, on the real axis, so, by definition, Y_n is bounded by some real constant. As in the previous proof, we check that $x'(X_n)$ converges a.s., hence Y_n converges a.s. By corollary 1 from [1] $(Y_n, C\cap F_n)$ is an amart, of course L^1-bounded. Thus, by theorem 5 from [1], X_n converges a.s. on C, so, by an obvious argument, it converges a.s. on Ω.

4. Examples.

1. If a L^1-bounded martingale $(X_n, F_n, n\geqq 1)$ takes values in a finite-dimensional subspace E_1 of E, it converges a.s. Indeed, as it is mentioned in [8], p. 108, the sequence $(\|X_n\|, F_n, n\geqq 1)$ is a real-valued L^1-bounded submartingale and thus $Z=\sup_n\|X_n\|<\infty$ a.s. This fact, in connection with compactness of a ball in E_1, yields (4).

2. In [8], p. 111, we can find an interesting example of a L^1-bounded martingale taking values in c_0 which diverges a.s. We shall see how our criterion works in that case.

Let Y_n denote a sequence of i.i.d real r.v.s such that $P[Y_n=\pm 1]=1/2$ and let $X_n=(Y_1, \cdots Y_n, 0, 0, \cdots)$.

Let us remark that if we put

$$A=\{y=(y_1, y_2, \cdots)\in c_0:\ \exists n_0(y)(\forall n>n_0(y)y_n=0\vee\forall n\leqq n_0(y)y_n=\pm 1\},$$

then for $a, b\in A$, $a\neq b$, we have $\|a-b\|\geqq 1$ and thus every compact subset of A is finite. Now it is obvious that in this case (4) cannot hold.

3. Let, for every $k\in N$, $(X_n^k, F_n^k, n\geqq 1)$ be a real martingale such that $P(|X_k^n|>1)=0$ for all $n\in N$ and σ-fields $F_\infty^k=\sigma(\bigcup_{n=1}^\infty F_n^k)$, $k\in n$, are independent on one another. Let $a_1, a_2, \cdots$ be a sequence of real numbers which converges to zero as $n\to\infty$. Put

$$X_n=(a_1X_n^1, a_2X_n^2, a_3X_n^3, \cdots) \quad\text{and}\quad F_n=\sigma\Big(\bigcup_{k=1}^\infty F_n^k\Big).$$

$(X_n, F_n, n\geqq 1)$ is a L^1-bounded martingale taking values in c_0. Indeed, X_n is F_n-measurable and $\|X_n\|\leqq\sup_m|a_m|<\infty$ for every n. It remains to verify the martingale property. Obviously $E^{F_n}X_{n+1}$ exists. The only question is whether or not it equals to X_n a.s.

Let $x_l'\in c_0'$, $x_l'((x_1, x_2, \cdots))=x_l$, be the coordinate mappings in c_0. We have

$$x_l'(E^{F_n}X_{n+1})=E^{F_n}x_l'(X_{n+1})=a_lE^{F_n}X_{n+1}^l. \tag{17}$$

But

$$E^{F_n}X_{n+1}^l=E^{F_n^l}X_{n+1}^l. \tag{18}$$

Indeed, $E^{F_n^l}X_{n+1}^k$ is F_n-measurable. It remains to check that

$$\forall A \in F_n \int_A X_{n+1}^l dP = \int_A E^{F_n^l} X_{n+1}^l dP. \tag{19}$$

It sufficies to verify (19) for sets $A=B\cap C$, where $B\in F_n^l$, $C\in\sigma(\bigcup_{m\neq l}F_n^m)$, because they form a π-system generating a λ-system F_n (see the Dynkin theorem e.g. in [2]). But

$$\int_{B\cap C} E^{F_n^l} X_{n+1}^l dP = P(C)\int_B E^{F_n^l} X_{n+1}^l dP = P(C)\int_B X_{n+1}^l dP = \int_{B\cap C} X_{n+1}^l dP, \tag{20}$$

(the first and the last equality follow from an easy to prove fact that if $B\in F_\infty^l$, X is a F_∞^l-measurable, integrable r.v. and $C\in\sigma(\bigcup_{m\neq l}F_n^m)$, then C is independent of F_∞^l and thus

$$\int_{B\cap C} X dP = P(C)\int_B X dP. \tag{21}$$

We have proved (18).

Thus, by (17) and (18), we have

$$x_l'(E^{F_n}X_{n+1}) = a_l E^{F_n^l} X_{n+1}^l = a_l X_n^l, \tag{22}$$

so $E^{F_n}X_{n+1}=(a_1X_n^1, a_2X_n^2, \cdots)=X_n$. We have proved the martingale property (in fact only integrability of X_n, the martingale property of their coordinates and independence of F_∞^k have been used).

It is easy to see that a set $K=\{x=(x_1, x_2, \cdots)\in c_0\colon \forall k\in N\ |x_k|\leqq|a_k|\}$ is compact in c_0, because $\lim_{n\to\infty}\sup_{x\in K}\|R_n x\|=0$, where $R_n((x_1, \cdots x_n, x_{n+1}, x_{n+2}, \cdots))=(0, \cdots 0, x_{n+1}, x_{n+2}, \cdots)$. Thus the martingale X_n converges a.s., becaue $P(X_n\in K)=1$ for every $n\in N$ and thus this sequence is strongly tight.

It is natural to pose a question: is strong tightness sufficient for a.s. convergence of an arbitrary asymptotic martingale (not necessarily L^1-bounded)? Unfortunately, the answer is negative even in the case of real martingales which are L^α-bounded for every $\alpha\in(0, 1)$; a counter-example is given below.

Let $(\Omega, A, P)=([0, 1], B([0, 1]), \mu)$, where μ is the Lebesgue measure on the unit interval and let $F_0=\{\emptyset, \Omega\}$. We construct the σ-field F_{n+1} from F_n by dividing each atom of F_n into two parts, one of which has $1/2^{n+1}$ of the mass of the previous one and the second one has $1-1/2^{n+1}$ of it.

Let $X_0=0$ a.s. and let $P(X_1=\pm1)=1/2$. We construct X_{n+1} from X_n in the following way. If n is even (odd), we take each atom of F_n, we divide it into parts as above, we put $X_{n+1}=1$ (0) on the bigger one and such (constant) number on the second one, that the martingale property is retained. In what follows we shall call this procedure balancing to 1 (0). E.g. $P(X_2=0)=3/4$ and $P(X_2=\pm4)=1/8$. $(X_n, F_n, n\geqq0)$ is a real martingale. It is easy to see that in the real case strong tightness of the sequence X_n is equivalent to

$$\sup_n |X_n| < \infty \quad \text{a.s.} \tag{23}$$

and that the martingale defined above fulfils this condition (it sufficies to use the Borel-Cantelli lemma to verify that $P(X_n \notin \{0, 1\}$ for infinitely many $n)=0$). But $X_{2n} \xrightarrow{a.s.} 0$ and $X_{2n+1} \xrightarrow{a.s.} 1$ as $n \to \infty$, so the martingale X_n does not even converge in law.

We shall show that X_n exhibits one more interesting property: for every $\alpha \in (0, 1)$ it is L^α-bounded, i.e.

$$\sup_n E|X_n|^\alpha < \infty. \tag{24}$$

Let $\alpha \in (0, 1)$. Consider the way in which we construct X_{n+1} from X_n. We have two types of atoms:

i) Atoms, for which $X_n=0$ (if n is even) or $X_n=1$ (if n is odd) by the basic construction principle. Probability of the sum of these atoms is equal to $1-1/2^n$.

ii) Atoms for which $|X_n|=a=const$, $|a|>1$ (in fact, X_n takes only integer values and thus $|a| \geqq 2$).

iii) Atoms for which $X_n=1$ if n is even) or $X_n=0$ (if n is odd). It is easy to see that on these atoms $X_{n+1}=X_n$ a.s., thus if we compare $E|X_{n+1}|^\alpha$ to $E|X_n|^\alpha$, we can take into account only atoms of type i) and ii).

You can ask whether $X_n=-1$ on some atom of F_n (none of the cases i)-iii) covers this situation). It really happens if $n=1$, but for $n \geqq 2$ it is impossible, because it is impossible to balance an integer on $1-1/2^{n+1}$ of an atom to another integer putting $X_{n+1}=-1$ on the remaining $1/2^{n+1}$ of it.

We shall estimate the change of $E|X_n|^\alpha$ on the atoms of types i) and ii).

i) a) n is even, so we balance 0 to 1.

Let B be an atom of F_n such that $X_n=0$ on B. We divide B into B_1 and B_2, $P(B_1)=(1-1/2^{n+1})P(B)$, $P(B_2)=1/2^{n+1}P(B)$.

We put $X_{n+1}=1$ on B_1 and thus we must put $X_{n+1}=1-2^{n+1}$ on B_2 to retain the martingale property. Thus

$$\begin{aligned} E|X_{n+1}|^\alpha I_B &= P(B)\Big[1^\alpha\Big(1-\frac{1}{2^{n+1}}\Big)+(2^{n+1}-1)^\alpha\frac{1}{2^{n+1}}\Big] \\ &= P(B)\Big[1-\frac{1}{2^{n+1}}+\frac{(2^{n+1}-1)^\alpha}{2^{n+1}}\Big]. \end{aligned} \tag{25}$$

Summing over all atoms B of type i) we obtain

$$\begin{aligned} S_1^{n+1} &= \sum_B E|X_{n+1}|^\alpha I_B = \Big(1-\frac{1}{2^n}\Big)\Big[1-\frac{1}{2^{n+1}}+\frac{(2^{n+1}-1)^\alpha}{2^{n+1}}\Big] \\ &< \Big(1-\frac{1}{2^n}\Big)\Big[1-\frac{1}{2^{n+1}}+\Big(\frac{1}{2^{n+1}}\Big)^{1-\alpha}\Big] < 2. \end{aligned} \tag{26}$$

Observe that on these atoms $S_1^n=E|X_n|^\alpha I_B=0$, so the total increase of S_1^n is in this case less than 2.

b) We balance 1 to 0, n is odd.

We take an atom $B\in F_n$ such that $X_n=1$ on B. We divide it into B_1 and B_2 like in the point a) and we put $X_{n+1}=0$ on B_1 and $X_{n+1}=2^{n+1}$ on B_2. Similarly like in a)

$$S_2^{n+1}=\sum_B E|X_{n+1}|^\alpha I_B=\left(1-\frac{1}{2^n}\right)\frac{1}{2^{(n+1)(1-\alpha)}} \tag{27}$$

and

$$S_2^n=\sum_B E|X_n|^\alpha I_B=\left(1-\frac{1}{2^n}\right)\cdot 1=1-\frac{1}{2^n}, \tag{28}$$

thus here S_2^n decreases.

Let us suppose that n is even and figure the increase from $E|X_n|^\alpha$ to $E|X_{n+2}|^\alpha$ only on the atoms of type i).

From X_n to X_{n+1} the increase of $E|\cdot|^\alpha$ is less than

$$\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{2^{n+1}}+\left(\frac{1}{2^{n+1}}\right)^{1-\alpha}\right) \tag{29}$$

and from X_{n+1} to X_{n+2} we have a decrease

$$\left(1-\frac{1}{2^{n+1}}\right)\left(1-\left(\frac{1}{2^{n+2}}\right)^{1-\alpha}\right). \tag{30}$$

Thus the increase of $E|\cdot|^\alpha$ from X_n to X_{n+2} is less than

$$\left(1-\frac{1}{2^{n+1}}\right)\left[\left(\frac{1}{2^{n+2}}\right)^{1-\alpha}-\frac{1}{2^n}\right]+\left(1-\frac{1}{2^n}\right)\left(\frac{1}{2^{n+1}}\right)^{1-\alpha} \tag{31}$$

$$<\left(\frac{1}{2^{1-\alpha}}\right)^{n+2}+\left(\frac{1}{2^{1-\alpha}}\right)^{n+1}-\frac{1}{2^n}.$$

The series $\sum(1/2^{1-\alpha})^n$ and $\sum 1/2^n$ both converge and thus the increase of $E|\cdot|^\alpha$ only on all atoms of type i) after an arbitrary even number of steps is bounded above by some integer M and so, by a), the increase after an odd (and thus an arbitrary) number of steps is bounded above by $M+2$.

ii) a) We balance from a to 0, n is odd, $|a|\geqq 2$.

We take an atom $B\in F_n$ such that $X_n=a$ on B, divide it into B_1 and B_2 like in i) a) and put $X_{n+1}=0$ on B_1 and $X_{n+1}=2^{n+1}a$ on B_2.

$$B|X_n|^\alpha I_B=|a|^\alpha P(B) \tag{32}$$

and

$$E|X_{n+1}|^\alpha I_B=P(B)\frac{1}{2^{n+1}}2^{(n+1)\alpha}|a|^\alpha, \tag{33}$$

$$\text{so}\quad \frac{E|X_{n+1}|^\alpha I_B}{E|X_n|^\alpha I_B}=\frac{1}{2^{(n+1)(1-\alpha)}}<1,$$

thus on these atoms $E|\cdot|^\alpha$ decreases.

b) We balance from a to 1, $|a|\geqq 2$, n is even.

We take an atom $B\in F_n$ such that $X_n=a$ on it, divide it into B_1 and B_2 as above and put $X_{n+1}=1$ on B_1, $X_{n+1}=2^{n+1}(a-1)+1$ on B_2.

$$E|X_n|^\alpha I_B=P(B)|a|^\alpha \tag{34}$$

and

$$E|X|_{n+1}^\alpha I_B=P(B)\left[|2^{n+1}(a-1)+1|^\alpha\frac{1}{2^{n+1}}+1^\alpha\left(1-\frac{1}{2^{n+1}}\right)\right] \tag{35}$$

$$\leqq P(B)\left[1-\frac{1}{2^{n+1}}+(2^{n+2}|a|)^\alpha\frac{1}{2^{n+1}}\right)\right]<P(B)[1+|a|^\alpha 2^{(n+1)(\alpha-1)+\alpha}],$$

the inequality follows from $2^{n+1}(a-1)+1<2^{n+1}a<2^{n+2}a$ and $|a-1|=-a+1<2(-a)=2|a|$ (and thus, because all the numbers used are integers, $2^{n+1}|a-1|+1<2^{n+2}|a|$) for $a\leqq -2$, so for every integer a such that $|a|\geqq 2$ we have $|2^{n+1}(a-1)+1|<2^{n+2}|a|$.

But $(n+1)(\alpha-1)+\alpha\to-\infty$ as $n\to\infty$, thus there exists n_0 such that for all $n\geqq n_0$ we have

$$\frac{E|X_{n+1}|^\alpha I_B}{E|X_n|^\alpha I_B}<\frac{1+|a|^\alpha(1-1/2^\alpha)}{|a|^\alpha}<\frac{1}{|a|^\alpha}+1-\frac{1}{2^\alpha}\leqq 1. \tag{36}$$

Thus for $n\geqq n_0$ (not dependent on a) $E|\cdot|^\alpha$ decreases.

Finally, taking into account i), ii) and iii) we can state that the sequence $E|X_n|^\alpha$ is bounded.

The above example shows that the well known Doob's theorem stating that every L^1-bounded real martingale converges a.s. is, in some sense, the best possible: it cannot be extended to any L^α, $\alpha\in(0, 1)$.

Using the martingale $(X_n, F_n, n\geqq 1)$ described above we can easily construct an example of a martingale which converges in law and does not converge in probability.

Let $(\Omega, A, P)=([0, 2], B([0, 2], \mu/2)$, i.e. $\mu/2(A)=\mu(A)/2$ for every $A\in B([0, 2])$, where μ is the Lebesgue measure, and let $Y_n(\omega)=X_n(\omega)$ for $\omega\in[0, 1]$ and $Y_n(\omega)=1-X_n(\omega-1)$ for $\omega\in(1, 2]$, $B_n=\sigma(F_n, F_n+1)$, where $F_n+1=\{A+1: A\in F_n\}$ and $A+1=\{\omega+1: \omega\in A\}$.

It is obvious that $(Y_n, B_n, n\geqq 0)$ is an integrable martingale. Let us remark that $Y_{2n+1}\longrightarrow Y_\infty$ and $Y_{2n}\longrightarrow 1-Y_\infty$, where $Y_\infty=1$ on $[0, 1]$ and $Y_\infty=0$ on $(1, 2]$. Thus Y_n clearly does not converge in probability, although it converges in law, because the laws of Y_∞ and $1-Y_\infty$ are equal.

Acknowledgements. The authors thank the referee for all his comments, especially for the idea of the second proof of Theorem 2.

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