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GEODESIC TUBES AND SPACES OF CONSTANT CURVATURE

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Summary. This is a contribution to the general problem of how the properties of geodesic tubes on a Riemannian manifold (M^n, g) determine the geometry of the ambient space. Using Jacobi vector fields and Fermi coordinates, we characterize spaces of constant sectional curvature by means of the shape operator of small enough geodesic tubes.

1. Introduction

This paper is a contribution to the general problem of investigating how the properties of small geodesic tubes of a Riemannian manifold determine the geometry of the manifold. More precisely, in this paper we give a characterization of the Riemannian manifolds (M^n, g) of constant sectional curvature, in two ways. First, by means of the shape operator on small geodesic tubes, provided that each tube about every geodesic σ of M is a quasi-umbilical hypersurface and the eigenvector fields of the corresponding shape operator are parallel along a unit speed geodesic γ of M meeting σ orthogonally. Secondly, by means of the shape operator of small geodesic tubes of M^n about every topologically embedded q-dimensional submanifold P of M, provided that the shape operator of every such tube has a parallel eigenspace of dimension $n-q-1$ along a geodesic γ meeting P orthogonally.

The paper is organized as following:

In section two, we recall some facts about Fermi coordinates and Fermi vector fields and give the relation which exists between Fermi vector fields and Jacobi vector fields. In section three, we give the definitions of tubes about a geodesic σ and a submanifold P of M. Then using the relationship between Jacobi and Fermi vector fields, we show how the shape operator of a tube can be expressed in terms of Jacobi vector fields. In section four we apply this technique to obtain the main results. For more details concerning tubes we refer to [10] and [13].

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2. Preliminaries

Let (M, g) be a n-dimensional manifold of class C^∞. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M. The metric tensor g gives rise to an inner product, which will be denoted by $\langle\, ,\rangle$, on each tangent space M_m. The curvature operator R of M is defined by

$$R_{XY}=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]} \tag{2.1}$$

for $X, Y\in\mathfrak{X}(M)$. Let $\sigma:(a, b)\to M$ be a curve in M of finite length. To describe the geometry of a Riemannian manifold M in a neighborhood of a curve σ we use Fermi coordinates [5]. To define a system of Fermi coordinates we need an open neighborhood U of σ for which every point of U can be joined to σ by a shortest-unit speed geodesic, meeting σ orthogonally.

Definition 2.1. The Fermi coordinates $(x_1, \cdots, x_n)$ of U centered at $m=\sigma(0)$, relative to a given orthonormal frame field $\{E_1, \cdots, E_n\}$, along the curve σ for which $\dot\sigma(t)=(E_1)_{\sigma(t)}$, are the real-valued functions defined by

$$x_1\Big(\exp_{\sigma(t)}\sum_{j=2}^{n}t_j(E_j)_{\sigma(t)}\Big)=t, \qquad x_i\Big(\exp_{\sigma(t)}\sum_{j=2}^{n}t_j(E_j)_{\sigma(t)}\Big)=t_i \tag{2.2}$$

provided that the numbers $t_2, \cdots, t_n$ are small enough in order the $\exp_{\sigma(t)}$ to be a diffeomorphism.

Since $\exp_{\sigma(t)}$ is a diffeomorphism on U, equations (2.2) define a coordinate system near m. Let $\{\partial/\partial x_1, \cdots, \partial/\partial x_n\}$ be the coordinate vector fields associated with the Fermi coordinate system $(x_1, \cdots, x_n)$. It is known then, [10], that the restrictions to σ of the coordinate vector fields $\{\partial/\partial x_2, \cdots, \partial/\partial x_n\}$ are orthogonal.

Now, if γ is a unit speed geodesic of M normal to σ with $\gamma(0)=m=\sigma(0)$ and $u=\gamma'(0)$, then there is a system of Fermi coordinates $(x_1, \cdots, x_n)$, such that for small s we have

$$\Big(\frac{\partial}{\partial x_2}\Big)_{\gamma(s)}=\gamma'(s), \quad \Big(\frac{\partial}{\partial x_1}\Big)_m=\{\dot\sigma(t)\}_m, \quad \Big(\frac{\partial}{\partial x_i}\Big)_m\in\{\dot\sigma(t)\}_m^\perp, \quad i=2, \cdots, n. \tag{2.3}$$

Furthermore,

$$(x_\alpha\circ\gamma)(s)=s\delta_\alpha^2, \qquad 1\leqq\alpha\leqq n \tag{2.4}$$

where δ_α^2 is the Kronecker's delta.

Let $\mathfrak{X}(U)$ be the Lie algebra of C^∞ vector fields on U. We introduce a certain finite dimensional Abelian subalgebra of the infinite dimensional Lie algebra $\mathfrak{X}(U)$.

Definition 2.2. Let $(x_1, \cdots, x_n)$ be a Fermi coordinate system of $U=U(\sigma)$

relative to the orthonormal frame field $\{E_1, \cdots, E_n\}$. We say that $X \in \mathfrak{X}(U)$ is a Fermi vector field relative to $(x_1, \cdots, x_n)$ provided

$$X = \sum_{i=2}^{n} c_i \frac{\partial}{\partial x_i} \tag{2.5}$$

where the c_i's are constants.

Two other simple objects, r and N, will be needed in the following. We determine them in terms of Fermi coordinates.

Definition 2.3. Let $(x_1, \cdots, x_n)$ be a system of Fermi coordinates for $U = U(\sigma)$. For $r > 0$ we put

$$r^2 = \sum_{i=2}^{n} x_i^2, \quad N = \sum_{i=2}^{n} \frac{x_i}{r} \frac{\partial}{\partial x_i}. \tag{2.6}$$

For $m \in \sigma$ it is easily proved that the definitions of r and N are independent of the choice of Fermi coordinates at m. In fact, for $m' \in M$ near σ, $r(m') = d(m', \sigma)$ where d is the distance function of M. Furthermore,

$$N_{\gamma(s)} = \left(\frac{\partial}{\partial x_2}\right)_{\gamma(s)} = \gamma'(s), \quad s > 0 \tag{2.7}$$

where γ is the unique geodesic from m' to σ which meets σ orthogonally at $\gamma(0) = m$.

In what follows we assume that σ is also a geodesic of M and put $A = \partial/\partial x_1$. The most important properties of N, r and the Fermi fields are included in the following lemma.

Lemma 2.1 [10]. *Let X be a Fermi vector field for $U = U(\sigma)$ and A, r, N as previously. Then we have*:

1. $\nabla_N N = 0$
2. $\|N\| = 1$
3. $N(r) = 1$
4. $A(r) = 0$
5. $[X, A] = [N, A] = 0$
6. $[N, X] = -\frac{1}{r} X + \frac{1}{r} X(r) N$
7. $[N, rX] = X(r) N$
8. $\nabla_N^2 U = R(N, U) N$

for any U of the form $U = A + rX$.

If ξ is a curve of M and Y is a vector field along ξ, write $Y' = \nabla_{\xi'} Y$ and $Y'' = \nabla_{\xi'}^2 Y$. Then a vector field Y along a geodesic ξ is called a Jacobi field if it satisfies the following second order differential equation

$$Y'' = R(\xi', Y)\xi'. \tag{2.8}$$

There is a simple but important relation between Fermi fields and Jacobi fields.

Corollary 2.1. *Let γ be a geodesic normal to σ at $m=\sigma(0)$ and let X be a Fermi vector field on $U=U(\sigma)$. Then the restrictions to γ of rX and A, i.e.*

$$rX|\gamma \quad \textit{and} \quad A|\gamma$$

are Jacobi vector fields.

Proof. This is an immediate consequence of (8) of Lemma 2.1.

We now generalize the previous ideas, substituting the curve σ by a q-dimensional submanifold of M. More precisely let P be a q-dimensional topologically embedded submanifold of a Riemannian manifold M^n. To describe the geometry of M^n in a neighborhood of P we use a generalization of Fermi's coordinates. We denote by ν the normal bundle of P in M. The exponential map of ν is defined on a neighborhood of the zero section of ν. Let $m\in P$ and $(y_1, \cdots, y_q)$ be an arbitrary system of coordinates for P defined on a neighborhood U of $m\in P$. Assume that this neighborhood U is sufficiently small so that $\nu|U$ is parallelizable. Let $E_{q+1}, \cdots, E_n$ be orthonormal sections of ν which effect this parallelization.

Definition 2.4. The Fermi coordinates $(x_1, \cdots, x_n)$ of $P\subset M$ centered at $m\in P$ (relative to a given coordinate system $(y_1, \cdots, y_q)$ on P at m and given orthonormal sections $E_{q+1}, \cdots, E_n$ of ν), are the real valued functions defined by

$$x_\alpha\Big(\exp_\nu\Big(\sum_{j=q+1}^{n} t_j E_j(m')\Big)\Big)=y_\alpha(m'), \quad \alpha=1, \cdots, q \tag{2.9}$$

$$x_i\Big(\exp_\nu\Big(\sum_{j=q+1}^{n} t_j E_j(m')\Big)\Big)=t_i, \quad i=q+1, \cdots, n \tag{2.10}$$

for $m'\in U$, provided the numbers $t_{q+1}, \cdots, t_n$ are small enough so that $\sum_{j=q+1}^{n} t_j E_j(m')$ is in the domain of definition $\exp_\nu$.

Similarly now if $(x_1, \cdots, x_n)$ is a system of Fermi coordinates centered at $m\in P$, then the restrictions to P of the coordinate vector fields $\partial/\partial x_{q+1}, \cdots, \partial/\partial x_n$ are orthonormal. Moreover, if we suppose that γ is a geodesic of M parametrised by its arc length normal to P with $\gamma(0)=m\in P$ and $u=\gamma'(0)$, then there is a system of Fermi coordinates $(x_1, \cdots, x_n)$ such that for small s we have

$$\Big(\frac{\partial}{\partial x_{q+1}}\Big)_{\gamma(s)}=\gamma'(s), \quad \Big(\frac{\partial}{\partial x_\alpha}\Big)_m\in P_m,$$

$$\Big(\frac{\partial}{\partial x_i}\Big)_m\in P_m^\perp, \quad 1\leqq\alpha\leqq q, \ q+1\leqq i\leqq n. \tag{2.11}$$

Furthermore,

$$(x_\alpha\circ\gamma)(s)=s\delta_\alpha^{q+1}, \quad 1\leqq\alpha\leqq n. \tag{2.12}$$

The relations now (2.6) may be written as

$$r^2 = \sum_{i=q+1}^{n} x_i^2, \qquad N = \sum_{i=q+1}^{n} \frac{x_i}{r}\frac{\partial}{\partial x_i} \tag{2.13}$$

where r may be expressed in terms of the distance function of M, and N in terms of velocities of geodesics. We close this section with a brief discussion of quasi-umbilicity. Normally this refers to the shape operator S of a hypersurface, as having at least $m-1$ eigenvalues equal, m being the dimension of the hypersurface.

3. The shape operator of tubes and other auxiliary results

Let $\sigma:(a, b)\rightarrow M$ be a curve of finite length in a manifold M. We now give the definition of a tube about the curve.

Definiiton 3.1. A solid tube of radius $r\geqq 0$ about a curve σ is the set of points of M given by

$$T(\sigma, r)=\{\exp_{\sigma(t)}(X)\,|\,X\in M_{\sigma(t)},\ \|X\|\leqq r,\ \langle X, \dot\sigma(t)\rangle=0,\ a<t<b\} \tag{3.1}$$

where $M_{\sigma(t)}$ denotes the tangent space of M at the point $\sigma(t)$.

For small s, $0<s\leqq r$, we call the hypersurface of the form

$$P_s=\{m'\in T(\sigma, r)\,|\,d(m', \sigma)=s\} \tag{3.2}$$

the tubular hypersurface at distance s from σ, or just tube.

If σ is a geodesic of M then the corresponding tubes are called geodesic tubes.

Similarly, if P is a q-dimensional embedded submanifold of M we have the following corresponding definitions:

Definition 3.2. A solid tube of radius $r\geqq 0$ about a topologically embedded submanifold P in a Riemannian manifold M is called the set

$$T(P, r)=\{\exp_m(X)\,|\,m\in P,\ X\in P_m^{\perp},\ \|X\|\leqq r\}. \tag{3.3}$$

We assume that r is less than the distance of P to its nearest focal point.

For small s the hypersurface P_s given by (3.2), with P instead of σ, is also closely related to tube and is called the tubular hypersurface at distance s from P.

The vector field N is now the unit normal to each of the tubular hypersurfaces s=constant, about the submanifold P of M.

Suppose now, we are given a unit speed geodesic σ in the manifold M

with $\sigma(0)=m$ and let u be a unit vector in M_m orthogonal to σ at m and denote by γ the geodesic such that $\gamma(0)=m$ and $\gamma'(0)=u$. Let $\{E_1, E_2, \cdots, E_n\}$ be an orthonormal basis of M_m with $E_1=\sigma'(0)$ and $E_2=u$ and let $(x_1, \cdots, x_n)$ be a corresponding Fermi coordinate system. Denote by $\{e_1, \cdots, e_n\}$ the parallel orthonormal frame field along γ, obtained by parallel translation of $\{E_1, \cdots, E_n\}$. Also let $Y_1(s), Y_3(s), \cdots, Y_n(s)$ the $n-1$ Jacobi vector fields along γ, uniquely determined by the initial conditions:

$$Y_1(0)=E_1,\quad Y_1'(0)=0,\quad Y_i(0)=0,\quad Y_i'(0)=E_i,\quad i=3, \cdots, n. \tag{3.4}$$

These vector fields Y_i, $i=1, 3, \cdots, n$ are perpendicular to E_2 at m, hence, with respect to the orthonormal frame $\{e_1; e_3, \cdots, e_n\}$, we have

$$Y_i(s)=(Y_{i1}(s), Y_{i3}(s), \cdots, Y_{in}(s))^t; \quad i=1, 3, \cdots, n \tag{3.5}$$

where "t" denotes the transpose of a matrix. We write

$$Y_i(s)=(Be_i)(s). \tag{3.6}$$

This defines an endomorphism-valued function $s\mapsto B(s)$ which satisfies the equation

$$B''(s)=R(s)\circ B(s) \tag{3.7}$$

where $R(s)$ denotes the endomorphism of $\{\gamma'(s)\}^{\perp}\subset M_{\gamma(s)}$ given by

$$R(s)X=R(N, X)N. \tag{3.8}$$

In fact, differentiate (3.6) to get

$$Y_i'(s)=(B'e_i)(s) \quad \text{and} \quad Y_i''(s)=(B''e_i)(s). \tag{3.9}$$

The Jacobi field equation along γ is

$$Y_i''(s)=R(N, Y_i)N, \quad i=1, 3, \cdots, n. \tag{3.10}$$

Now relation (3.7) is an immediate consequence of (3.6), (3.9) and (3.10) and, applying Corollary 2.1, we conclude that the vector fields

$$X_1(s)=\left(\frac{\partial}{\partial x_1}\right)_{\gamma(s)}; \quad sX_3(s)=s\left(\frac{\partial}{\partial x_3}\right)_{\gamma(s)}, \cdots, sX_n(s)=s\left(\frac{\partial}{\partial x_n}\right)_{\gamma(s)} \tag{3.11}$$

are Jacobi vector fields along the geodesic and satisfy the same initial conditions (3.4). Hence $Y_1(s)=X_1(s)$, $Y_3(s)=sX_3(s), \cdots, Y_n(s)=sX_n(s)$, where $X_1(s), X_3(s), \cdots, X_n(s)$ are the coordinate, Fermi fields along γ. Using (3.4) the endomorphism $B(s)$ relative to the basis $\{E_1; E_3, \cdots, E_n\}$ satisfies the following initial conditions:

$$B(0)=\begin{pmatrix}1 & 0\\ 0 & 0_{n-2}\end{pmatrix} \quad B'(0)=\begin{pmatrix}0 & 0\\ 0 & I_{n-2}\end{pmatrix}.$$

Denote now by $S(s)$ the shape operator, with respect to N as normal vector field, of the tubular hypersurface P_s, defined by $S(s)=-\nabla N$. Then

$$Y_i'(s)=-S(s)Y_i(s), \quad i=1, 3, \cdots, n \tag{3.12}$$

since $[N, Y_i]=0$, $i=1, \cdots, n$ and $Y_i'(s)=\nabla_N Y_i(s)$. Hence, using (3.6) and (3.9) we obtain

$$S(s)=-B'(s)B^{-1}(s). \tag{3.13}$$

This relation is an expression of the shape operator of tubes in terms of the Jacobi vector fields.

Next, we are going to prove the same equation (3.13), when the tube is considered about a specific q-dimensional submanifold of the Riemannian manifold M, instead about a geodesic of M. More precisely let $m\in M$, it is known that for each small positive number r there exists a neighborhood $N(m; r)$ at m (more precisely the zero vector at m) in M_m, which is mapped diffeomorphically onto a neighborhood $U(m, r)$ of m in M, by the exponential mapping. Let V be a q-dimensional subspace of M_m and consider the set $V\cap N(m; r)$. Denote by P the q-dimensional connected embedded submanifold of M obtained by $\exp_m(V\cap N(m; r))$. This manifold P has a Riemannian structure induced by that of M. In what follows, we will refer to this specific submanifold P.

Following the same notations as previously, let γ be a unit speed geodesic in M meeting P orthogonally at a point m. Assume that $\gamma(0)=m$ and $\gamma'(0)\in P_m^\perp$. Then, [12], there is a system of Fermi coordinates $(x_1, \cdots, x_n)$ such that for small s we have

$$\left(\frac{\partial}{\partial x_{q+1}}\right)_{\gamma(s)}=\gamma'(s), \quad \left(\frac{\partial}{\partial x_\alpha}\right)_m\in P_m, \quad \left(\frac{\partial}{\partial x_i}\right)_m\in P_m^\perp, \quad \alpha=1, \cdots, q, \; i=q+1, \cdots, n.$$

Furthermore, let $\{E_1, \cdots, E_n\}$ be an orthonormal basis of M_m such that the E_α, $\alpha=1, \cdots, q$ span P_m and

$$\left(\frac{\partial}{\partial x_\alpha}\right)_m=E_\alpha, \quad \left(\frac{\partial}{\partial x_i}\right)_m=E_i, \quad i=q+2, \cdots, n. \tag{3.14}$$

Assume that $\{e_1, \cdots, e_n\}$ be the parallel orthonormal frame field along γ obtained by parallel translation of $\{E_1, \cdots, E_n\}$. Then the following are Jacobi vector fields:

$$\left(\frac{\partial}{\partial x_1}\right)_{\gamma(s)}, \cdots, \left(\frac{\partial}{\partial x_q}\right)_{\gamma(s)}; \; s\left(\frac{\partial}{\partial x_{q+2}}\right)_{\gamma(s)}, \cdots, s\left(\frac{\partial}{\partial x_n}\right)_{\gamma(s)}. \tag{3.15}$$

Now consider, as previously, the $n-1$ Jacobi vector fields $Y_1(s), \cdots, Y_q(s)$; $Y_{q+2}(s), \cdots, Y_n(s)$ along γ, uniquely determined by the initial conditions

$$(3.16)\qquad \begin{aligned} &Y_\alpha(0)=E_\alpha, \quad Y'_\alpha(0)=0\,; \quad \alpha=1, \cdots, q\,; \\ &Y_i(0)=0, \quad Y'_i(0)=E_i, \quad i=q+2, \cdots, n. \end{aligned}$$

Then it is obvious that

$$(3.17)\qquad Y_\alpha(s)=\left(\frac{\partial}{\partial x_\alpha}\right)_{\gamma(s)}=X_\alpha(s), \quad Y_i(s)=s\left(\frac{\partial}{\partial x_i}\right)_{\gamma(s)}=sX_i(s)$$

since $X_\alpha(s)$, $sX_i(s)$ satisfy the initial conditions (3.16), where $X_\alpha(s)$, $X_i(s)$ are the coordinate Fermi vector fields along γ.

Now define the endomorphism-valued function $s\mapsto B(s)$ by

$$(3.18)\qquad Y_\alpha(s)=(Be_\alpha)(s), \quad \alpha=1, \cdots, q\,; \, q+2, \cdots, n.$$

Since the Y_α's are Jacobi vector fields, B satisfies

$$(3.19)\qquad B''(s)=R(s)\circ B(s)$$

where $R(s)$ is defined by (3.8). Using now (3.16) the endomorphism $B(s)$ relative to the basis $\{E_1, \cdots, E_q\,; E_{q+2}, \cdots, E_n\}$ satisfies the following initial conditions

$$(3.20)\qquad B(0)=\operatorname{diag}(I_q, 0), \quad B'(0)=\operatorname{diag}(0, I_{n-q-1})\,.$$

Denote by $S(s)$ the shape operator of the tubular hypersurface P_s. Then

$$S(s)Y_\alpha(s)=-Y'_\alpha(s), \quad \alpha=1, \cdots, q\,; \, q+2, \cdots, n$$

and using (3.18) we obtain again

$$(3.21)\qquad S(s)=-B'(s)B^{-1}(s)\,.$$

This is the required relation which expresses the shape operator of a tube P_s, in terms of Jacobi vector fields.

4. The main results

Theorem 4.1. *Let σ be a geodesic of finite length of a connected Riemannian manifold (M, g) of dimension $n>2$ and of constant sectional curvature c. Then every sufficiently small tube about σ of radius s is a quasi-umbilical hypersurface of M.*

Proof. Suppose that the Riemannian manifold M has constant sectional curvature c. Then [14]

$$R(X, Y)Z=c(g(Y, Z)X-g(X, Z)Y)$$

for $X, Y, Z\in\mathfrak{X}(M)$.

Following the same notation as previously, setting $X=N$ (which is the tangent vector field along γ) and $Y=Y_\alpha(s)$, $\alpha=1, 3, \cdots, n$ (which are orthogonal to γ), we get

$$R(N, Y_\alpha)N=-cY_\alpha, \quad \alpha=1, 3, \cdots, n. \tag{4.1}$$

As a consequence, now of this equation and (3.10) we get for the Jacobi vector fields along γ, the following differential equation:

$$Y''_\alpha(s)=-cY_\alpha(s), \quad \alpha=1, 3, \cdots, n. \tag{4.2}$$

Suppose that $c>0$. Then this equation can be solved explicitly to give

$$Y_\alpha=Y_\alpha(s)=\sum_{\substack{b=1\\ b\neq 2}}^{n}(c_{\alpha b}\cos\sqrt{c}s+c'_{\alpha b}\sin\sqrt{c}s)e_b(s), \quad \alpha, b=1, 3, \cdots, n. \tag{4.3}$$

Where $c_{\alpha b}$, $c'_{\alpha b}$ are constants of integration. Applying now the initial conditions (3.4) we have easily

$$c_{\alpha b}=\begin{cases} 1, & \text{if } \alpha=b=1 \\ 0, & \text{otherwise,}\end{cases} \quad c'_\alpha=\begin{cases} \dfrac{1}{\sqrt{c}}, & \text{if } \alpha=b=3, \cdots, n \\ 0, & \text{otherwise.}\end{cases}$$

Hence

$$\begin{aligned} &Y_1(s)=(\cos\sqrt{c}s, 0, \cdots, 0)^t, \\ &Y_3(s)=\left(0, \frac{\sin\sqrt{c}s}{\sqrt{c}}, 0, \cdots, 0\right)^t, \cdots, Y_n(s)=\left(0, \cdots, 0, \frac{\sin\sqrt{c}s}{\sqrt{c}}\right)^t. \end{aligned} \tag{4.4}$$

Therefore, the representation of $B(s)$ with respect to the orthonormal frame $\{e_1; e_3, \cdots, e_n\}$ is

$$B(s)=\operatorname{diag}\left(\cos\sqrt{c}s, \frac{1}{\sqrt{c}}\sin\sqrt{c}s, \cdots, \frac{1}{\sqrt{c}}\sin\sqrt{c}s\right) \tag{4.5}$$

Now applying the relation (3.13), we get, for the shape operator $S(s)$ of the tube P_s, the following $(n-1)\times(n-1)$ matrix:

$$S(s)=\operatorname{diag}(\sqrt{c}\tan\sqrt{c}s, -\sqrt{c}\cot\sqrt{c}s, \cdots, -\sqrt{c}\cot\sqrt{c}s). \tag{4.6}$$

Therefore, the shape operator has two distinct eigenfunctions:

$$k_1=\sqrt{c}\tan\sqrt{c}s$$

of multiplicity one, and

$$k_2=-\sqrt{c}\cot\sqrt{c}s$$

of multiplicity $n-2$. Now, since the dimension of the tube is $n-1$, we conclude that it is a quasi-umbilical hypersurface.

For the negative curvature case ($c<0$), in order to take the shape operator $S(s)$, it suffices in (4.6) to change the trigonometric functions into the corresponding hyperbolic functions, instead of c put $|c|$ and, instead of $\sqrt{c}\tan\sqrt{c}s$

put $-\sqrt{|c|}\tanh\sqrt{|c|}s$. So we again obtain the same conclusion.

For the zero curvature case ($c=0$), one now easily obtains that the shape operator $S(s)$ also has two distinct eigenfunctions, $k_1=0$ of multiplicity one and $k_2=-1/s$ of multiplicity $n-2$ and, therefore, we also get the same result and the proof of the Theorem is complete.

We now prove the converse of the above theorem, namely:

Theorem 4.2. *Suppose that, for every geodesic σ of finite length of a connected Riemannian manifold (M, g) of dimension >2, every sufficiently small tube about σ of radius s is a quasi-umbilical hypersurface of M, and the eigenspace of dimension $n-2$ of the corresponding shape operator is parallel along every unit speed geodesic γ meeting σ orthogonally. Then M has constant sectional curvature.*

Proof. Let $m=\sigma(0)$ be a point of M and denote by γ the unit speed geodesic of M which meets σ orthogonally at m, with $\gamma(0)=m$. Now our hypothesis of quasi-umbilicity of every sufficiently small tube about σ, gives that the shape operator $S(s)$, for every small s, of each tube will have two smooth distinct eigenfunctions, say $k_1=k_1(s)$ of multiplicity one and $k_2=k_2(s)$ of multiplicity $n-2$. Let $\{\varepsilon_1(s), \varepsilon_2(s), \cdots, \varepsilon_n(s)\}$ be the parallel orthonormal frame field along γ such that

$$S(s)\varepsilon_1(s)=k_1(s)\varepsilon_1(s),\quad S(s)\varepsilon_i(s)=k_2(s)\varepsilon_i(s),\quad i=3, \cdots, n. \tag{4.7}$$

and let $\varepsilon_2(s)=\gamma'(s)=N(s)$.

Now using the equation (3.13), we obtain

$$B'(s)=-S(s)B(s).$$

Differentiate and use this relation again to get

$$B''(s)=(S^2(s)-S'(s))B(s)$$

or equivalently

$$Y_i''(s)=(S^2(s)-S'(s))Y_i(s),\quad i=1, \cdots, n$$

where Y_i, $i=1, 3, \cdots, n$ are the Jacobi vector fields along γ perpendicular to N. But the Y_i's also satisfy the Jacobi vector field equation

$$Y_i''(s)=R(N(s), Y_i(s))N(s).$$

Therefore, we get on $\gamma-\{m\}$

$$(S^2(s)-S'(s))Y_i(s)=R(N(s), Y_i(s))N(s). \tag{4.8}$$

From (4.8) we conclude that along $\gamma-\{m\}$ for every Y perpendicular to N

$$(S^2(s)-S'(s))Y=R(N, Y)N. \tag{4.8a}$$

Using the equation (4.7), we now obtain

$$(4.9)\qquad (k_1^2-k_1')\varepsilon_1(s)=R(N,\ \varepsilon_1)N,\quad (k_2^2-k_2')\varepsilon_i(s)=R(N,\ \varepsilon_i)N\,;\quad i=3,\ \cdots,\ n\,.$$

From the last equations it is obvious that $\varepsilon_1(s)$ and $\varepsilon_i(s)$, $i=3, \cdots, n$ are eigen-vector fields of the mapping $R(N, -)N$ along $\gamma-\{m\}$, corresponding to the eigen-functions $k_1^2(s)-k_1'(s)$ of multiplicity one and $k_2^2(s)-k_2'(s)$ of multiplicity $n-2$.

By continuity, at the point m we now have

$$(4.10)\qquad R(E_2,\ E_1)E_2=k(E_2,\ E_1)E_1,\quad R(E_2,\ E_i)E_2=k(E_2,\ E_i)E_i,\quad i=3,\ \cdots,\ n.$$

Since we can take any geodesic of M for σ and any geodesic meeting orthogonally to σ for γ, as a consequence of (4.10), we have

$$(4.11)\qquad R(Y,\ X)Y=k(Y,\ X)X$$

for any orthogonal pair of unit tangent vectors $\{X, Y\}$. In the following, we will show that the function $k(Y, X)$ does not depend on the choice of $\{X, Y\}$ and is a function on M. First we fix Y. Let X_1, X_2 be orthogonal vectors perpendicular to Y. It follows from

$$(4.12)\qquad \begin{aligned} R(Y,\ X_1+X_2)Y&=R(Y,\ X_1)Y+R(Y,\ X_2)Y\\ &=k(Y,\ X_1)X_1+k(Y,\ X_2)X_2 \end{aligned}$$

and

$$(4.13)\qquad R(Y,\ X_1+X_2)Y=k(Y,\ X_1+X_2)(X_1+X_2)$$

that

$$(4.14)\qquad k(Y,\ X_1)=k(Y,\ X_2)=k(Y,\ X_1+X_2)\,.$$

This shows that the function $k(Y, X)$ does not depend on X and (4.11) can be written as

$$(4.15)\qquad R(Y,\ X)Y=k(Y)X\,.$$

Since $g(R(Y, X)Y, X)=g(R(X, Y)X, Y)$, we have $k(Y)=k(X)$, which shows that k is a function on M. By Schur's theorem ([14]), k must be constant on M and the proof is completed.

Theorem 4.3. *Let (M, g) be a connected n-dimensional $(n>2)$ Riemannian manifold of constant sectional curvature c and P be a q-dimensional submanifold of M obtained by $\exp_m(V\cap N(m\,;r))$, where $m\in M$, V is a q-dimensional subspace of M_m, $N(m\,;r)$ is a neighborhood at m in M_m and r is a positive number. Then the shape operator of every sufficiently small tube about P of radius s has two distinct eigenfunctions, of multiplicity q and $n-q-1$ respectively.*

Proof. Since the Riemannian manifold M has constant sectional curvature c, we have

$$R(X, Y)Z=c(g(Y, Z)X-g(X, Z)Y) \tag{4.17}$$

for $X, Y, Z\in\mathfrak{X}(M)$.

Following the same notation, as previously, setting $X=\gamma'(s)$ and $Y=Y_\alpha(s)$, $\alpha=1, \cdots, q; q+2, \cdots, n$ we get

$$R(\gamma'(s), Y_\alpha(s))\gamma'(s)=-cY_\alpha(s). \tag{4.18}$$

Now, as a consequence of this equation, and that the Y_α's are Jacobi fields along γ, we deduce that

$$Y''_\alpha(s)=-cY_\alpha(s), \quad \alpha=1, \cdots, q; q+2, \cdots, n. \tag{4.19}$$

Suppose now that $c>0$. Then this equation can be solved explicitly, to give

$$Y_\alpha(c) \sum_{\substack{b=1\\ b\neq q+1}}^{n} (c_{\alpha b}\cos\sqrt{c}s+c'_{\alpha b}\sin\sqrt{c}s)e_b, \quad 1\leqq\alpha\neq q+1\leqq n \tag{4.20}$$

where each e_b is parallel along the geodesic γ.

Now using the initial conditions (3.20), we get

$$Y_1(s)=X_1(s)=(\cos\sqrt{c}s, 0, \cdots, 0)^t, \cdots, Y_q(s)=X_q(s)=(0, \cdots, \cos\sqrt{c}s, \cdots, 0)^t$$

$$Y_{q+2}(s)=sX_{q+2}(s)=\left(0, \cdots, \frac{\sin\sqrt{c}s}{\sqrt{c}}, \cdots, 0\right)^t, \cdots, Y_n(s)=sX_n(s)$$

$$=\left(0, \cdots, 0, \frac{\sin\sqrt{c}s}{\sqrt{c}}\right)^t.$$

Hence, the representation of $B(s)$ with respect to the orthonormal frame field $\{e_1, \cdots, e_q; e_{q+2}, \cdots, e_n\}$ is

$$B(s)=\mathrm{diag}\left(\cos\sqrt{c}s, \cdots, \cos\sqrt{c}s, \frac{1}{\sqrt{c}}\sin\sqrt{c}s, \cdots, \frac{1}{\sqrt{c}}\sin\sqrt{c}s\right).$$

Now, easily evaluating $B'(s)$ and $B^{-1}(s)$, and using (3.21) we get, for the shape operator $S(s)$, of each tube P_s:

$$S(s)=\mathrm{diag}(\sqrt{c}\tan\sqrt{c}s, \cdots, \sqrt{c}\tan\sqrt{c}s, -\sqrt{c}\cot\sqrt{c}s, \cdots, -\sqrt{c}\cot\sqrt{c}s). \tag{4.21}$$

Therefore, the shape operator has two distinct eigenfunctions, k_1, k_2 such that

$$k_1=\sqrt{c}\tan\sqrt{c}s$$

of multiplicity $q\geqq 2$ and

$$k_2=-\sqrt{c}\cot\sqrt{c}s$$

of multiplicity $n-q-1<n-2$.

For the negative curvature case ($c<0$) and for the zero curvature case ($c=0$)

the proof is similar to that of Theorem 4.1, so we omit it.

Next, we state the converse of the above theorem, namely:

Theorem 4.4. *Let P be a q-dimensional submanifold of a connected n-dimensional ($n>2$) Riemannian manifold M obtained, for $m\in M$, by* $\exp_m(V\cap N(m;r))$, *where V is a q-dimensional subspace of M_m, $N(m;r)$ is a neighborhood at m in M_m, and r a small positive number. Assume that the shape-operator of every sufficiently small tube about P of radius s for every q-dimensional submanifold P has a $(n-q-1)$-dimensional parallel eigenspace, along the corresponding unit speed geodesic γ of M meeting P orthogonally at M. Then M has constant curvature.*

Proof. Let $m=\sigma(0)$ be a point of M and denote by γ the unit speed geodesic of M which meets P orthogonally at m, with $\gamma(0)=m$. Let $k_1=k_1(s)$ and $k_2=k_2(s)$, be the distinct eigenfunctions of multiplicity q and $n-q-1$ respectively, of the shape operator $S(s)$ for every small s of each tube P_s. Let $\{\varepsilon_1(s),\cdots,\varepsilon_q(s);\varepsilon_{q+2}(s),\cdots,\varepsilon_n(s)\}$ be the parallel orthonormal frame field along γ such that

$$S(s)\varepsilon_\alpha(s)=k_1(s)\varepsilon_\alpha(s),\quad S(s)\varepsilon_i(s)=k_2(s)\varepsilon_i(s);\quad \alpha=1,\cdots,q;\ i=q+2,\cdots,n \tag{4.22}$$

and let $\varepsilon_{q+1}(s)=\gamma'(s)$. Now using the equation (3.21) we obtain

$$B'(s)=-S(s)B(s)$$

from which by differentiation we have

$$B''(s)=(S^2(s)-S'(s))B(s)$$

or equivalently

$$Y_i''(s)=(S^2(s)-S'(s))Y_i(s);\quad i=1,\cdots,q;\ q+2,\cdots,n$$

where Y_i are the Jacobi vector fields along γ. But the Y_i's also satisfy the Jacobi field equation

$$Y_i''(s)=R(N(s),Y_i(s))N(s)$$

where $N(s)=\gamma'(s)$. Therefore, we get on $\gamma-\{m\}$

$$(S^2(s)-S'(s))Y_i(s)=R(N(s),Y_i(s))N(s). \tag{4.23}$$

From this equation one concludes that along $\gamma-\{m\}$ for every Y perpendicular to N

$$(S^2(s)-S'(s))Y=R(N,Y)N.$$

Hence, along $\gamma-\{m\}$, since the ε_α's, $\alpha=1,\cdots,n$, $\alpha\neq q+1$, are parallel, from this equation and (4.22) we get

(4.24) $$(k_1^2(s)-k_1'(s))\varepsilon_\alpha(s)=R(N,\ \varepsilon_\alpha(s))N\,;\quad \alpha=1,\ \cdots,\ q$$

(4.25) $$(k_2^2(s)-k_2'(s))\varepsilon_i(s)=R(N,\ \varepsilon_i(s))N\,;\quad i=q+2,\ \cdots,\ n\,.$$

From these equations, it is obvious that $\varepsilon_\alpha(s)$ and $\varepsilon_i(s)$ are eigenvector fields of the mapping $R(N, -)N$, along $\gamma-\{m\}$ with corresponding eigenfunctions $k_1^2(s)-k_1'(s)$ of multiplicity q and $k_2^2(s)-k_2'(s)$ of multiplicity $n-q-1$. By continuity, at the point m we will now have:

(4.26) $$R(E_2,\ E_\alpha)E_2=k(E_2,\ E_\alpha)E_\alpha,\quad R(E_2,\ E_i)E_2=k(E_2,\ E_i)E_i\,.$$

Now, following the same method which we developed in the proof of Theorem 4.2, we conclude that the functions k are constant and hence the manifold is of constant curvature and the proof of the Theorem is complete.

Remark 1. Now it seems that this method of tubes may be applied to the Kähler manifolds of constant holomorphic sectional curvature, as well as possibly, to locally symmetric spaces and to Sasakian manifolds of constant φ-sectional curvature.

Remark 2. After this work had been completed I was informed by Professor L. Vanhecke, that his colleague, L. Gheysens, proved similar results in his Ph. D., which as far as I know, he hasn't published yet.

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