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ON VON NEUMANN REGULARITY, INJECTIVITY AND FLATNESS

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Summary. It is still unknown whether a SF-ring A (every simple left or right A-module is flat) is von Neumann regular. The following non-trivial generalization of regular rings is considered: write "A satisfies(*)" if, for any maximal right ideal M of A, every $y\in M$, A/yM is a flat right A-module. Conditions are given for such rings to be (a) von Neumann regular; (b) left self-injective regular; (c) right Artinian; (d) right Kasch; (e) ELT regular; (f) strongly regular.

Introduction.

It is well-known that A is von Neuman regular if, and only if, every left (right) A-module is flat. Von Neumann regular rings and their generalizations have drawn the attention of many authors since several years (cf. for example, [1], [5], [7], [9]-[23]).

Motivated by SF-rings (rings whose simple modules are flat), we study the following generalization of regular rings:

We say that "A satisfies (*)" if A has the following property: for any maximal right M of A, every $y\in M$, A/yM is a flat right A-module. This note contains the following results: (1) Let A satisfy (*). Then (i) if A is left self-injective, then A is either von Neuman regular or a left pseudo-Frobeniusean local ring with the square of the Jacobson radical zero; (ii) If every principal left ideal of A is projective, then A is von Neumann regular; (iii) A is right Artinian if, and only if, A is right Noetherian; (iv) If A contains an injective maximal left ideal, then A is left self-injective regular; (2) If A is a MELT fully idempotent ring whose essential left ideals are idempotent, then A is regular; (3) A left or right quasi-due fully idempotent ring satisfying (*) is strongly regular.

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Throughout, A denotes an associative ring with identity and A-modules are unital. J will denote the Jacobson radical of A. $Z=\{z\in A\setminus Lz=0$ for some essential left ideal $L\}$ is called the lef singular ideal of A and A is called left non-singular if $Z=0$. Following G. O. MICHLER-O. E. VILLAMAYOR, A is called semi-simple if $J=0$. As in previous papers, we write:

A is ELT (resp. MELT) if every essential (resp. maximal essential, if it exists) left ideal of A is an ideal. (An ideal of A will always mean a two-sided ideal of A). Similarly, ERT and MERT rings are defined on the right side. As usual, (1) A is called left (resp. right) quasi-duo if every maximal left (resp. right) ideal of A is an ideal; (2) A is called fully idempotent (resp. (a) fully left idempotent, (b) fully right idempotent) if every ideal (resp. (a) left ideal, (b) right ideal) of A is idempotent. Note that if A is fully idempotent, it is not necessary that $J=0$ (i.e. A semi-simple) (cf. [20, p. 1068]). Consequently, [18, Proposition 1(5)] should be modified as follows:

Remark 1. A is ELT regular iff every factor ring of A is a semisimple MELT left p-injective ring.

Question: Is a MELT fully idempotent left self-injective ring von Neumann regular?

Remark 2. The ring constructed by ZHANG in [22, Theorem 1] is neither ELT nor ERT.

Thus MELT rings effectively generalize ELT and left quasi-duo rings (cf. Proposition 9 and Corollary 10).

The question whether right SF-rings are regular is still open. Obviously, von Neumann regular rings satisfy (*). Also, if A is a local ring with non-zero J such that $J^2=0$, then A also satisfies (*). Consequently, a ring satisfying (*) needs not be von Neumann regular. Recall that A is a left (resp. right) Kasch ring if every maximal left (resp. right) ideal of A is an annihilator [7]. It is well-known that left self-injective left Kasch rings are left pseudo-Frobeniusean.

Proposition 1. *The following conditions are equivalent for a left self-injective ring A:*

(1) *A is either regular or A is a left pseudo-Frobeniusean local ring with $J\neq 0$, $J^2=0$;*

(2) *A satisfies* (*).

Proof. Obviously, (1) implies (2).

Assume (2). If A is not regular, then $J\neq 0$. Let $0\neq y\in J$. For any maximal right ideal M of A, $y \in M$ and since A/yM_A is flat, then for any $b \in M$, $yb = ycyb$ for some $c \in M$ (cf. [3, Proposition 2.1]). Therefore $(1-yc)yb=0$ and since $yc \in J$, there exists $w \in A$ such that $w(1-yc)=1$ which yields $yb = w(1-yc)yb=0$, showing that $M=r(y)$. This proves that A is a local ring and J is the unique maximal left (or right) ideal of A. The proof also shows that $J^2=0$. Thus A is left Kasch which implies that A is left pseudo-Frobeniusean. Therefore (2) implies (1).

The proof of Proposition 1 yields the right injective analogue and a similar result for semi-perfect rings.

Proposition 2. *If A is a right self-injective ring satisfying* (*), *then either A is regular or A is a right pseudo-Frobeniusean local ring.*

Proposition 3. *A semi-perfect ring satisfying* (*) *is either completely primary or semi-simple Artinian.*

We now give a nice characterization of regular rings.

Theorem 4. *The following conditions are equivalent:*
(1) *A is von Neumann regular;*
(2) *A is left pp ring satisfying* (*).

Proof. Since every principal left ideal in a regular ring is projective, then (1) implies (2).

Assume (2). Let $F=b_1A+b_2A+ \cdots +b_nA$ be a finitely generated proper right ideal of A, M a maximal right ideal containing F. First suppose that for every $a \in M$, aA is generated by an idempotent. Then any finitely generated right ideal of A contained in M is generated by an idempotent. In particular, $F=eA$, $e=e^2 \in A$. Since F is a proper right ideal, $l(F)=A(1-e) \neq 0$. Now suppose there exists $c \in M$ such that cA is not generated by an idempotent. If $cM=0$, then $c \in l(F)$ implies that $l(F) \neq 0$. If $cM \neq 0$, A/cM_A is flat by hypothesis. Given $cb_1, cb_2, \cdots, cb_n \in cM$, by [3, Proposition 2.2], there exists a right A-homomorphism $f: A \to cM$ such that $f(cb_i)=cb_i$ for each i, $1 \leqq i \leqq n$. If $f(1)=d \in cM$, $d=cu$ for some $u \in M$, and $cb_i=dcb_i$ for each i, $1 \leqq i \leqq n$. Therefore, $(1-d)cF=0$. If $(1-d)c=0$, then $c=dc=cuc$ implies that cA is generated by an idempotent, contradicting our hypothesis. Thus $(1-d)c \neq 0$ which again shows that $l(F) \neq 0$. We have just proved that any proper finitely generated right ideal of A has a non-zero left annihilator and by [2, Theorem 5.4], a finitely geneerated projective submodule of any projective left A-module is a direct summand. Since A is left pp, every principal left ideal is a direct summand of ${}_AA$ and hence (2) implies (1).

Corollary 5. *The following conditions are equivalent:*

(1) *A is left self-injective regular;*

(2) *A is left non-singular ring satisfying* (*) *such that every finitely generated non-singular left A-module is projective.*

Proof. (1) implies (2) by [23, Corollary 6].

Assume (2). Then A is left semi-hereditary satisfying (*) and by Theorem 4, A is regular. If ${}_AH$ is the injective hull of ${}_AA$, for any $h \in H$, $F = A + Ah$ is a finitely generated non-singular left A-module which is therefore projective. Since every finitely generated proper right ideal of A has non-zero left annihilator by [2, Theorem 5.4], ${}_AA$ is a direct summand of ${}_AF$. But ${}_AA$ is essential ${}_AF$ which implies that $A = F$, whence $A = H$. Thus (2) implies (1).

Rings whose essential right ideals are idempotent two-sided ideals need not be regular (cf. [12, p. 701]).

Proposition 6. *Let A satisfy* (*). *Then A is right Artinian iff A is right Noetherian.*

Proof. One implication is obvious.

Now assume that A is right Noetherian satisfying (*). If $J = 0$, then A satisfies the maximum condition on left and right annihilators. By [6, Theorem], any proper essential right ideal E of A contains a non-zero-divisor c. Let M be a maximal right ideal of A containing E. Since A/cM_A is flat, then from [3, Proposition 2.1], $c^2 = cdc^2$ for some $d \in M$. Now $l(c) = 0$ implies that $1 = cd \in M$, contradicting $M \neq A$. This proves that A contains no proper essential right ideal which implies that A is semi-simple Artinian. If $J \neq 0$, the proof of Proposition 1 shows that A is a local ring with $J^2 = 0$. Then A is right Artinian by [7, Corollary 11.6.4].

Remark 3. A is right Artinian iff A is right Noetherian such that for every prime factor ring B of A, either B satisfies (*) or every essential right ideal of B is an idempotent two-sided ideal.

We mention another case when Noetherian rings are Artinian.

Proposition 7. *Let A be either MELT or MERT. If A is right Noetherian and every right ideal of A is a right annihilator, then A is right Artinian.*

Proof. Since A is right Noetherian whose right ideals are right annihilators, then A/V is a right V-ring [4, Theorem 2.3]. Therefore A/J is a MELT or MERT right V-rings which is von Neumann regular in any case (cf.

[1]). It follows that A/J is semi-simple Artinian and by [4, Proposition 3.3(iii)], A is right Artinian.

Applying [8, Proposition 1], we get

Corollary 8. *If A is either MELT or MERT, then A is quasi-Frobeniusean iff A is right Noetherian such that every right ideal is a right annihilator and every minimal left ideal is a left annihilator.*

If all maximal and all essential left ideals of A are idempotent two-sided ideals, A needs not be regular.

Example. Let $A=\begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$, where K is a field. Then $M=\begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$, $N=\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$ are maximal left ideals which are idempotent two-sided. N is the unique proper essential left ideal of A. ${}_AM$ is injective but ${}_AA$ is not injective (cf. [12], [22]).

Remark 4. In the above example, A/J is a strongly regular ring.

Although MELT fully idempotent rings need not be regular [22], the following holds.

Proposition 9. *Let A be a MELT fully idempotent ring whose essential left ideals are idempotent. Then A is regular.*

Proof. Let B be a prime factor ring of A. Then every essential left ideal of B is idempotent. For any $0\neq t\in B$, if $T=BtB$, K a complement left subideal of T such that $Bt\oplus K$ is an essential left subideal of T. Since ${}_BT$ is essential in ${}_BB$, then ${}_BBt\oplus K$ is essential in ${}_BB$ which implies that $(Bt\oplus K)^2=Bt\oplus K$. Now $t=\sum_{i=1}^n (b_it+k_i)(c_it+s_i)$, $b_i, c_i\in B$, $k_i, s_i\in K$, and therefore $t-\sum_{i=1}^n (b_it+k_i)c_it=\sum_{i=1}^n (b_it+k_i)s_i\in Bt\cap K=0$. Thus $t\in Tt$ which proves that B is fully left idempotent. Since B is MELT fully left idempotent, then B is regular by [1, Theorem 3.1].

By [5, Corollary 1.18], A is regular.

The next corollary improves [12, Theorem 3].

Corollary 10. *If A is ELT fully idempotent, then A is regular.*

Proposition 11. *If A is a ring satisfying (*) such that the right singular ideal is non-zero, then A is right Kasch.*

Proof. Let Y denote the right singular ideal of A. For any maximal right ideal M of A, since Y cannot contain a non-zero idempotent, then $Y\cap M\neq 0$. If $0\neq y\in Y\cap M$, since A/yM_A is flat, for any $b\in M$, $yb=ycyb$ for some $c\in M$. Since $r(yc)\cap ybA=0$, then $yb=0$ which yields $yM=0$, whence $M=r(y)$. This proves that A is right Kasch.

Rings containing an injective maximal left ideal need not be left self-injective (cf. above example).

Theorem 12. *Let A be a ring satisfying* (*) *and containing an injective maximal left ideal. Then A is left self injective regular.*

Proof. Let M be an injective maximal left ideal of A. Then $A=M\oplus V$, where V is a minimal left ideal of A. If $M=Ae$, $e=e^2\in A$, $V=Av$, $v=1-e$, then $r(M)=vA$. Suppose that $J\neq 0$. Since A satisfies (*), then $J^2=0$ and A is a local ring (cf. Proposition 1). In that case, $J=M=Ae$, which contradicts the fact that J cannot contain a non-zero idempotent. This proves that $J=0$. Therefore A is semi prime and $r(M)=vA$ must be a minimal right ideal of A.

By [19, Theorem 1.2], A is left self-injective.

Since $J=0$, A is von Neumann regular.

Proposition 13. *If A satisfies* (*) *and is either left or right SF, then $J=0$.*

Proof. Suppose that $J\neq 0$. Then A is a local ring with $J^2=0$ (cf. Proposition 1). Since A is left and right quasi-duo, whether A is left or right SF, A must be strongly regular by [14, Theorem 1.7], which contradicts $J\neq 0$. This proves the proposition.

Following [16], A is called a left GQ-injective ring if, for any left ideal C isomorphic to a complement left ideal of A, every left A-homomorphism of C into A extends to an endomorphism of ${}_AA$.

Applying [16, Proposition 1], we get

Corollary 14. *Let A be left GQ-injective ring. The following conditions are then equivalent:*

(1) *A is von Neumann regular;*

(3) *A is a left SF-ring satisfying* (*);

(3) *A is a right SF-ring satisfying* (*).

Strongly regular rings are now characterized in terms of rings satisfying (*).

Proposition 15. *The following conditions are equivalent:*

(1) *A is strongly regular;*

(2) *A is left quasi-duo, fully idempotent ring satisfying* (*);

(3) *A is right quasi-duo, fully idempotent ring satisfying* (*).

Proof. (1) implies (2) and (3) evidently.

Assume (2). Let B be any factor ring of A. Then B is a semi-prime left quasi-duo ring satisfying (*). Since B is semi-prime, the proof of Proposition 1 shows that $J(B)$, the Jacobson radical of B, must be zero. Since $J(B)=0$ and B is left quasi-duo, then B must be a reduced ring (cf. the proof of "(2) implies (3)" in [15, Theorem 2.1]). For any prime factor ring P of A, P (being reduced) is an integral domain by [13, Proposition 6]. Suppose there exists $0\neq u\in P$ such that $uP\neq P$. Let R be a maximal right ideal of P containing uP. Since P satisfies (*), P/uRu_P is flat. If $uR=0$, $R=r(u)$ and uP ($\approx P/r(u)$) is a minimal right ideal of P which is generated by an idempotent e, whence $e=1$ (P being an integral domain) which contradicts $uP\neq P$. If $ur\neq 0$ for some $r\in R$, $ur=utur$, $t\in R$, and again $1=ut\in R$, which contradicts $R\neq P$. This proves that P is a division ring. Since A is fully idempotent, then A is regular by [5, Corollary 1.18] and hence (2) implies (1).

Similarly, (3) implies (1).

Corollary 16. *If A is a MELT fully idempotent ring satisfying* (*), *then A is ELT and regular.*

Proof. Since A is semi-prime, the left (and right) socle S of A is a fully left (and right) idempotent ring. Since A/S is a left quasi-duo fully idempotent ring satisfy (*), then A/S is strongly regular by Proposition 15. Therefore A is ELT and von Neumann regular [17, Remark 1(b)].

Remark 5. If A is a fully idempotent ring satisfying (*), then any factor ring of A is semi-simple.

References

[1] G. Baccella, Generalized V-rings and von Neumann regular rings. Rend. Sem. Mat. Univ. Padova **72** (1984), 117–133.

[2] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings. Trans. Amer. Math. Soc. **95** (1960), 466–488.

[3] S.U· Chase, Direct product of modules. Trans. Amer. Math. Soc. **97** (1960), 457–473.

[4] C. Faith and P. Menal, A counter-example to a conjecture of Johns. Proc. Amer. Math. Soc. **116** (1992), 21–26.

[5] K.R. Gooderl, Von Neumann regular rings. Pitman (1979).

[6] R.E. Johnson and L.S. Levy, Regular elements in semi-prime rings. Proc. Amer. Math. Soc. **19** (1968), 961–963.

[7] F. Kasch, Modules and rings. London Math. Soc. Monograph **17** (translated by D.A.R. Wallace) (1982).

[8] H.H. Storrer, A note on quasi-Frobenius rings and ring epimorphisms. Canad. Math. Bull. **12** (1969), 287-292.
[9] R. Wisbauer, Foundations of module and ring theory. Gordon and Breach (1991).
[10] Xiao Yufei, One sided SF-rings with certain conditions. Canad. Math. Bull. **37** (1994), 272-277.
[11] Xiao Yufei, SF-rings and excellent rings. Comm. Alg. **22** (1994), 2463-2471.
[12] Xue Weimin, Three questions on strongly π-regular rings and regular rings. Comm. Algebra **21** (1993), 699-704.
[13] R. Yuechiming, On V-rings and prime rings. J. Algebra. **62** (1980), 13-20.
[14] R. Yuechiming, On von Neumann regular rings, V. Math. J. Okayama Univ. **22** (1980), 151-160.
[15] R. Yuechiming, On von Neumann regular rings, VI. Rend. Sem. Mat. Univ. Torino **39** (1981), 75-84.
[16] R. Yuechiming, On quasi-injectivity and von Neumann regularity. Mh. Math. **95** (1982), 25-32.
[17] R. Yuechiming, Annihilators and strongly regular rings. Rend. Sem. Fac. Sci. Univ. Cagliari **57** (1987), 51-59.
[18] R. Yuechiming, On von Neumann regular rings, XV. Act a Math. Vietnamica **13** (1988), 71-79.
[19] R. Yuechiming, A note on injective rings. Hokkaido Math. J. **21** (1992), 231-238.
[20] Zhang Jule, Fully idempotent rings whose every maximal left ideal in a ideal. Chinese Sc. Bull. **37** (1992), 1065-1068.
[21] Zhang Jule, Von Neumann regularity of SF-rings. Comm. Algebra **21** (1993), 2445-2451.
[22] Zhang Jule, Hereditary rings containing an injective maximal left ideal. Comm. Algebra **21** (1993), 4473-4479.
[23] J. Zelmanowitz, Injective hulls of torsionfree modules. Canad. J. Math. **23** (1971), 1094-1101.

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