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SOME CONDITIONS FOR UNIVALENCE DEFINED BY RUSCHEWEYH'S DERIVATIVE

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Abstract. For a function $f(z)=z+a_2z^2+\cdots$ analytic in the unit disc we give certain criteria for univalence in terms of $(D^nf(z))/z$, where D^nf denotes the Ruscheweyh derivative.

1. Introduction

Let A denote the class of functions analytic in the unit disc $U=\{z: |z|<1\}$ with $f(0)=f'(0)-1=0$.

In his paper [2] Ruscheweyh proved that if $f\in A$ and

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^nf(z)}\right\}>\frac{1}{2},\ z\in U,\ n\in N_0=N\cup\{0\}=\{0, 1, 2, \cdots\}$$

where $D^nf=(z/(1-z)^{n+1})*f$ means the Hadamard product or convolution of the functions $z/(1-z)^{n+1}$ and f, then $f\in S^*(1/2)$, i.e. f is starlike function of order 1/2.

We note that for D^nf we have the identity

$$z(D^nf)'=(n+1)D^{n+1}f-nD^nf,\ n\in N_0. \tag{1}$$

In this paper we give some conditions for univalence in U by using $(D^nf)/z$. For the proofs of our results we need the following well-known lemma.

Lemma 1 (Jack [1]). *Let ω be nonconstant and analytic in U with $\omega(0)=0$. If $|\omega|$ attains its maximum value at a point z_0 on the circle $|z|=r<1$, we have $z_0\omega'(z_0)=k\omega(z_0)$, $k\geqq 1$.*

2. Some conditions for univalence

Theorem 1. *Let $f\in A$ and*

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(2)
$$\operatorname{Re}\left\{\frac{D^n f(z)}{z}\right\}>\alpha_n, \qquad z\in U,\ n\in N,$$

where α_n *is defined by*

(3)
$$\alpha_n=\frac{1}{2n}[(2n+1)\alpha_{n-1}-1], \qquad n\geqq 2, \text{ and } \alpha_1=0,$$

then f *is univalent in* U *and* $\operatorname{Re}\{f'(z)\}>0$ $(z\in U)$.

Proof. Let us show that the following implication

(4)
$$\operatorname{Re}\left\{\frac{D^n f(z)}{z}\right\}>\alpha_n \Rightarrow \operatorname{Re}\left\{\frac{D^{n-1} f(z)}{z}\right\}>\alpha_{n-1} \ (n\geqq 2),$$

is true, where α_n is defined by (3). In that sense, let (2) be satisfied and let's put

(5)
$$\frac{D^{n-1}f(z)}{z}=\frac{1-(2\alpha_{n-1}-1)\omega(z)}{1-\omega(z)}.$$

Then ω is analytic in U and $\omega(0)=0$. From (5) by multiplying with z, differentiation, by using the identity (1) and some simple transformations we get

(6)
$$\frac{D^n f(z)}{z}=\frac{1-(2\alpha_{n-1}-1)\omega(z)}{1-\omega(z)}+\frac{2(1-\alpha_{n-1})}{n}\cdot\frac{z\omega'(z)}{(1-\omega(z))^2}.$$

We want to show that $|\omega(z)|<1$, $z\in U$. If not $|\omega(z)|<1$, then there exists a point z_0, $|z_0|<1$, such that $|\omega(z_0)|=1$ and $z_0\omega'(z_0)=k\omega(z_0)$, $k\geqq 1$ (where we use Lemma 1). If we put $\omega(z_0)=e^{i\theta}$, then $z_0\omega'(z_0)=ke^{i\theta}$ and from (6) for such z_0 we obtain

$$\begin{aligned}\operatorname{Re}\left\{\frac{D^n f(z_0)}{z_0}\right\}&=\operatorname{Re}\left\{\frac{1-(2\alpha_{n-1}-1)e^{i\theta}}{1-e^{i\theta}}+\frac{2(1-\alpha_{n-1})}{n}\frac{ke^{i\theta}}{(1-e^{i\theta})^2}\right\}\\&=\alpha_{n-1}+\frac{2(1-\alpha_{n-1})k}{n}\left(-\frac{1}{4\sin^2(\theta/2)}\right)\\&\leqq\alpha_{n-1}-\frac{1-\alpha_{n-1}}{2n}=\alpha_n,\end{aligned}$$

which is the contradiction to (2). From the implication (4) we conclude that $\operatorname{Re}\{D^n f(z)/z\}>\alpha_n$ implies $\operatorname{Re}\{D^1 f(z)/z\}=\operatorname{Re}\{f'(z)\}>0$, $z\in U$, i.e. f is univalent (close-to-convex) in U.

For example, for $n=2$ from Theorem 1 we have

Corolary 1. *If* $f\in A$ *and*

$$\operatorname{Re}\left\{f'(z)+\frac{1}{2}zf''(z)\right\}>-\frac{1}{4},\ z\in U,$$

then f is univalent in U with $\operatorname{Re}\{f'(z)\}>0$, $z\in U$.

Theorem 2. *Let $f\in A$ and let*

$$\left|\frac{D^n f(z)}{z}-1\right|<\frac{n+1}{2},\ z\in U,\ n\in N, \tag{7}$$

then f is univalent in U and $|f'(z)-1|<1$, $z\in U$.

Proof. Since (7) is equivalent to

$$\frac{D^n f(z)}{z}\prec 1+\frac{n+1}{2}z, \tag{8}$$

let show that (8) implies that $D^{n-1}f(z)/z\prec 1+(n/2)z$. If we put

$$\frac{D^{n-1}f(z)}{z}=1+\frac{n}{2}\omega(z), \tag{9}$$

then $\omega(0)=0$ and $\omega(z)$ is analytic in U. By using the same method as in Theorem 1, from (9) we obtain

$$\frac{D^n f(z)}{z}=1+\frac{n}{2}\omega(z)+\frac{1}{2}z\omega'(z). \tag{10}$$

We want to show that $|\omega(z)|<1$, $z\in U$. If not $|\omega(z)|<1$, then there exists (by Lemma 1) a point z_0, $|z_0|<1$, such that $|\omega(z_0)|=1$ and $z_0\omega'(z_0)=k\omega(z_0)$, $k\geqq 1$. If we put $\omega(z_0)=e^{i\theta}$, then $z_0\omega'(z_0)=ke^{i\theta}$ and from (10) for such z_0 we have

$$\left|\frac{D^n f(z_0)}{z_0}-1\right|=\left|\frac{n+k}{2}e^{i\theta}\right|=\frac{n+k}{2}\geqq\frac{n+1}{2},$$

which is the contradiction to (7). In that sense we conclude that (7) implies

$$\left|\frac{D^1 f(z)}{z}-1\right|<1,\ z\in U,$$

which is the same as $|f'(z)-1|<1$, $z\in U$, and the proof is complete.

For $n=2$ from Theorem 2 we get

Corollary 2. *If $f\in A$ and if*

$$\left|f'(z)+\frac{1}{2}zf''(z)-1\right|<\frac{3}{2},\ z\in U,$$

then f is univalent and $|f'(z)-1|<1$, $z\in U$.

Remark 1. By induction we easily conclude that the disc $|w-1|<(n+1)/2$ doesn't belongs to the half-plane defined by $\mathrm{Re}\{w\}>\alpha_n$, where α_n is defined by (3). It means that the result of Theorem 2 is not a consequence of Theorem 1.

Theorem 3. *Let* $f\in A$ *and*

$$\left|\arg\frac{D^n f(z)}{z}\right|<\frac{\pi}{2}\alpha_n,\ n\in N,\ z\in U, \tag{11}$$

where

$$\begin{cases} \alpha_n\leqq\alpha_{n-1}+\dfrac{2}{\pi}\arctan\dfrac{\alpha_{n-1}}{n},\ n\geqq 2,\ \alpha_1=1 \\ \alpha_n+\alpha_{n-1}\leqq 3 \end{cases} \tag{12}$$

Then f *is univalent in* U *and* $|\arg f'(z)|<\pi/2,\ z\in U.$

Proof. First, let show that the following implication

$$\left|\arg\frac{D^{n+1}f(z)}{z}\right|<\alpha_{n+1}\Rightarrow\left|\arg\frac{D^n f(z)}{z}\right|<\alpha_n \tag{13}$$

is true, where α_n is defined by (12). Suppose that

$$\left|\arg\frac{D^{n+1}f(z)}{z}\right|<\alpha_{n+1} \tag{14}$$

and put

$$\frac{D^n f(z)}{z}=\left(\frac{1+\omega(z)}{1-\omega(z)}\right)^{\alpha_n}. \tag{15}$$

Then $\omega(z)$ is analytic in U and $\omega(0)=0$ (we take the principal values). From (15), as in the proofs of previous theorems, we can get

$$\frac{D^{n+1}f(z)}{z}=\left(\frac{1+\omega(z)}{1-\omega(z)}\right)^{\alpha_n}\left[1+\frac{\alpha_n}{n+1}\frac{2z\omega'(z)}{1-\omega^2(z)}\right]. \tag{16}$$

We want to show that $|\omega(z)|<1,\ z\in U$. If not then by Jack's lemma there exists a z_0, $|z_0|<1$, such that $|\omega(z_0)|=1$, i.e. $\omega(z_0)=e^{i\theta}$, and $z_0\omega'(z_0)=k\omega(z_0)=ke^{i\theta}$, $k\geqq 1$. From (16) for such z_0 we have

$$\begin{aligned}\frac{D^{n+1}f(z_0)}{z_0}&=\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^{\alpha_n}\left[1+\frac{\alpha_n}{n+1}\frac{2ke^{i\theta}}{1-e^{2i\theta}}\right]\\ &=\left(i\,\mathrm{ctg}\frac{\theta}{2}\right)^{\alpha_n}\left[1+\frac{\alpha_n k}{n+1}\frac{1+\mathrm{ctg}^2(\theta/2)}{2\,\mathrm{ctg}(\theta/2)}i\right]\\ &=(it)^{\alpha_n}\left[1+\frac{\alpha_n k}{2(n+1)}\left(t+\frac{1}{t}\right)i\right],\end{aligned} \tag{17}$$

where we put $\mathrm{ctg}(\theta/2)=t$. If take $t>0$, then from (17) we have

$$\arg\frac{D^{n+1}f(z_0)}{z_0}=\frac{\pi}{2}\alpha_n+\arctan\left[\frac{\alpha_n k}{2(n+1)}\left(t+\frac{1}{t}\right)\right],$$

and from there

$$\frac{\pi}{2}\alpha_{n+1}\leqq\frac{\pi}{2}\alpha_n+\arctan\frac{\alpha_n}{n+1}\leqq\arg\frac{D^{n+1}f(z_0)}{z_0}\leqq\frac{\pi}{2}\alpha_n+\frac{\pi}{2}.$$

If we consider the case $t<0$, then in the both cases we get

$$\frac{\pi}{2}\alpha_{n+1}\leqq\left|\arg\frac{D^{n+1}f(z_0)}{z_0}\right|\leqq\frac{\pi}{2}\alpha_n+\frac{\pi}{2}\leqq 2\pi-\frac{\pi}{2}\alpha_{n+1}.$$

which implies that $(D^{n+1}f(z_0))/z_0$ lies outside the angle $|\alpha|<(\pi/2)\alpha_{n+1}$. This is the contradiction to (14). Now by induction from (13) we conclude that (11) implies

$$\left|\arg\frac{D^1f(z)}{z}\right|=|\arg f'(z)|<\frac{\pi}{2},\ z\in U,$$

i.e. f is univalent.

For $n=2$, from Theorem 3 we derive

Corollary 3. *If $f\in A$ and if*

$$\left|\arg\left(f'(z)+\frac{1}{2}zf''(z)\right)\right|<\frac{\pi}{2}\alpha_2,\ z\in U,$$

where $\alpha_2=1+(2/\pi)\arctan(1/2)=1.2951\geqq$, *then f is univalent in U and* $|\arg f'(z)|<\pi/2$, $z\in U$.

Remark 2. From the result of the previous corollary we conclude that it doesn't follow from Theorem 1 or Theorem 2.

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