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DEGREE THEORETIC SOLVABILITY OF INCLUSIONS INVOLVING PERTURBATIONS OF NONLINEAR M-ACCRETIVE OPERATORS IN BANACH SPACES

By

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Abstract. Various mapping results are given involving perturbations of accretive operators in a Banach space X. The inclusions studied are mainly of the form

$$Tx+Cx\ni p, \qquad (*)$$

where $T: X\supset D(T)\rightarrow 2^{X}$ is m-accretive and $C: \overline{D(T)}\rightarrow X$ is compact. It is shown that recent results of Yang and Morales can be improved without using the concept of a generalized topological degree. A Leray-Schauder boundary condition is also considered for the sum $T+C$, and various results of of Morales involving (*) with $C=0$ are extended.

1. Introduction-Preliminaries

In what follows, the symbol X stands for a real Banach space with norm $\|\cdot\|$ and (normalized) duality mapping J. An operator $T: X\supset D(T)\rightarrow 2^{X}$ is called "accretive" if for every $x, y\in D(T)$ there exists $j\in J(x-y)$ such that

$$\langle u-v, j\rangle\geqq 0$$

for all $u\in Tx, v\in Ty$. An accretive operator T is "m-accretive", if $R(T+\lambda I)=X$ for all $\lambda\in(0, \infty)$. We denote by $B_{r}(0)$ the open ball of X with center at zero and radius $r>0$.

For an m-accretive operator T, the "resolvents" $J_{\lambda}: X\rightarrow D(T)$ of T are defined by $J_{\lambda}=(I+\lambda T)^{-1}$ for all $\lambda\in(0, \infty)$. The "Yosida approximants" T_{λ}: $X\rightarrow X$ of T are defined by $T_{\lambda}=(1/\lambda)(I-J_{\lambda})$.

Some of the main properties of J_{λ} and T_{λ} are given below:

1. $\|J_{\lambda}x-J_{\lambda}y\|\leqq\|x-y\|$ for all $x, y\in X$.
2. $\|J_{\lambda}-x\|=\lambda\|T_{\lambda}x\|\leqq\lambda\inf\{\|y\|; y\in Tx\}$ for all $x\in D(T)$.

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3. T_λ is m-accretive on X and $\|T_\lambda x - T_\lambda y\| \leqq (2/\lambda)\|x-y\|$ for all $\lambda>0$, x, $y \in X$.

4. $T_\lambda x \in TJ_\lambda x$ for all $x \in X$.

In what follows, "continuous" means "strongly continuous" and the symbol "$\to$" ("$\rightharpoonup$") means strong (weak) convergence. The symbol R (R_+) stands for the set $(-\infty, \infty)$ ($[0, \infty)$) and the symbols ∂D, $\mathrm{int} D$, $\bar{D}$ denote the strong boundary, interior and closure of the set D, respectively. An accretive operator T is called "strongly accretive" if there exists a constant $\alpha>0$ such that: for each x, $y \in D(T)$ there exists $j \in J(x-y)$ such that

$$\langle u-v, j\rangle \geqq \alpha\|x-y\|^2$$

for all $u \in Tx$, $v \in Ty$. An operator $T: X \supset D(T) \to X$ is "bounded" if it maps boundep subsets of $D(T)$ onto bounded sets. It is "compact" if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact sets. It is called "demicontinuous" ("completely continuous") if it is strong-weak (weak-strong) continuous on $D(T)$.

For facts involving accretive operators, and other related concepts, the reader is referred to Barbu [3], Browder [4], Cioranescu [6] and Lakshmikantham and Leela [20]. We cite the books of Lloyd [21], Petryshyn [25], Rothe [27] and the paper of Nagumo [24] as references to the degree theories discussed herein. For a survey article on recent mapping theorems involving compactness and accretiveness, we refer to [15].

In this paper we first show (Theorem 1) how Theorem 1 of Yang [29] and Corollary 2 of Morales [23] can be improved. Yang was unaware of Morales' result, but gave an interesting extension of it in [29] by using the concept of generalized degree introduced by Chen in [5]. In our approach, we use two homotopy equations, $H_i(t, x)=0$, $i=1, 2$, whose solvability, for $t=0$ or $t=1$, leads eventually to the solvability of our target equation

$$Tx + Cx \ni p, \tag{$*$}$$

where p is a fixed point in X. We make use of specific homotopies related to those used by Yang in [29] and [30].

Applications of results involving compact perturbations and compact resolvents of accretive operators can be found in the papers [11], [17] and [18]. Various results involving sums of three operators can be found in [8].

2. Main Results

For a set $A \subset X$ we set $|A| = \inf\{\|x\| : x \in A\}$. Morales gave in [23] the following result.

Theorem A. *Let $T: X \supset D(T) \to 2^X$ be m-accretive and $C: \overline{D(T)} \to X$ compact. Assume that there exists a positive constant r and $z \in D(T)$ such that*

$$\|Cz\| < r \leqq \liminf_{\substack{\|x\| \to \infty \\ x \in D(T)}} |Tx + Cx|.$$

Assume, further, that there exists a constant $r_1 > 0$ such that for every $x \in D(T)$ with $\|x\| \geqq r_1$ there exists $j \in J(x)$ such that

$$\langle u + Cx - Cz, j \rangle \geqq 0, \qquad \text{for every} \quad u \in Tx.$$

Then $B_\mu(0) \subset \overline{R(T+C)}$, where $\mu = (r - \|Cz\|)/2$.

Yang gave in [29] the following theorem.

Theorem B. *Let $T: X \supset D(T) \to 2^X$ be m-accretive, with $0 \in D(T)$ and $0 \in T(0)$, and $C: X \to X$ compact. Assume that there exists a positive constant r such that*

$$\|C(0)\| < r \leqq \liminf_{\substack{\|x\| \to \infty \\ x \in D(T)}} |Tx + Cx|.$$

Assume, further, that there exists a constant $r_1 > 0$ such that for every $x \in D(T)$ with $\|x\| \geqq r_1$, there exist $j \in J(x)$ such that

$$\langle u + Cx - C(0), j \rangle \geqq 0, \qquad \text{for every} \quad u \in Tx.$$

Then $\overline{B_r(0)} \subset \overline{R(T+C)}$.

One of our intentions here is to give a theorem, Theorem 1 below, that unifies and improves the above two results. We would like to point out that our method does not involve the complicated homotopy argument in [23, proof of Theorem 6] and does not make use of Chen's topological degree as in [29]. The reader should note that the assumption that $0 \in T(0)$ cannot be omitted in the proof of Yang [23], because of the generalized degree-theoretic approach. Also, in Young's result the operator C is defined on all of X. In addition, Yang's result is considerably stronger than Morales' result in many cases. In fact, if $C(0) = 0$ in the two results and the assumptions of Yang hold also in Morales' result, Yang's result says that the entire closed ball $\overline{B_r(0)}$ lies in $\overline{R(T+C)}$, while Morales' result says that only the closed ball with half that radius lies in the same set.

Theorem 1. *Let $T: X \supset D(T) \to 2^X$ be m-accretive and $C: \overline{D(T)} \to X$ compact. Let $z_0 \in X$ and the positive constant r be snch thaț*

$$\|z_0\| < r \leqq \liminf_{\substack{\|x\| \to \infty \\ x \in D(T)}} |Tx + Cx|.$$

Assume, further, that there exists a constant $r_1 > 0$ such that for every $x \in D(T)$

with $\|x\| \geqq r_1$ *there exists* $j \in J(x)$ *such that*

$$\langle u + Cx - z_0, j \rangle \geqq 0, \quad \textit{for every} \quad u \in Tx.$$

Then $\overline{B_r(0)} \subset \overline{R(T+C)}$.

Proof. Fix $x_0 \in D(T)$ and consider the mappings $\tilde{T}: x \to T(x+x_0) - v_0$, $\tilde{C}: x \to C(x+x_0) + v_0$, $x \in D(\tilde{T}) \equiv D(T) - x_0$, where v_0 is a fixed vector in Tx_0. It is easy to see that $0 \in D(\tilde{T})$, $0 \in \tilde{T}(0)$ and

$$\|z_0\| < r \leqq \liminf_{\substack{\|x\| \to \infty \\ x \in D(\tilde{T})}} |\tilde{T}x + \tilde{C}x|.$$

Also, for every $x \in D(\tilde{T})$ with $\|x + x_0\| \geqq r_1$ there exists $j \in J(x+x_0)$ such that

$$\langle \tilde{u} + \tilde{C}x - z_0, j \rangle \geqq 0, \quad \text{for every} \quad \tilde{u} \in \tilde{T}x. \tag{1}$$

We are planning to solve the approximate equation

$$\tilde{T}x + \tilde{C}x + (1/n)(x + x_0) \ni p, \tag*{$(*)_n$}$$

where $p \in B_r(0)$ is a fixed vector. To this end, we consider the homotopy equations

$$H_1(t, x) \equiv x - (t\tilde{T} + (1/n)I)^{-1}(-t(\tilde{C}x - z_0)) = 0, \tag{2}$$

and

$$H_2(t, x) \equiv x - (\tilde{T} + (1/n)I)^{-1}(-\tilde{C}x + tz_0 + (1-t)(p - (1/n)x_0)) = 0, \tag{3}$$

for a fixed positive integer n and $t \in [0, 1]$. We choose a number $\varepsilon > 0$ and a positive integer n_0 so that

$$r - 2\varepsilon > \max\{\|p\| + (1/n_0)\|x_0\|, \|z_0\|\}. \tag{4}$$

We also note that there exists $Q = Q(\varepsilon) > r_1 + 2\|x_0\|$ such that: for every $x \in D(\tilde{T})$ with $\|x\| = Q$,

$$|\tilde{T}x + \tilde{C}x| \geqq r - \varepsilon \tag{5}$$

and there exists $j \in J(x+x_0)$ such that (1) is satisfied. Since (4) holds for any $n > n_0$ instead of n_0, we may choose n_0 further so that $Q/n_0 < \varepsilon$. We consider only values of n such that $n \geqq n_0$. Since we are only interested in the range of the operator C on the ball $\overline{B_Q(0)}$ and the set $\overline{B_Q(0)} \cap \overline{D(\tilde{T})}$ is closed and bounded, we may extend the operator $\tilde{C}$ to the whole space X by Lemma 31 in Rothe's book [27]. We use the same symbol, $\tilde{C}$, for this extension. Before we proceed with the homotopy arguments, we should note that the mapping $H_1(t, x)$ is actually a homotopy of compact transformations. In fact, as in Theorem 4 of the author in [14], we have

$$\|H_1(t, x) - H_1(t_0, x)\| \leqq \frac{2|t - t_0|}{t_0} \|nt(\tilde{C}x - z_0)\| + n|t - t_0| \|\tilde{C}x - z_0\|,$$

for t_0, $t \in (0, 1]$, and

$$\|H_1(t, x)-x\| \leqq nt\|\tilde{C}x-z_0\|,$$

for all $t \in [0, 1]$. These two inequalities show the continuity of the $H_1(t, x)$ w.r.t. t uniformly for x lying in any bounded set.

We want to show that

$$d(H_1(1, \cdot), B_Q(0), 0)=1 \tag{6}$$

and

$$d(H_2(0, \cdot), B_Q(0), 0)=d(H_1(1, \cdot), B_Q(0), 0)=1, \tag{7}$$

where the degree function $d=d(\cdot, \cdot, \cdot)$ denotes the Leray-Schauder degree, provided that 0 is not in the image of $\partial B_Q(0)$ by the mappings $H_1(1, \cdot)$, $H_2(0, \cdot)$. To show (6), let us assume that the homotopy equation (2) has a solution $x_t \in \partial B_Q(0)$. We have

$$t(\tilde{y}_t+\tilde{C}x_t-z_0)+(1/n)x_t=0, \tag{8}$$

for some $\tilde{y}_t \in \hat{T}x_t$. Obviously, $t=0$ implies $x_t=0$, i.e., a contradiction. Thus $t \in (0, 1]$ and, after dividing (8) by t, we obtain, for an appropriate $j_t \in J(x_t+x_0)$,

$$\begin{aligned}
0&=\langle \tilde{y}_t+\tilde{C}x_t-z_0, j_t\rangle+\langle (1/(nt))x_t, j_t\rangle \\
&\geqq [1/(nt)]\langle x_t+x_0, j_t\rangle-[1/(nt)]\langle x_0, j_t\rangle \\
&\geqq [1/(nt)]\|x_t+x_0\|^2-[1/(nt)]\|x_0\|\|x_t+x_0\| \\
&\geqq [1/(nt)](\|x_t\|-2\|x_0\|)\|x_t+x_0\| \\
&\geqq [1/(nt)](\|x_t\|-2\|x_0\|)(\|x_t-2\|x_0\|) \\
&=[1/(nt)](Q-2\|x_0\|)^2 \\
&>0,
\end{aligned}$$

i.e., a contradicition. Consequently, the Leray-Schauder degree $d(H_1(t, \cdot)$, $B_Q(0), 0)$ is well-defined for all $t \in [0, 1]$ and equals 1 because we have $0 \in B_Q(0)$ and $d(H_1(0, \cdot), B_Q(0), 0)=d(I, B_Q(0), 0)=1$.

To show (7), let (3) have a solution $x_t \in \partial B_Q(0)$. Then we have

$$\tilde{y}_t+\tilde{C}x_t-tz_0+(1-t)(-p+(1/n)x_0)+(1/n)x_t=0,$$

where $\tilde{y}_t \in \hat{T}x_t$. Since $\|x_t\|=Q$, we also have

$$\begin{aligned}
0&=\|\tilde{y}_t+\tilde{C}x_t+(1/n)x_t-tz_0+(1-t)(-p+(1/n)x_0)\| \\
&\geqq |\hat{T}x_t+\tilde{C}x_t|-(1/n)\|x_t\|-(t\|z_0\|+(1-t)\|-p+(1/n)x_0\|) \\
&\geqq |\hat{T}x_t+\tilde{C}x_t|-(1/n)Q-\max\{\|p\|+(1/n)\|x_0\|, \|z_0\|\}
\end{aligned}$$

$$\geqq r-2\varepsilon-\max\{\|p\|+(1/n_0)\|x_0\|, \|z_0\|\}$$
$$>0.$$

This contradiction says that $d(H_2(t, \cdot), B_Q(0), 0)$ is well-defined for all $t\in[0, 1]$ and equals the degrees $d(H_1(1, \cdot), B_Q(0), 0)$ and $d(H_2(0, \cdot), B_Q(0), 0)$. Since we have already established (6), we conclude that $d(H_2(0, \cdot), B_Q(0), 0)=1$, which implies that

$$x-(\tilde{T}+(1/n)I)^{-1}(-\tilde{C}x+p-(1/n)x_0)=0,$$

for some $x\in B_Q(0)$. Thus we have the solvability of $(*)_n$ for each $n\geqq n_0$, i.e., the solvability of the inclusion

$$Tx+Cx+(1/n)x\ni p \tag{9}$$

with solution $x_n\in B_Q(x_0)$, $n\geqq n_0$. Since $\|x_n-x_0\|<Q$, it is easy to see that Inclusion (9) implies $p\in\overline{(T+C)(B_Q(x_0)\cap D(T))}$. The proof is complete. ∎

Evidently, Theorems A and B are special cases of Theorem 1. We can actually generalize Theorems A, B and 1 by using a localized version of Theorem 1 as follows.

Theorem 2. *Let $T: X\supset D(T)\to 2^X$ be m-accretive and $C:\overline{D(T)}\to X$ compact. Assume that there exist $x_0\in D(T)$ and a bounded open set G containing x_0 such that:*

$$|Tx+Cx|\geqq r>\|z_0\|, \qquad x\in\partial G\cap D(T),$$

and for each $x\in\partial G\cap D(T)$ there exists $j\in Jx$ such that

$$\langle u+Cx-z_0, j\rangle\geqq 0, \qquad \text{for every } u\in Tx.$$

Then $\overline{B_r(0)}\subset\overline{(T+C)(G\cap D(T))}$.

Proof. We consider the operators $\tilde{T}$, $\tilde{C}$ and the domain $D(\tilde{T})$ as in the proof of Theorem 1. We also set $\tilde{G}=G-x_0$. In order to imitate the proof of Theorem 1, we let $Q=\sup\{\|x\|: x\in\partial\tilde{G}\}$ and fix ε, n_0 such that $Q/n_0<\varepsilon$ and

$$r-2\varepsilon>\max\{\|p\|+(1/n_0)\|x_0\|, \|z_0\|\}.$$

Then

$$|\tilde{T}x+\tilde{C}x|\geqq r>r-\varepsilon, \qquad x\in\partial\tilde{G}\cap D(\tilde{T}),$$

with $\partial\tilde{G}=\partial(G-x_0)=\partial G-x_0$. The proof now follows as in Theorem 1 with $B_Q(0)$ replaced by $\tilde{G}$. It is therefore omitted. ∎

Naturally, if $0\in D(T)$ in the proof of Theorem 1, we may take $x_0=0$ and $v_0\in T(0)$. If $0\in T(0)$, then we may also choose $v_0=0$. A similar remark holds for Theorem 2.

Proposition 1. *Let* $T: X\supset D(T)\to 2^X$ *be m-accretive and* $C: X\supset \bar{G}\to X$ *compact, where G is open and bounded. Assume that there exists* $x_0\in G\cap D(T)$ *such that*

$$Tx+Cx\not\ni\mu(x-x_0), \quad \text{for every } (\mu, x)\in(-\infty, 0)\times(\partial G\cap D(T)). \tag{10}$$

Then $0\in\overline{(T+C)(G\cap D(T))}$. *Moreover,* $0\in(T+C)(\bar{G}\cap D(T))$ *under one of the following conditions:*

(i) *X is uniformly convex,* $\bar{G}$ *is convex and C is completely continuous;*

(ii) *T is strongly accretive on* $G\cap D(T)$;

(iii) *the compactness assumption on C is replaced by: C is bounded, continuous and the resolvent* $(T+I)^{-1}$ *is compat.*

Proof. Again, we may assume that $x_0=0\in T(0)$. If this is not true, then we consider instead the operators $\tilde{T}x\equiv T(x+x_0)-v_0$, $\tilde{C}x\equiv C(x+x_0)+v_0$ on $D(\tilde{T})=D(T)-x_0$, and replace G by $\tilde{G}=G-x_0$. In this case, Relation (10) should be replaced by

$$\tilde{T}x+\tilde{C}x\not\ni\mu x, \qquad \text{for} \quad (\mu, x)\in(-\infty, 0)\times(\partial\tilde{G}\cap D(\tilde{T})).$$

Now, we consider the homotopy equation

$$H(t, x)\equiv x-(tT+(1/n)I)^{-1}(-tCx)=0, \qquad \text{for} \quad (t, x)\in[0, 1]\times\bar{G}, \tag{11}$$

$n=1, 2, \cdots$. As we mentioned in the proof of Theorem 1, the author has shown in [14, Theorem 4] that the mapping $H(t, x)$ in (11) is a homotopy of compact transformations. Because of this, (11) has a zero $x\in G$ for $t=1$ if it does not possess any solutions $x_t\in\partial G$ for any $t\in[0, 1]$. We know that $H(0, x)=0$ has no solution $x\in\partial G$. Thus, we may assume that $t\in(0, 1]$. Let $x_t\in\partial G$ solve (11) for some $t\in(0, 1]$. Then

$$t(Tx_t+Cx_t)+(1/n)x_t\ni 0,$$

or

$$Tx_t+Cx_t+[1/(nt)]x_t\ni 0,$$

i.e., a contradiction to Relation (10) (with $x_0=0$). It follows that

$$Tx+Cx+(1/n)x\ni 0 \tag{12}$$

is solvable for all sufficiently large n. We let x_n denote a solution of (12). Since $\{x_n\}\subset G$, it is bounded. Thus, $0\in\overline{(T+C)(G\cap D(T))}$.

Let us assume (i). Then $x_n\to$ (some) $x_0\in\bar{G}$ and $Cx_n\to Cx_0$. Now, we can apply a "multi-valued" version of Lemma 1 in [13] (cf. also [8, Lemma 1]) in order to conclude that $x_0\in D(T)$ and $Tx_0+Cx_0\ni 0$.

In the case of Assumption (ii), we have that the sequence $\{x_n\}$ is Cauchy. This follows easily from the strong accetiveness of T. If we let $x_0\in\bar{G}$ denote the strong limit of $\{x_n\}$, we have $Cx_n \to Cx_0$. Since T is closed, $x_0 \in D(T)$ and $Tx_0 \ni -Cx_0$.

In Case (iii) is satisfied, the proof of the theorem goes through up to the point of the existence of $\{x_n\}$ because (10) defines still a homotopy of compact transformations. Now, we may add dx to both sides of (12) and then invert $T+dI$ to obtain

$$x_n=(T+dI)^{-1}(-(C+(1/n)I)x_n+dx_n),$$

which shows that $\{x_n\}$ lies inside a compact set. Thus, there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that $x_n \to x_0 \in \bar{G}$. We have $Cx_n \to Cx_0$. Since T is closed, $x_0 \in D(T)$ and $Tx_0 \ni -Cx_0$. ■

Obviously, a condition like one of (i)-(iii) of Proposition 1 ensures the fact that $B_r(0) \subset R(T+C)$ in Theorems 1 and 2 as well. Proposition 1 is a significant improvement of Yang's Theorem 2 in [29]. Yang assumed that $T: X \to X$ is a (single-valued) bounded, demicontinuous and strongly accretive operator on all of X with X^* uniformly convex in order to show that $T+C$ has a zero in $\bar{G}$. It is well.known that such an operator T is m-accretive.

For other applications of Condition (10), the reader is referred to the book of Petryshyn [25] (especially pages 193-195) and several of the references therein. In particular, let us examine the Leray-Schauder fixed point theorem on page 191 of [25]. It says that, for an open, bounded set $G \subset X$ containing the point x_0 and $C: \bar{G} \to X$ compact, the condition

$$Cx-x_0 \neq \eta(x-x_0), \qquad \text{for all } x \in \partial G, \ \eta>1, \tag{LS}$$

implies that C has a fixed point in $\bar{G}$. This condition (LS) is exactly Condition (10) with $T=I$ and $\mu=(1-\eta)$. In fact, (LS) is equivalent to $x-Cx+x_0 \neq x-\eta(x-x_0)$, or $(I-C)x \neq (1-\eta)(x-x_0)$, i.e., Condition (10).

An application of the above theorem is Corollary 1 below, which extends Theorem 2 of He [9] to m-accretive operators in general Banach spaces. He's result (actually the main part of it) is given in Theorem C below. He assumes in [9] that T and C are only defined on the domain G below, but he needs to apply Corollary 4 there. Therefore, T, C should be defined on $\bar{G}$.

Theorem C. *Let X^* be uniformly convex and let let $G \subset X$ be open with $0 \in G$. Let $T: X \supset \bar{G} \to X$ be continuous, accretive and $C: \bar{G} \to X$ compact. Assume that there exist positive constants b, r such that for every $x \in G$ with $\|x\| \geq b$ we have*

$$\|Tx+Cx\| \geq \|Tx+Cx-x\| \geq r. \tag{13}$$

Then $\overline{B_r(0)} \subset \overline{(T+C)(B_b(0) \cap G)}$.

Corollary 1. *Let* $T: X\supset D(T)\to 2^X$ *be m-accretive, with* $0\in D(T)$ *and* $C: \bar{G}\to X$ *compact, where* $G\subset X$ *is open, bounded and such that*: $0\in G$ *and there exists* $r>0$ *such that*

$$\|y+Cx\|\geqq\|y+Cx-x\|\geqq r, \quad \text{for every} \quad x\in\partial G\cap D(T), \quad y\in Tx. \tag{14}$$

Then $\overline{B_r(0)}\subset\overline{(T+C)(G\cap D(T))}$.

Proof. We first note that we may assume that $0\in T(0)$, otherwise we replace the operators T, C by $\hat{T}x\equiv T(x)-v_0$, $\tilde{C}x\equiv Cx+v_0$, respectively, where $v_0\in T(0)$ is fixed. With this substitution, Inequality (14) becomes

$$\|y+\tilde{C}x\|\geqq\|y+\tilde{C}x-x\|\geqq r, \quad \text{for all} \quad x\in\partial G\cap D(T), \quad y\in\hat{T}x.$$

We consider the homotopy equations

$$H_1(t, x)\equiv x-(tT+(1/n)I)^{-1}(-tCx)=0 \tag{15}$$

and

$$H_2(t, x)\equiv x-(T+(1/n)I)^{-1}(-(Cx-(1-t)p))=0, \tag{16}$$

where $p\in B_r(0)$ is fixed. We note first that $d(H_1(1, \cdot), G, 0)=1$. To see this, it suffices to show that (15) has no solutions $x_t\in\partial G$. In fact, this is obviously true for $t=0$. We assume that $t\in(0, 1]$ and lethx_t be such a solution. Then

$$ty_t+tCx_t+(1/n)x_t=0, \tag{17}$$

or

$$y_t+Cx_t=\mu_0 x_t, \tag{18}$$

where $\mu_0=-1/(nt)$, $y_t\in Tx_t$. To show that (18) is impossible, let $Sx_t\equiv y_t+Cx_t$ and $\nu_0=-\mu_0$, and observe that (14) implies

$$\|Sx_t\|^2\geqq\|Sx_t-x_t\|^2-\|x_t\|^2. \tag{19}$$

Since $Sx_t=-\nu_0 x_t$, we use (19) to get

$$\begin{aligned}\nu_0^2\|x_t\|^2&\geqq\|-\nu_0 x_t-x_t\|^2-\|x_t\|^2\\&=(\nu_0+1)^2\|x_t\|^2-\|x_t\|^2.\end{aligned}$$

Since $\|x_t\|\neq 0$, we obtain

$$\nu_0^2\geqq(\nu_0+1)^2-1,$$

which is a contradiction. Thus, $d(H_1(1, \cdot), G, 0)=1$.

We now examine the homotopy equation (16). We fix $\varepsilon\in(0, r)$, $p\in B_{r-\varepsilon}(0)$ and assume that n is sufficiently large so that $(1/n)\|x\|<\varepsilon/2$, $x\in\partial G$, and consider from this point on only such values of n. We want to show that $d(H_2(0, \cdot), G, 0)=1$. To this end, we assume that (16) has a solution $x_t\in\partial G$. Then, for some $y_t\in Tx_t$, we have

$$0=\|y_t+Cx_t+(1/n)x_t-(1-t)p\|$$
$$\geqq\|y_t+Cx_t\|-(1/n)\|x_t\|-\|p\|$$
$$>r-\varepsilon/2-(r-\varepsilon)=\varepsilon/2>0,$$

i.e., a contradiction. It follows that

$$d(H_2(t,\cdot),G,0)=\text{const.}=d(H_2(1,\cdot),G,0)=d(H_1(1,\cdot),G,0)=1.$$

This implies the solvability of the equation $H_2(0,u)=0$, or the inclusion

$$Tx+Cx+(1/n)x\ni p,$$

for $p\in B_{r-\varepsilon}(0)$. It is easy to see, as before, that $p\in\overline{R(T+C)}$. ■

3. Discussion

It should not be surprising that the boundary condition (10) (with C=0) is actually necessary and sufficient for the existence of a zero of the m-accretive operator T in pretty general Banach spaces. In order to further elaborate on this item, we cite a relevant result of Reich and Torrejón [26, Theorems 3, 4]. The space X is called a (BCC) ((BUC)) space if every nonempty, bounded, closed and convex set $M\subset X$ (the closed unit ball of X) has the fixed point property for nonexpansive self-mappings.

Theorem D. *Let $T: X\supset D(T)\to 2^X$ be m-accretive. Then the following two statements are true.*

(I) *Let X be a (BCC) space. Then $0\in R(T)$ if and only if there exists an open, bounded set $G\subset X$ and a point $x_0\in G\cap\overline{D(T)}$ such that $\langle y, x-x_0\rangle_+\geqq 0$ for every $x\in\partial G\cap D(T)$ and $y\in Tx$.*

(II) *Let X be a (BUC) space. Then $0\in R(T)$ if and only if there exists $r>0$ and $x_0\in\overline{D(T)}$ such that $\langle y, x-x_0\rangle_+\geqq 0$ for every $x\in\partial B_r(x_0)\cap D(T)$ and every $y\in Tx$.*

The symbol $\langle y, x\rangle_+$ stands for $\max_{j\in Jx}\{\langle y, j\rangle\}$. It is easy to see that "$x_0\in G\cap\overline{D(T)}$" can be replaced in Theorem D by "$x_0\in G\cap D(T)$" and "$x_0\in\overline{D(T)}$" can be replaced in Theorem D by "$x_0\in G\cap D(T)$" and "$x_0\in\overline{D(T)}$" by "$x_0\in D(T)$". This can be done by careful examination of the proofs of Theorems 3 and 4 in [26]. If we let $\mu=-(1/t)$ in Condition (10), with $C=0$, $t>0$, we have

$$tTx+x\not\ni x_0,\qquad (t,x)\in(0,\infty)\times(\partial G\cap D(T)).$$

Thus, the existence of such an open set G and a point $x_0\in G\cap D(T)$, in a (BCC) space X, is equivalent to the statement that there exists a point $x_0\in$

$G\cap D(T)$ such that $J_t x_0 \notin \partial G$, $t\in(0, \infty)$. This is nothing more than saying that the continuous function $t\to J_t x_0$ is bounded because it never leaves the bounded set G. By Theorem 1 of Kirk and Schöneberg [19], the boundedness of the set $\{J_t x_0 : t>0\}$ is equivalent to the existence of a zero of T. This result is practically included in the proof of Theorem 3 of [26]. It is also included in Theorem 1 of Morales [22], who was not aware of the Reich and Torrejón paper. The analogous statement for (BUC) spaces can be found in Torrejón's paper [28].

Inequality (19), with x instead of x_t, is called "Altman's Condition" and is used in Altman's fixed point theorem in [1] and [2]. It would be interesting to see Altman's fixed point theorem extended to so that it includes the existence of zeros of mappings $S=T+C$, where $T: \bar{G}\to X$ is demicontinuous accretive, with X^* uniformly convex, and $C: \bar{G}\to X$ is compact. The same problem is open for continuous accretive operators T in general Banach spaces. Here, G is an open and bounded subset of X.

References

[1] M. Altman, *A fixed point theorem in Hilbert space,* Bull. Acad. Polon. Sci. Cl. III **5** (1957), 89-92.

[2] M. Altman, *A fixed point theorem in Banach space,* Bull. Acad. Polon. Sci. Cl. III **5** (1957), 17-22.

[3] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces,* Noordhoff Int. Publ., Leyden (The Netherlands), 1975.

[4] F. Browder, *Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces,* Proc. Symp. Pure Appl. Math., **18**, Part 2, Providence, 1976.

[5] Y.Z. Chen, *The generalized degree for compact perturbations of m-accretive operators and applications,* Nonl. Anal. TMA **13** (1989), 393-403.

[6] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems.,* Kluwer Acad. Publ., Boston, 1990.

[7] K. Deimling, *Zeros of accretive operators,* Manuscr. Math. **13** (1974), 365-374.

[8] Z. Guan and A.G. Kartsatos, *Solvability of nonlinear equations with coercivity generated by compact perturbations of m-accretive operators,* Houston J. Math. (to appear).

[9] Z. He, *Some mapping theorems involving the perturbations of m-accretive operators,* Nonl. Anal. TMA **19** (1992), 345-351.

[10] V.I. Istrătescu, *Fixed point theory,* D. Reidel Publ. Co., Boston, 1981.

[11] D.R. Kaplan and A.G. Kartsatos, *Ranges of sums and the control of nonlinear evolutions with pre-assigned responses,* J. Opt. Th. Appl. **81** (1994), 121-141.

[12] A.G. Kartsatos, *Zeros of demicontinuous accretive operators in reflexive Banach spaces,* J. Int. Eq. **8** (1985), 175-184.

[13] A.G. Kartsatos, *On the solvability of abstract operator equations involving compact perturbations of m-accretive operators,* Nonl. Anal. TMA **11** (1987), 997-1004.

[14] A.G. Kartsatos, *On compact perturbations and compact resolvents of nonlinear m-accretive operators in Banach spaces,* Proc. Amer. Math. Soc., **119** (1993), 1189-1199.

[15] A.G. Kartsatos, *Recent results involving compact perturbations and compact resolvents of accretive operators in Banach spaces,* Proceedings of the World Congress of Nonlinear Analysts, Tampa, Florida, 1992, Walter De Gruyter, New York, (to appear).

[16] A.G. Kartsatos, *Aduanced Ordinary Differential Equations,* Mancorp Publ., Tampa, 1993.

[17] A.G. Kartsatos, *On the construction of methods of lines for functional evolutions in general Banach spaces,* Nonl. Anal. TMA (to appear).

[18] A.G. Kartsatos and R.D. Mabry, *Controlling the space with pre-assigned responsess,* J. Opt. Th. Appl. **54** (1987), 517-540.

[19] W.A. Kirk and R. Schöneberg, *Zeros of m-accretive operators in Banach spaces,* Israel J. Math. **35** (1980), 1-8.

[20] V. Lakshmikantham and S. Leela, *Nonlinear Differential Equations in Abstract Spaces,* Pergamon Press, Oxford, 1981.

[21] N.G. Lloyd, *Degree Theory,* Cambridge University Press, London, 1978.

[22] C. Morales, *Nonlinear equations involving m-accretive operators,* J. Math. Anal. Appl. **97** (1983), 329-336.

[23] C. Morales, *Remarks on compact perturbations of m-accretive operators,* Nonl. Anal. TMA **16** (1991), 771-780.

[24] M. Nagumo, *Degree of mapping in convex linear topological spaces,* Amer. J. Math. **73** (1951), 497-511.

[25] W.V. Petryshyn, *Approximation-solvability of nonlinear functional and differential equations,* Marcel Dekker, New York, 1993.

[26] S. Reich and R. Torrejón, *Zeros of accretive operators,* Comment. Math. Univ. Carolinae **21** (1980), 619-625.

[27] E.H. Rothe, *Introduction to Various Aspects of Degree Theory in Banach Spaces,* Math. Surveys and Monographs, No. 23, A.M.S., Providence, 1986.

[28] R. Torrejón, *Some remarks on nonlinear functional equations,* Cont. Math. **18** (1983), 217-246.

[29] G.H. Yang, *Some theorems for the ranges of perturbed m-accretive operators,* J. Math. Res. Exp. **12** (1992), 28-32.

[30] G.H. Yang, *The generalized topological degree for the perturbations of m-accretive operators and its applications,* Nonl. Anal. TMA **18** (1992), 403-412.

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