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OPTIMIZATION OF PARAMETRIC CONTROLLED NONLINEAR EVOLUTION EQUATIONS

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Abstract. In this paper we study optimization problems for systems monitored by parametric nonlinear controlled evolution equations. First we solve a min-max problem and then we prove the wellposedness (variational stability) of a parametric optimal control problem. Three examples of parabolic control systems are presented in detail.

1. Introduction.

In this paper we study optimization problems involving parametric controlled nonlinear evolution equations.

So let $T=[0, b]$ and let (X, H, X^*) be an evolution triple of spaces (see section 2). We consider the following parametric nonlinear control system:

$$\left\{\begin{array}{c} \dot{x}(t)+A(t, x(t), \lambda)=f(t, x(t), \lambda)u(t) \text{ a.e. on } T \\ x(0)=x_0(\lambda),\ u(t)\in U(t) \text{ a.e. on } T. \end{array}\right\} \tag{1}$$

Here E is a complete metric space and $\lambda\in E$ models noise, disturbances and inaccuracy of measurement, which interfere with the control of the system. The control space is a separable reflexive Banach space Y and $A: T\times X\times E\rightarrow X^*$, $f: T\times H\times E\rightarrow \mathcal{L}(Y, H)$, $U: T\rightarrow 2^Y\setminus\{\emptyset\}$ (the control constraint set). Precise hypotheses on these items will be provided in the sequel. A control function $u: T\rightarrow Y$ is said to be admissible, if it is measurable and $u(t)\in U(t)$ a.e. We will denote the set of admissible controls by S_U. Given $[\lambda, u]\in E\times S_U$ under reasonable hypotheses on the data, we can guarantee the existence of a unique solution $x(\lambda, u)(\cdot)\in C(T, H)$ of (1). Then for this triple $[\lambda, u, x(\lambda, u)]$, the performance of the system is measured by the integral cost functional

$$J(\lambda, u)=\int_0^b L(t, x(\lambda, u)(t), \lambda, u(t))dt.$$

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Since it is not known a priori which disturbance element $\lambda\in E$ is in force, the best that the system analyst can do is to minimize the maximum cost. Thus our optimization problem is the following "min-max" problem:

$$\inf_{u\in S_U}\sup_{\lambda\in E} J(\lambda, u)=\beta. \tag{P_1}$$

Let $m(u)=\sup_{\lambda\in E} J(\lambda, u)$ (the maximum cost associated with a given admissible control $n\in S_U$). Our goal is to find $\hat{u}\in S_U$ such that

$$\beta=m(\hat{u}).$$

Such an admissible control will be called "optimal".

In addition to (P_1), we also examine the parametric objective functional

$$m_1(\lambda)=\inf_{u\in S_U} J(\lambda, u) \tag{P_2}$$

and determine verifiable conditions on the data that guarantee the solvability of (P_2) (i.e. the existence for every $\lambda\in E$ of a $\hat{u}\in S_U$ such that $m_1(\lambda)=J(\lambda, \hat{u})$) and the continuity of $\lambda\rightarrow m_1(\lambda)$.

In the last section, we present three examples of parabolic distributed parameter systems which illustrate the applicability of our abstract results.

2. Preliminaries

Let H be a separable Hilbert space with norm $|\cdot|$. Let X be a separable, reflexive Banach space, which embeds into H continuously and densely. Identifying H with its dual (pivot space), we have $X\rightarrow H\rightarrow X^*$, with all embeddings being continuous and dense. Such a triple of spaces (X, H, X^*) is said to be an "evolution triple". Here we will also assume that the embeddings are also compact. The norms of X and X^* will be denoted by $\|\cdot\|$ and $\|\cdot\|_*$ respectively. By $(\cdot, \cdot)$ we will denote the inner product in H and by $\langle\cdot, \cdot\rangle$ the duality brackets for the pair (X, X^*). The two are compatible in the sense that $\langle\cdot, \cdot\rangle|_{X\times H}=(\cdot, \cdot)$. Let $T=[0, b]$ and $1<p, q<\infty$, $(1/p)+(1/q)=1$ and define $W_{pq}(T)=\{x\in L^p(T, X): \dot{x}\in L^q(T, X^*)\}$. In this definition the derivative is understood in the sense of vector valued distributions. Furnished with the norm $\|x\|_{W_{pq}(T)}=(\|x\|^2_{L^p(T,X)}+\|\dot{x}\|^2_{L^q(T,X^*)})^{1/2}$, $W_{pq}(T)$ becomes a separable reflexive Banach space, which embeds continuously into $C(T, H)$. Furthermore since we have assumed that X embeds compactly into H, we have that $W_{pq}(T)$ embeds compactly into $L^p(T, H)$. For details we refer to Zeidler to Zeidler [12], proposition 23.23, p. 422 and p. 450. When $p=q=2$, we write $W_{pq}(T)=W(T)$ and in this case $W(T)$ is a separable Hilbert space with inner product $(x, y)_{W(T)}=(x, y)_{L^2(T,Y)}+(\dot{x}, \dot{y})_{L^2(T,X^*)}$. We model the control space by a separable reflexive Banach space Y. By $P_{fc}(Y)$ we will denote the family of all nonempty closed and convex subsets of Y. A multifunction (set-valued function) $U: T\to P_{fc}(Y)$ is said to be measurable, if for all $y\in Y$ the $\boldsymbol{R}_+$-valued function $t\to d(y, F(t))=\inf\{\|y-v\|_Y: v\in F(t)\}$ is measurable. Other equivalent definitions of measurability can be found in Wagner [10], theorem 4.2.

Following Kolpakov [6], we say that a sequence of operators $A_n: X\to X^*$, G-converges to an operator $A: X\to X^*$, if for all $n\geq 1$, A_n^{-1}, $A^{-1}: X^*\to X$ are defined and for every $x^*\in X^*$, $A_n^{-1}x^*\to A^{-1}x^*$ in X (hence strongly in H). We will use the symbol G to indicate G-convergence. This is a nonlinear abstract extension of a notion first introduced by Spagnolo [9] for linear parabolic and elliptic equations and extended to abstract linear evolution equations by Zhikov-Kozlov-Oleinik-Ngoan [13]. Also this notion is closely connected to the sequential Γ-convergence of certain related integral functionals. For details we refer to the well-written monograph of DalMaso [3].

3. Main results

We will need the following hypotheses on the data of (1):

$H(A)$: $A: T\times X\times E\to X^*$ is an operator such that

(1) $\|A(t+\tau, x, \lambda)-A(t, x, \lambda)\|_*\leqq O(\tau)(1+\|x\|^{p-1})$ for all t, $t+r\in T$, $x\in X$, $\lambda\in E$ and with $O(\tau)$ being independent of λ and x,

(2) $x\to A(t, x, \lambda)$ is hemicontinuous (i.e. $r\to\langle A(t, x+ry, \lambda), z\rangle$ is continuous from $[0, 1]$ to $\boldsymbol{R}$, for every $x, y, z\in X$),

(3) $\langle A(t, x, \lambda)-A(t, y, \lambda), x-y\rangle\geqq c_{1B}\|x-y\|^p$ for all $t\in T$, x, $y\in X$, $\lambda\in B\subseteq E$ compact and with $c_{1B}>0$ and $2\leqq p<\infty$ (i.e. $A(t, \cdot, \lambda)$ is strongly monotone, uniformly for $\lambda\in B$),

(4) $\|A(t, x, \lambda)\|_*\leqq c_{2B}(1+\|x\|^{p-1})$ for all $t\in T$, $x\in X$, $\lambda\in B\subseteq E$ compact and with $c_{2B}>0$,

(5) if $\lambda_n\to\lambda$, then for all $t\in T A(t, \cdot, \lambda_n)\xrightarrow{G} A(t, \cdot, \lambda)$.

$H(f)$: $f: T\times H\times E\to\mathcal{L}(Y, H)$ is a map such that

(1) $t\to f(t, x, \lambda)$ is measurable,

(2) $\|f(t, x, \lambda)\|_{\mathfrak{L}}\leqq a_B(t)+b_B|x|^{2/q}$ $a.e.$ for all $\lambda\in B\subseteq E$ compact and with $a_B(\cdot)\in L^q(T)$,

(3) $\|f(t, x, \lambda)-f(t, y, \lambda)\|_{\mathfrak{L}}\leqq k_B(t)|x-y|$ $a.e.$ for all $\lambda\in B\subseteq E$ compact and with $k_B(\cdot)\in L^1(T)$,

(4) $\lambda\to f(t, x, \lambda)u$ and $\lambda\to f(t, x, \lambda)^*u$ are both continuous (here $u\in Y$, $u^*\in Y^*$).

$H(U)$: $U: T\to P_{fc}(Y)$ is a measurable multifunction such that $|U(t)|=\sup\{\|u\|: u\in U(t)\}\leqq M$ with $M>0$.

H_0: $\lambda\to x_0(\lambda)$ is continuous from E into H.

Also our hypothesis on the cost integrand $L(t, x, \lambda, u)$ is the following:

$H(L)$: $L: T\times H\times E\times Y\to\bar{\boldsymbol{R}}:=\boldsymbol{R}\cup\{+\infty\}$ is an integrand such that

(1) $(t, x, \lambda, u)\to L(t, x, \lambda, u)$ is measurable,

(2) $(x, \lambda, u)\to L(t, x, \lambda, u)$ is lower semicontinuous ($l.s.c.$),

(3) $\varphi_B(t)-c_{3B}(|x|+\|u\|_Y)\leqq L(t, x, \lambda, u)$ for all $\lambda\in B\subseteq E$ compact, with $\varphi_B(\cdot)\in L^1(T)$ and $c_{3B}>0$.

Given $[\lambda, u]\in E\times S_U$ (recall $S_U=\{u: T\to Y$ measurable such that $u(t)\in U(t)$ $a.e.\}$), under hypotheses $H(A)$, $H(f)$ and H_0, we know (cf. Papageorgiou [8], theorem 3.1), that problem (1) admits a solution $x(\lambda, u)(\cdot)\in W_{pq}(T)\subseteq C(T, H)$. Furthermore using the monotonicity of $A(t, \cdot, \lambda)$ (hypothesis $H(A)$(3)) and the Lipschitzness of $f(t, \cdot, \lambda)$ (hypothesis $H(f)$(3)), we can easily check that $x(\lambda, u)(\cdot)$ is unique. So we can define the map $p: E\times S_U\to L^p(T, H)$ by $p(\lambda, u)(\cdot)=x(\lambda, u)(\cdot)$ (the solution map). In what follows on $S_U\subseteq L^\infty(T, Y)$ we consider the relative w^*-topology. Since $L^\infty(T, Y)=L^1(T, Y^*)^*$ and $L^1(T, Y^*)$ is separable (recall that Y is separable reflexive), it is well known that S_U topologized as above is compact and metrizable.

Our first result establishes the continuity properties of $p(\cdot, \cdot)$:

Proposition 1. *If hypotheses $H(A)$, $H(f($, $H(U)$ and H_0 hold, then $p: E\times S_U\to L^p(T, H)$ is continuous.*

Proof. Let $[\lambda_n, u_n]\to[\lambda, u]$ in $E\times S_U$ and let $x_n(\cdot)=p(\lambda_n, u_n)(\cdot)$ and $x(\cdot)=p(\lambda, u)(\cdot)$. First we establish some a priori bounds for the states $\{x_n\}_{n\geq1}$. To this end, with $B=\{\lambda_n, \lambda\}_{n\geq1}\subseteq E$ we have:

$$\dot{x}_n(t)+A(t, x_n(t), \lambda_n)=f(t, x_n(t), \lambda_n)u_n(t) \ a.e.$$

$$\Rightarrow \langle\dot{x}_n(t), x(t)\rangle+\langle A(t, x_n(t), \lambda_n), x_n(t)\rangle=(f(t, x_n(t), \lambda_n)u_n(t), x_n(t)) \ a.e.$$

$$\Rightarrow\frac{1}{2}\frac{d}{dt}|x_n(t)|^2+c_{4B}\|x_n(t)\|^p\leqq|f(t, x_n(t), \lambda_n)u_n(t)|\cdot|x_n(t)|+c_{5B} \ a.e.$$

(for some c_{4B}, $c_{5B}>0$; cf. hypotheses $H(A)$(3) and (4)),

$$\Rightarrow\frac{1}{2}\frac{d}{dt}|x_n(t)|^2+c_{4B}\|x_n(t)\|^p\leqq|f(t, x_n(t), \lambda_n)u_n(t)|\gamma\|x_n(t)\|+c_{tB} \ a.e.$$

(for some $\gamma>0$ such that $|\cdot|\leqq\gamma\|\cdot\|$; recall X embeds into H continuously),

$$\Rightarrow\frac{d}{dt}|x_n(t)|^2+2c_{4B}\|x_n(t)\|^p\leqq2|f(t, x_n(t), \lambda_n)u_n(t)|\gamma\|x_n(t)\|+2c_{5B} \ a.e.$$

Using the elementary inequality $ab\leqq(\varepsilon^q/q)a^q+(1/\varepsilon^p p)b^p$, $a, b, \varepsilon>0$ (Young's inequality), we get

$$\frac{d}{dt}x_n(t)|^2+2c_{4B}\|x_n(t)\|^p\leqq2\gamma\left(\frac{\varepsilon^q}{q}|f(t, x_n(t), \lambda_n)u_n(t)|^q+\frac{1}{\varepsilon^p p}\|x_n(t)\|^p\right)+2c_{5B} \quad a.e.$$

Choose $\varepsilon>0$ so that $2\gamma/\varepsilon^p p=2c\Rightarrow\varepsilon=(\gamma/cp)^{1/p}$, to get

$$\frac{d}{dt}|x_n(t)|^2 \leqq c_{6B}|f(t, x_n(t), \lambda_n)u_n(t)|^q + 2c_{5B}$$

$$\left(\text{with } c_{6B} = \frac{2\gamma}{q}\left(\frac{\gamma}{c_{5B}p}\right)^{p-1} > 0\right),$$

$$\Rightarrow |x_n(t)|^2 \leqq |x_0(\lambda_n)|^2 + \int_0^t M^q c_{6B}(a_B(s) + b_B|x_n(s)|^{2/q})^q ds + 2bc_{5B}$$

$$\leqq M_1^2 + 2^{q-1}M^q a_{6B}\|a_B\|_q^q + 2^{q-1}M^q c_{6B} b_B^q \int_0^t |x_n(s)|^2 ds + 2b_B c_{5B}$$

$$(\text{with } M_1 > 0 \text{ such that } \sup_{n\geq 1}|x_0(\lambda_n)| \leqq M_1).$$

Invoking Gronwall's inequality, we deduce that there exists $M_2 > 0$ such that for all $n \geqq 1$ and all $t \in T$, we have

$$|x_n(t)| \leqq M_2.$$

So we can write

$$\frac{d}{dt}|x_n(t)|^2 + 2c_{4B}\|x_n(t)|^p \leqq 2|f(t, x_n(t), \lambda_n)u_n(t)|M_2 + 2c_{5B} \ a.e.$$

$$\Rightarrow 2c_{4B}\int_0^b \|x_n(t)\|^p dt \leqq M_1^2 + 2MM_2\int_0^b \|f(t, x_n(t), \lambda_n)\|_* dt + 2bc_{5B}$$

$$\leqq M_1^2 + 2MM_2\int_0^b (a_B(t) + b_B M_2^{2/q})dt + 2bc_{5B}.$$

From this inequality, we deduce that there exists $M_3 > 0$ such that for all $n \geqq 1$, we have

$$\|x_n\|_{L^p(T,X)} \leqq M_3.$$

Finally recall that $\dot{x}_n(t) + A(t, x_n(t), \lambda_n) = f(t, x_n(t), \lambda_n)u_n(t)$ $a.e.$ Combining the above bounds with the growth hypotheses $H(A)(4)$ and $H(f)(2)$, we get that there exists $M_4 > 0$ such that for all $n \geqq 1$

$$\|\dot{x}_n\|_{L^q(T,X^*)} \leqq M_4.$$

Therefore we have shown that the sequence $\{x_n(\cdot)\}_{n\geq 1}$ is bounded in $W_{pq}(T)$ and since the latter is reflexive, by the Eberlein-Smulian theorem (cf. Lakshmikantham-Leela [7], theorem 1.1.12, p. 7), we have that $\{x_n(\cdot)\}_{n\geq 1}$ is relatively sequentially weakly compact. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} \hat{x}$ in $W_{pq}(T)$. Since $W_{pq}(T)$ embeds compactly in $L^p(T, H)$, we have $x_n \to \hat{x}$ in $L^p(T, H)$.

Next consider the following evolution equation:

$$\left\{ \begin{array}{c} \dot{y}_n(t) + A(t, y_n(t), \lambda_n) = f(t, \hat{x}(t), \lambda)u(t) \ a.e. \\ y_n(0) = x_0(\lambda). \end{array} \right\}$$

From theorem 1 of Kolpakov [6] (see also Lemma 5 of that paper), we have that $y_n \to y$ in $L^p(T, H)$, with $y \in W_{pq}(T, H) \subseteq L^p(T, H)$ being the unique solution of the evolution equation

$$\left\{ \begin{array}{c} \dot{y}(t)+A(t, y(t), \lambda)=f(t, \hat{x}(t), \lambda)u(t) \ a.e. \\ y(0)=x_0(\lambda). \end{array} \right\}$$

Let $g_n(\cdot)=\hat{f}(x_n, \lambda_n)(\cdot)=f(\cdot, x_n(\cdot), \lambda_n)u_n(\cdot)$ and $g(\cdot)=\hat{f}(\hat{x}, \lambda)(\cdot)=f(\cdot, \hat{x}(\cdot), \lambda)u(\cdot)$. Our claim is that $g_n(\cdot) \to g(\cdot)$ in $L^p(T, H)$. Indeed let $h \in L^p(T, H)$. Then we have:

$$((g_n, h))_{pq}=\int_0^b (f(t, x_n(t), \lambda_n)u_n(t), h(t))dt$$

$$=\int_0^b (u_n(t), f(t, x_n(t), \lambda_n)^* h(t))_{YY^*} dt.$$

From hypothesis $H(f)$ (4) and the dominated convergence theorem, we get

$$\int_0^b (u_n(t), f(t, x_n(t), \lambda_n)^* h(t))_{YY^*} dt \to \int_0^b (u(t), f(t, x(t), \lambda)^* h(t))_{YY^*} dt$$

$$\Rightarrow ((g_n, h))_{pq} \longrightarrow ((g, h))_{pq}.$$

Now note that

$$\langle \dot{x}_n(t)-\dot{y}_n(t), x_n(t)-y_n(t)\rangle+\langle A(t, x_n(t), \lambda_n)-A(t, y_n(t), \lambda_n), x_n(t)-y_n(t)\rangle$$

$$+(g_n(t)-g(t), x_n(t)-y_n(t)) \ a.e.$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}|x_n(t)-y_n(t)|^2 \leqq (g_n(t)-g(t), x_n(t)-x(t)) \ a.e.$$

(since by hypothesis $H(A)$, $A(t, \cdot, \lambda_n)$ is monotone for all $n \geqq 1$)

$$\Rightarrow \frac{1}{2}|x_n(t)-y_n(t)|^2 \leqq \frac{1}{2}|x_0(\lambda_n)-x_0(\lambda)|^2+((g_n-g, x_n-y_n))_{pq}.$$

But since $g_n \xrightarrow{w} g$ in $L^q(T, H)$, we have $((g_n-g, x_n-y_n))_{pq} \to 0$ as $n \to \infty$. So for all $t \in T$, we have

$$|x_n(t)-y_n(t)|^2 \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, we know that $x_n \to \hat{x}$ and $y_n \to y$ in $L^p(T, H)$. Therefore we conclude that $\hat{x}=y$. So we have

$$\left\{ \begin{array}{c} \dot{\hat{x}}(t)+A(t, \hat{x}(t), \lambda)=f(t, \hat{x}(t), \lambda)u(t) \ a.e. \\ x(0)=x_0(\lambda), \ u(t) \in U(t) \ a.e. \end{array} \right\}$$

Hence $\hat{x}(\cdot)=p(\lambda, u)(\cdot)=x(\cdot) \Rightarrow x_n(\cdot)=p(\lambda_n, u_n)(\cdot) \to p(\lambda, u)(\cdot)=x(\cdot)$ in $L^p(T, H) \Rightarrow p(\cdot, \cdot)$ is continuous as claimed. Q.E.D.

Next we examine the cost functional $[\lambda, u] \to J(\lambda, u)$:

Proposition 2. *If hypotheses $H(A)$, $H(f)$, $H(U)$, H_0 and $H(L)$ hold, then $[\lambda, u] \to J(\lambda, u)$ is l.s.c. from $E \times S_U$ into $\bar{\boldsymbol{R}} = \boldsymbol{R} \cup \{+\infty\}$.*

Proof. We need to show that for every $\theta \in \boldsymbol{R}$, the level set

$$K_\theta = \{[\lambda, u] \in E \times : J(\lambda, u) \leqq \theta\}$$

is closed. To this end let $[\lambda_n, u_n] \in K_\theta$ $n \geqq 1$ and assume that $[\lambda_n, u_n] \to [\lambda, u]$ in $E \times S_U$. If we set $x_n(\cdot) = p(\lambda_n, u_n)(\cdot)$ and $x(\cdot) = p(\lambda, u)(\cdot)$, from proposition 1 we have that $x_n \to x$ in $L^p(T, H)$. Recall that E being a complete metric space is isometrically isomorphic to a closed subset of a Banach space V. Let $B = \{\lambda_n, \lambda\}_{n \geq 1} \subseteqq E$ compact and let V_B be the closed subspace of V generated by the isometric image of B. Then V_B is separable (in fact, compactly generated). Hence due to hypothesis $H(L)$, we can apply theorem 2.1 of Balder [1] and get that

$$\int_0^b L(t, x(t), \lambda, u(t))dt \leqq \varliminf_{n \to \infty} \int_0^b L(t, x_n(t), \lambda_n, u_n(t))dt$$

$$\Rightarrow J(\lambda, u) \leqq \varliminf_{n \to \infty} J(\lambda_n, u_n) \leqq \theta$$

$$\Rightarrow [\lambda, u] \in K_\theta .$$

So indeed K_θ is closed in $E \times S_U$ and so $J(\cdot, \cdot)$ is *l.s.c.* Q.E.D.

Now we are ready to establish the existence of an optimal control for problem (P_1). Since our cost functional is $\bar{\boldsymbol{R}}$-valued, we will also need the following hypothesis:

H_1: there exists at least one $u \in S_U$ such that for all $\lambda \in E$, $J(\lambda, u) \leqq \hat{M}$, with $\hat{M} > 0$.

This hypothesis, together with $H(L)$, guarantees that the value β of the problem is finite. Using the a priori bounds obtained in the process of the proof of proposition 1, we can easily see that hypothesis H_1 is satisfied if for example E is compact and for all $\lambda \in E$, $x \in H$ with $|x| \leqq r$ and $u \in Y$ with $\|u\|_Y \leqq M$, we have $L(t, x, \lambda, u) \leqq \psi_r(t)$ *a.e.* with $\psi_r(\cdot) \in L^1(T)$.

Theorem 3. *If hypothesis $H(A)$, $H(f)$, $H(U)$, H_0, $H(L)$ and H_1 hold, then problem (P_1) admits an optimal control.*

Proof. Since by proposition 2, $J(\cdot, \cdot)$ is *l.s.c.*, from theorem 4, p. 122 of Berge [2], we know that $u \to m(u) = \sup_{\lambda \in E} J(\lambda, u)$ is *l.s.c.* from S_U into $\boldsymbol{R}$. So $\inf_{u \in S_U} m(u)$ admits a solution, i.e. there exists $\hat{u} \in S_U$ such that $\beta = m(\hat{u})$. Q.E.D.

Now we turn our attention to the parametric optimal control problem (P_2).

To analyze this problem and establish the continuity of the value $\lambda \to m_1(\lambda) = \inf[J(\lambda, u): u \in S_U]$ (robustness or well-posedness of the optimization problem), we will need the following stronger hypothesis on the cost integrand $L(t, x\lambda, u)$:

$H(L)_1$: $L: T \times H \times E \times Y \to \boldsymbol{R}$ is an integrand such that
(1) $(t, u) \to L(t, x, \lambda, u)$ is measurable,
(2) $(x, \lambda, u) \to L(t, x, \lambda, u)$ is *l.s.c.*,
(3) $(x, \lambda) \to L(t, x, \lambda, u)$ is continuous,
(4) $L(t, x, \lambda, \cdot)$ is convex,
(5) for every $B \subseteq E$ compact and $r > 0$, there exists $\psi_{rB}(\cdot) \in L^1(T)$ such that $|L(t, x, \lambda, u)| \leq \psi_{rB}(t)$ *a.e.* for all $\lambda \in B$, all $x \in H$ with $|x| \leq r$ and all $u \in Y$ with $\|u\|_Y \leq M$.

Then we have the following existence and stability result concerning problem (P_2):

Theorem 4. *If hypotheses $H(A)$, $H(f)$, $H(U)$, H_0 and $H(L)_1$ hold, then for every $\lambda \in E$, (P_1) has a solution and $\lambda \to m_1(\lambda)$ is continuous.*

Proof. The existence part of the theorem is an immediate consequence of propositions 1 and 2. So we need to prove the stability part.

Let $\lambda_n \to \lambda$ in E. For each $n \geq 1$, let $u_n \in S_U$ such that $m_1(\lambda_n) = J(u_n, \lambda_n)$. By passing to a subsequence if necessary, we may assume that $u_n \to u$ in S_U (recall on S_U we consider the relative $w^* - L^\infty(T, Y)$ topology). Hence proposition 1 tells us that $x_n(\cdot) = p(\lambda_n, u_n)(\cdot) \to x(\cdot) = p(\lambda, u)(\cdot)$ in $L^p(T, H)$. Then as in the proof of proposition 2, via theorem 2.1 of Balder [1], we get that

$$m_1(\lambda) \leq J(\lambda, u) \leq \underline{\lim}\, J(\lambda_n, u_n) = \underline{\lim}\, m_1(\lambda_n). \tag{2}$$

Next let $u \in S_U$ such that $m_1(\lambda) = J(\lambda, u)$. Let $x_n(\cdot) = p(\lambda_n, u)(\cdot)$. Then from proposition 1, we have $x_n(\cdot) = p(\lambda_n, u)(\cdot) \to x(\cdot) = p(\lambda, u)(\cdot)$ in $L^p(T, H)$. Also because of hypotheses $H(L)_1$ (3) and (5) and the dominated convergence theorem, we get that

$$J(\lambda_n, u) \longrightarrow J(\lambda, u) = m_1(\lambda)$$
$$\Rightarrow \overline{\lim}\, m_1(\lambda_n) \leq m_1(\lambda). \tag{3}$$

From (2) and (3) we conclude that $m_1(\lambda_n) \to m_1(\lambda)$; i.e. $m_1(\cdot)$ is continuous.

Q.E.D.

4. Applications

In this section we present three examples of distributed parameter systems, which illustrate the applicability of this work.

(A) Let $Z\subseteqq \boldsymbol{R}^N$ be a bounded domain with smooth boundary $\Gamma=\partial Z$. We consider the following nonlinear parabolic control system:

$$\left\{\begin{array}{l}\dfrac{\partial x}{\partial t}-div\ a(z,\ Dx(z),\ \lambda)=f(t,\ z,\ x(t,\ z),\ \lambda)+B(t,\ z,\ \lambda)u(t,\ z)\ a.e.\ \text{on}\ T\times Z\\ x(0,\ z)=x_0(z,\ \lambda)\ a.e.\ \text{on}\ Z,\ x|_{T\times\Gamma}=0,\ \|u(t,\ \cdot)\|_{L^2(Z)}\leqq r(t)\ a.e.\end{array}\right\} \tag{4}$$

The cost criterion is given by

$$J(\lambda,\ u)=\int_0^b\int_Z L(t,\ z,\ x(t,\ z),\ \lambda,\ u(t,\ z))dzdt.$$

Let $S_U=\{u\in L^2(T\times Z):\ \|u(t,\ \cdot)\|_{L^2(Z)}\leqq r(t)\ a.e.\}$. As before our problem is:

$$\inf_{u\in S_U}\sup_{\lambda\in E} J(\lambda,\ u)=\beta. \tag{P_3}$$

We will need the following hypotheses on the data:

$H(a)$: $a(z,\ \xi,\ \lambda)=\partial_\xi\varphi(z,\ \xi,\ \lambda)$ with $\varphi:\ Z\times\boldsymbol{R}^n\times E\rightarrow\boldsymbol{R}$ is a function such that

(1) $z\rightarrow\varphi(z,\ \xi,\ \lambda)$ is measurable,

(2) $\varphi(z,\ \cdot,\ \lambda)$ is strongly convex (cf. Zeidler [12], p. 264) and positivively homogeneous of degree $p\in[2,\ \infty)$ (hence $\partial_\xi\varphi(z,\ \xi,\ \lambda)$ denotes the Frechet derivative of $\varphi(z,\ \cdot,\ \lambda)$),

(3) $c_{1B}\|\xi\|^p-c_{2B}\leqq\varphi(z,\ \xi,\ \lambda)\leqq c_{3B}(1+\|\xi\|^p)$ for all $(z,\ \xi)\in Z\times\boldsymbol{R}^N$, all $\lambda\in B\subseteqq E$ compact and with $c_{1B},\ c_{2B},\ c_{3B}<0$,

(4) if $\lambda_n\rightarrow\lambda$ in E, then $\varphi(z,\ \xi,\ \lambda_n)\rightarrow\varphi(z,\ \xi,\ \lambda)$ $a.e.$ on Z.

$H(f)_1$: $f:\ T\times Z\times\boldsymbol{R}\times E\rightarrow\boldsymbol{R}$ is a function such that

(1) $(t,\ z)\rightarrow f(t,\ z,\ x,\ \lambda)$ is measurable,

(2) $|f(t,\ z,\ x,\ \lambda)-f(t,\ z,\ y,\ \lambda)|\leqq k_B(t,\ z)|x-y|$ $a.e.$ for all $\lambda\in B\subseteqq E$ compact and with $k_B\in L^\infty(T\times Z)$,

(3) $\lambda\rightarrow f(t,\ z,\ x,\ \lambda)$ is continuous,

(4) $|f(t,\ z,\ 0,\ \lambda)|\leqq a_B(t,\ z)$ $a.e.$ for all $\lambda\in B\subseteqq E$ compact and with $a_B(\cdot,\ \cdot)\in L^\infty(T,\ L^2(Z))$.

$H(B)$: $B:\ T\times Z\times E\rightarrow\boldsymbol{R}$ is a function such that

(1) $(t,\ z)\rightarrow B(t,\ z,\ \lambda)$ is measurable,

(2) $\lambda\rightarrow B(t,\ z,\ \lambda)$ is continuous,

(3) $B(t,\ \cdot,\ \lambda)\in L^\infty(Z)$ and $t\rightarrow\|B(t,\ \cdot,\ \lambda)\|_{L^\infty(Z)}\in L^2(T)$.

$H(r)$: $r\in L^\infty(T)$.

H_0': $\lambda\rightarrow x_0(\lambda)(\cdot)$ is continuous from E into $L^2(Z)$.

$H(L)_2$: $L:\ T\times Z\times\boldsymbol{R}\times E\times\boldsymbol{R}\rightarrow\bar{\boldsymbol{R}}=\boldsymbol{R}\cup\{+\infty\}$ is an integrand such that

(1) $(t,\ z,\ x,\ \lambda,\ u)\rightarrow L(t,\ z,\ x,\ \lambda,\ u)$ is measurable,

(2) $(x,\ \lambda,\ u)\rightarrow L(t,\ z,\ x,\ \lambda,\ u)$ is $l.s.c.$,

(3) $L(t,\ z,\ x,\ \lambda,\ \cdot)$ is convex,

(4) $\psi_B(t, z) - c_B(z)(|x|+|u|) \leqq L(t, z, x, u)$ *a.e.* for all $\lambda \in B \subseteq E$ compact and with $\psi_B \in L^1(T \times Z)$ and $c_B \in L^\infty(Z)$.

Theorem 5. *If hypotheses $H(a)$, $H(f)_1$, $H(B)$, $H(r)$, H'_0, $H(L)$ and H_1 hold, then problem (P_3) admits an optimal control.*

Proof. From hypothesis $H(a)$, we have that

$$z \longrightarrow a(z, \xi, \lambda) \text{ is measurable}$$

and $\xi \to a(z, \xi, \lambda)$ is strongly monotone (cf. Zeidler [12], p. 264).

Let $X = W_0^{1,p}(Z)$, $H = L^2(Z)$, $X^* = W^{-1,q}(Z)$ and $Y = L^2(Z)$. From the Sobolev embedding theorem, we know that (X, H, X^*) is an evolution triple with all embeddings being compact. Let $\alpha: W_0^{1,p}(Z) \times W_0^{1,p}(Z) \times E \to \boldsymbol{R}$ be the Dirichlet form defined by

$$\alpha(x, y, \lambda) = \int_Z a(z, Dx(z), \lambda) Dy(z) dz .$$

Then using the properties of $a(z, \xi, \lambda)$ (derived easily from those of $\varphi(z, \xi, \lambda)$; cf. hypothesis $H(a)$), we have for all $\lambda \in B \subseteq E$ compact

$$|\alpha(x, y, \lambda)| \leqq \hat{c}_{1B} \|x\|^{p-1} \|y\|$$

and

$$\alpha(x, x-y, \lambda) - \alpha(y, x-y, \lambda) \geqq \hat{c}_{2B} \|x-y\|^{p-1}$$

with $\hat{c}_{1B}, \hat{c}_{2B} > 0$. Then define $A: W_0^{1,p}(Z) \times E \to W^{-1,q}(Z)$ by

$$\langle A(x, \lambda), y \rangle = \alpha(x, y, \lambda) .$$

Clearly then $A(\cdot, \lambda)$ satisfies hypotheses $H(A)(1) \to (4)$. Furthermore, if we set

$$\Phi(x, \lambda) = \int_Z \varphi(z, Dx(z), \lambda) dz \qquad [x, \lambda] \in W_0^{1,p}(Z) \times E ,$$

from hypothesis $H(a)(4)$ and theorem 5.14, p. 51 of DalMaso [3], for $\lambda_n \to \lambda$ in E we have

$$\Gamma_{seq}(w - W_0^{1,p}(Z)) \Phi(\cdot, \lambda_n) = \Phi(\cdot, \lambda) .$$

So invoking theorems 3.3 and 2.17 of Defranceschi [4], we get

$$A(\cdot, \lambda_n) \xrightarrow{G} A(\cdot, \lambda) .$$

Thus we have satisfied hypothesis $H(A)$.

Also set $\hat{f}(t, x, \lambda)(\cdot) = f(t, \cdot, x(\cdot), \lambda)$, $\hat{B}(t, \lambda)(\cdot) = B(t, \cdot, \lambda)$ and $\hat{L}(t, x, \lambda, u) = \int_Z L(t, z, x(z), \lambda, u(z)) dz$ and using hypotheses $H(f)_1$, $H(B)$ and $H(L)_2$, we can easily see that $H(f)$ and $H(L)$ hold.

Now rewrite (4) in the following equivalent abstract form

$$\left\{\begin{array}{c} \dot{x}(t)+A(x(t),\lambda)=\hat{f}(t,x(t),\lambda)+\hat{B}(t,\lambda)u(t)\ a.e. \\ x(0)=\hat{x}_0(\lambda),\ u(t)\in U(t)\ a.e. \end{array}\right\}$$

where $\hat{x}_0(\lambda)(\cdot)=x_0(\cdot,\lambda)\in L^2(Z)$ and $U(t)=\{u\in L^2(Z)=Y : \|u\|_2\leqq r(t)\}$. Also rewrite the cost functional as $J(\lambda,u)=\int_0^b \hat{L}(t,x(t),\lambda,u(t))dt$. Thus problem (P_3) is a particular case of problem (P_1). Applying theorem 3, we get the desired optimal control. Q.E.D.

This example incorporates the generalized nonlinear heat equation. More precisely, consider the following system:

$$\left\{\begin{array}{l} \frac{\partial x}{\partial t}-\sum_{k=1}^{N} D_k(a(z,\lambda)|D_k x(z)|^{p-2}D_k x(z))=f(t,z,x(t,z),\lambda) \\ \qquad\qquad\qquad +B(t,z,\lambda)u(t,z)\ a.e.\text{ on } T\times Z \\ x(0,z)=x_0(z,\lambda)\ a.e.\text{ on } Z,\ x|_{T\times\Gamma}=0,\quad \|u(t,\cdot)\|_2\leqq r(t)\ a.e. \end{array}\right\} \tag{5}$$

This system is a particular instant of (4). Indeed in this case

$$a(z,\xi,\lambda)=\partial_\xi\varphi(z,\xi,\lambda)$$

with $\varphi(z,\xi,\lambda)=\frac{1}{p}\sum_{k=1}^{N} a(z,\lambda)|\xi_k|^p$.

So if we assume, for example, that $a(z,\lambda_n)\xrightarrow{G} a(z,\lambda)$ $r.e.$ on Z, then we have $A(\cdot,\lambda_n)\to A(\cdot,\lambda)$ (in this case $A(\cdot,\lambda)$, $\lambda\in E$ is the pseudo-Laplacian) and so theorem 5 holds.

(B) Again let $Z\subseteq \boldsymbol{R}^n$ be a bounded domain with smooth boundary $\Gamma=\partial Z$. We consider the following semilinear parabolic control system, parametrized by $\lambda\in E$:

$$\left\{\begin{array}{l} \frac{\partial x}{\partial t}-\sum_{i,j=1}^{N} D_j(a_{ij}(z,\lambda)D_i x(t,z))=f(t,z,x(t,z),\lambda) \\ \qquad\qquad\qquad +B(t,z,\lambda)u(t,z)\ a.e.\text{ on } T\times Z \\ x(0,z)=x_0(z,\lambda)\ a.e.\text{ on } Z,\ x|_{T\times\Gamma}=0 \text{ and } \|u(t,\cdot)\|_2\leqq r(t)\ a.e. \end{array}\right\} \tag{6}$$

In this case the cost functional is the following quadratic criterion:

$$J(\lambda,u)=\frac{1}{2}\int_0^b\int_Z \theta_1(z,\lambda)|x(t,z)-\hat{x}(t,z)|^2dzdt+\frac{\sigma}{2}\int_0^b\int_Z \theta_2(z,\lambda)|u(t,z)|^2dzdt$$

with $\sigma>0$. As before we consider the problem

$$\inf_{u\in S_U}\sup_{\lambda\in E} J(\lambda,u)=\beta. \tag{P_4}$$

We will need the following hypotheses on the data:

$H(a)_1$: for each $i, j=1, \cdots, N, (t, z)\to a_{ij}(t, z)$ is measurable, $\gamma_B^1\|\xi\|^2 \leqq \sum_{i,j=1}^N a_{ij}(t, z, \lambda)$ $\leqq \gamma_B^2\|\xi\|^2$ for all $(t, z)\in T\times Z$ and all $\lambda\in B\subseteqq E$ compact and with $\gamma_B^1, \gamma_B^2>0$, $a_{ij}=a_{ji}$ and finally $|a_{ij}(t, z, \lambda)-a_{ij}(t', z, \lambda)|\leqq \hat{k}_B(z)|t-t'|$ *a.e.* with $\hat{k}_B(\cdot)\in L^\infty(Z)$ and $\lambda\in B\subseteqq E$ compact.

$H(\theta)$: $0<d_{1B}\leqq \theta_1(z, \lambda), \theta_2(z, \lambda)\leqq d_{2B}$ for all $(z, \lambda)\in Z\times B$, $B\subseteqq E$ compact, $z\to \theta_1(a, \lambda), \theta_2(z, \lambda)$ are measurable and $\lambda\to\theta_1(z, \lambda), \theta_2(z, \lambda)$ are continuous.

H_2: If $\lambda_n\to\lambda$ in E, then $a_{ij}(t, \cdot, \lambda_n)\overset{w}{\to} a_{ij}(t, \cdot, \lambda)$ in $L^2(Z)$ and $D_t a_{ij}(t, \cdot, \lambda_n)\to D_t a_{ij}(t, \cdot, \lambda)$ in $H^{-1}(Z)$ for every $j=1, 2, \cdots, N$.

Remark. If $N=1$, then the above hypothesis is equivalent to saying that $a(t, \cdot, \lambda_n)\to a(t, \cdot, \lambda)$ in $L^2(Z)$. In this case hypothesis H_2 takes the following more general form: $1/a(t, \cdot, \lambda_n)\overset{w}{\to} 1/a(t, \cdot, \lambda)$ in $L^2(Z)$ for all $t\in T$ (see Zhikov et al. [13]).

Note that in this case, hypothesis H_1 is automatically satisfied.

Theorem 6. *If hypothese $H(a)_1$, $H(\theta)$, $H(f)_1$, $H(B)$, $H(r)$ and H_2 hold, then problem (P_4) admits an optimal control.*

Proof. In this case the evolution triple is $(H_0^1(Z), L^2(Z), H^{-1}(Z))$. So if we define $A: T\times H_0^1(Z)\times E\to H^1(Z)$ by

$$\langle A(t, x, \lambda), y)=\int_Z \sum_{j,j=1}^N a_{ij}(t, z, \lambda)D_i x(z)Dy(z)dz,$$

then clearly this operator satisfies hypotheses $H(A)(1)\to(4)$. Furthermore if we set

$$\Phi(t, x, \lambda)=\int_Z \sum_{i,j=1}^N a_{ij}(t, z, \lambda)D_i x(z)D_j x(z)dz$$

then from hypothesis H_2, we have that if $\lambda_n\to\lambda$ in E, then

$$\Gamma_{seq}(w-H_0^1(Z))-\Phi(t, \cdot, \lambda_n)=\Phi(t, \cdot, \lambda) \text{ (cf. DalMaso [3])}$$

$$\Rightarrow A(t, \cdot, \lambda_n)\xrightarrow{G} A(t, \cdot, \lambda) \text{ (cf. Defranceschi [4])}.$$

So we can apply theorem 3 and get an optimal control for (P_4). Q.E.D.

(C) The abstract framework of this paper allows us to consider also homogenization problems. As before let $Z\subseteqq \mathbf{R}^N$ be a bounded domain with smooth boundary $\Gamma=\partial Z$. Let $E=[0, 1]$. For $\lambda\in(0, 1]$, we consider

$$\left\{\begin{array}{l} \dfrac{\partial x}{\partial t}-div\, a\Big(\dfrac{z}{\lambda}, Dx(t, z)\Big)=f(t, z, x(t, z), \lambda) \\ \qquad\qquad\qquad +B(t, z, \lambda)u(t, z) \ a.e. \text{ on } T\times Z \\ x(0, z)=x_0(z, \lambda) \ a.e. \text{ on } Z,\ x|_{T\times\Gamma}=0 \text{ and } \|u(t, \cdot)\|_2\leqq r(t) \ a.e. \end{array}\right\} \qquad (6)_\lambda$$

For $\lambda=0$, we consider the homogenized system

$$\left\{\begin{array}{l} \dfrac{\partial x}{\partial t}-div\,\hat{a}(Dx(t,z))=f(t,z,x(t,z),\lambda)+B(t,z)u(t,z)\ a.e.\ \text{on}\ T\times Z \\ x(0,z)=x_0(z,\lambda)\ a.e.\ \text{on}\ Z,\ x|_{T\times\Gamma}=0\ \text{and}\ \|u(t,\cdot)\|_2\leq r(t)\ a.e. \end{array}\right\} \quad (6)$$

Our cost functional is the quadratic cost functional of example (B). In this case we examine the following parametric optimal control problem:

$$\inf[J(u,\lambda):\ u\in S_U]=m_1(\lambda). \qquad (P_5)$$

In what follows, K denotes the unit cube in $\boldsymbol{R}^N$.

$H(a)_2$: Hypothesis $H(a)$ holds and in addition $a(\cdot,\xi,\lambda)$ is K-periodic and satisfies

$$|a(z,\xi_1,\lambda)-a(z,\xi_2,\lambda)|\leq\hat{c}(\|\xi_1\|+\|\xi_2\|)^{p-2}\|\xi_1-\xi_2\|.$$

Then from Fusco-Moscariello [5], we know that $\hat{a}(\xi)$ in the homogenized state equation is given by

$$\hat{a}(\xi)=\int_K a(\theta,Dv(\theta))d\theta,$$

with $v(\cdot)$ being the unique solution of

$$\left\{\begin{array}{l} \int_K a(\theta,Dv(\theta))D\eta(\theta)d\theta=0 \quad \text{for all}\quad \eta\in W^{1,p}_{per}(K) \\ v(\theta)\in\xi\cdot\theta+W^{1,p}_{per}(K) \end{array}\right\}$$

where $W^{1,p}_{per}(K)=\{v\in W^{1,p}(K):\ u(\cdot)$ has the same trace on opposite faces of $K\}$. Then combining corollary 3.3 of Fusco-Moscariello [5], with theorem 4 of this paper we get:

Theorem 7. *If hypotheses $H(a)_2$, $H(f)_1$, $H(B)$, $H(r)$, $H(\theta)$ and H_0 hold, then for every $\lambda\in E$, problem (P_5) has a solution and if $\lambda_n\to 0$ in E then $m_1(\lambda_n)\to m_1(0)$ (i.e. $m_1(\cdot)$ is continuous at $\lambda=0$).*

Homogenization allows us to analyze the properties of inhomogeneous materials with periodic structure by considering a homogeneous (limit) material whose overall response is close to that of the original periodic material.

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