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THE SPECIAL LAGRANGIAN CONES OVER E_6/F_4 AND $SU(9)/Sp(3)$

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Abstract. We prove the existence of two new special lagrangian surfaces which are conical varieties over certain representations of the homogeneous spaces E_6/F_4 and $SU(6)/Sp(3)$.

1. Introduction

Lagrangian surfaces are of great interest not only to those working on dynamical systems, but also to the minimal surface specialists. In particular, it is well known that a special Lagrangian (SLAG) cone is an absolutely area-minimizing surface everywhere smooth except for a singular point at the origin [3]. In [1], the author mentioned the idea of looking into the various representations of the group $SU(n)$ to aid in the search for new SLAG cones. This has resulted in the discovery of the following SLAG cones: $SU(n)/O(n)$[1], $S(U(n)\times\cdots\times U(n))$ [1][2]. In this article, the search for the classification of SLAG cones continues with the addition of two new cones.

2. The homogeneous space E_6/F_4

Let $\boldsymbol{O}$ denote the Cayley algebra (octonions) over the reals, and denote by $M(3, \boldsymbol{O})$ the set of all 3×3 matrices with entries in $\boldsymbol{O}$. Let $\mathfrak{T}(3, \boldsymbol{O})$ be the subset of $M(3, \boldsymbol{O})$ consisting of all 3×3 Hermitian matrices. By Theorem 2 of [4], the tangent space at the identity point of the compact symmetric space E_6/F_4 is isomorphic to

$$\mathfrak{T}=\{T\in\mathfrak{J}(3, \boldsymbol{O})\,|\,\mathrm{tr}(T)=0\}.$$

Let $M(3, \boldsymbol{O})^C$ denote the complexification of the space $M(0, \boldsymbol{O})$. This induces a natural embedding of E_6/F_4 into the space $\mathfrak{J}(3, \boldsymbol{O})^C\cong \boldsymbol{C}^{27}$ (cf([4] Theorem 6).

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Theorem 1. *The cone over E_6/F_4 in C^{27} is SLAG.*

Proof. It is clear that for any pair of matrices $X, Y \in \mathfrak{T} \oplus i\boldsymbol{R}I$ ($I = 3\times 3$ identity matrix),

$$\mathrm{Re}\langle iX, Y\rangle = \omega(X, Y) = 0,$$

where i is the complex structure induced by the complexification, $\mathrm{Re}(z)$ denotes the real part of the complex number z, and ω is the standard Kähler 2-form on C^{27}. Thus the 27-dimensional space $\mathfrak{T} \oplus i\boldsymbol{R}I$ is a Lagrangian plane. Observe that E_6/F_4 in $C^{27} \cong \mathfrak{T}(3, \boldsymbol{O})^C$ is an orbit of E_6 through the point iI and the corresponding action of the Lie algebra $\mathcal{E}_6$ of E_6 on iI gives

$$\mathcal{E}_6 \cdot iI = \{X \cdot iI \mid X \in \mathcal{E}_6\} = \mathfrak{T}.$$

Furthermore, E_6 is an isometry of C^{27}, being a subgroup of $SU(27)$ [4]. Therefore, the cone is calibrated by the standard SLAG form on C^{27}, and this completes the proof. □

3. The homogeneous space $SU(6)/Sp(3)$

In this case, we replace the Cayley algebra in section 1 with the quaternion algebra $\boldsymbol{H}$ [4]. We note that here, the symmetric space $SU(6)/Sp(3)$ is now embedded in the space $\mathfrak{T}(3, \boldsymbol{H})^C \cong C^{15}$. Applying Proposition 15 in [4], the tangent space at the identity point of $SU(6)/Sp(3)$ is given by

$$\mathfrak{T} = \{T \in \mathcal{J}(3, \boldsymbol{H}) \mid \mathrm{tr}(T) = 0\}.$$

Theorem 2. *The cone over $SU(6)/Sp(3)$ in C^{15} is SLAG.*

Proof. In this case $\mathfrak{T} \oplus i\boldsymbol{R}I$ is a 15-dimensional vector space over the reals. The arguments are similar as in the proof of Theorem 1, and hence omitted. □

Remarks. As seen above, complexifying a linear space results in the original linear space being a Lagrangian plane with respect to the induced complex structure. If it also happens that the linear space is the tangent space of some manifold invariant under an isometry subgroup of the complexified space, then the manifold is SLAG. It would be interesting to know if there any other SLAG cones which can be produced in this manner.

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