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SUFFICIENTLY DECOMPOSABLE SURFACES IN THE 3-SPHERE

By

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1. Introduction

Waldhausen [W] showed that any Heegaard surface H of the 3-dimensional sphere S^3 is decomposed into a connected sum of unknotted tori. As its generalization, Tsukui [T1] and Suzuki [Su1] formulated a prime decomposition theorem for any pair $(F\subset S^3)$ of a connected, closed (=compact, without boundary), oriented surface F in S^3 with a fixed orientation, and discuss among others that whether such prime decompositions are unique. We refer the reader to Tsukui [T2], Suzuki [Su2], [Su3], and Motto [M] for some related topics.

In this paper, we give an affirmative answer to a question raised by Tsukui [T1, Conjecture (7.2)]; that is,

Theorem. *Let $(F\subset S^3)$ be a pair of a connected, closed, oriented surface F of genus $g(F)=n$ in S^3, and V_F and W_F be the closures of components of S^3-F. If both V_F and W_F have ∂-prime decompositions with n factors, then $(F\subset S^3)$ is not prime.*

In [T2], Tsukui gave the proof of this theorem for the case $n=2$. As a corollary to Theorem, we have the following by induction on n.

Corollary 1. *Under the hypothesis of Theorem, $(F\subset S^3)$ has a prime decomposition with n factors.*

Combining this with the knot complement theorem due to Gordon and Luecke [GL] and the uniqueness theorem of ∂-prime decomposition for a compact, orientable 3-manifold with connected boundary ([G], [Sw]), we have the following.

Corollary 2. *Under the hypothesis of Theorem, the prime decomposition for $(F\subset S^3)$ is unique, and the knot type of $(F\subset S^3)$ is determined by its complement*

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(V_F, W_F).

Throughout the paper, we shall only concern with the combinatorial category, consisting of simplicial complexes and piecewise-linear maps.

After establishing two systems of proper disks in 3-manifolds in § 1, we prove Theorem in § 2.

2. Preliminaries

We will make free use of notation and definitions which were introduced in the paper [Su1].

We use a complete disk system for a compact, connected and orientable 3-manifold with nonvoid connected boundary, which is a generalization of a complete disk system for a compression body (see Casson and Gordon [CG]).

Definition 1.1. Let $E_1, \cdots, E_k$ be exteriors of non-trivial knots in S^3, and $E_{k+1}, \cdots, E_n$ be solid tori ($\cong D^2 \times S^1$).

(1) Let M be a 3-manifold homeomorphic to the disk-sum $E_1 \natural \cdots \natural E_n$. A disjoint union $\boldsymbol{D}=D_1 \cup \cdots \cup D_{n-1}$ of proper disks in M is said to be a *decomposition disk system* for M iff $cl(M-N(\boldsymbol{D}: M))=E_1 \cup \cdots \cup E_k$, provided that $k \geqq 1$. If $k=0$, then M is a handlebody $n(D^2 \times S^1)$ of genus n, and a disjoint union $\boldsymbol{D}=D_1 \cup \cdots \cup D_n$ of proper disks in M is said to be a *decomposition disk system* iff $cl(M-N(\boldsymbol{D}\,;\, M)) \cong D^3$, which will be sometimes called a *complete meridian-disk system*.

(2) Let F be a connected, closed and orientable surface, and let $d_1, \cdots, d_n$ be mutually disjoint disks in one boundary component $F \times 1$ of $F \times I$. Let d_i' be a disk in ∂E_i for $i=1, \cdots, n$. Now let M be a 3-manifold obtained from $F \times I$ and $E_1 \cup \cdots \cup E_n$ by identifying d_i and d_i' for $i=1, \cdots, n$. Then, ∂M consists of two components, and we denote one which corresponds to $F \times 0$ by $\partial_- M$ and the other by $\partial_+ M$. $\partial_+ M$ is a closed orientable surface of genus $g(F)+n$.

A disjoint union $\boldsymbol{D}=D_1 \cup \cdots \cup D_n$ of n proper disks in M is said to be a *complete disk system* for M iff $\partial \boldsymbol{D} \subset \partial_+ M$ and $cl(M-N(\boldsymbol{D}\,;\, M))=(F \times I) \cup E_1 \cup \cdots \cup E_k$. If $k=0$, then M is a compression body and $\boldsymbol{D}$ is a *complete disk system* for M in a sense of Casson and Gordon [CG].

3. Proof of Theorem

Let $V_F \cong A_1 \natural \cdots \natural A_n$ and $W_F \cong B_1 \natural \cdots \natural B_n$ be ∂-prime decompositions for V_F and W_F, respectively. It will be noted that each A_i and each B_i are exteriors of knots. If both V_F and W_F are handlebodies (i.e. $A_i \cong B_i \cong D^2 \times S^1$ for $i=1, \cdots, n$), Theorem is true by Waldhausen [W] (see also [T1] and [Su2]). Thus, we may assume that V_F is not a handlebody and thus that $A_i \not\cong D^2 \times S^1$ for $i=1, \cdots, r$, and $A_j \cong D^2 \times S^1$ for $j=r+1, \cdots, n$, and $r \geqq 1$.

Let $\boldsymbol{D}_V$ be a decomposition disk system for V_F, and let $cl(V_F - N(\boldsymbol{D}_V; V_F)) = V_1 \cup \cdots \cup V_r$ with $V_i \cong A_i$ for $i=1, \cdots, r$. Now let $U = N(\partial V_1; V_1) \cup N(\boldsymbol{D}_V; V_F) \cup V_2 \cup \cdots \cup V_r$. It will be noticed that $\boldsymbol{D}_V$ is a complete disk system for U and $\partial_+ U = F$. We can easily see that $W_F \cup U = (S^3 - {}^\circ V_1) \cup N(\partial V_1; V_1) \cong S^3 - {}^\circ V_1$ is a solid torus by [F] or [Ho]. (See Figure 1.)

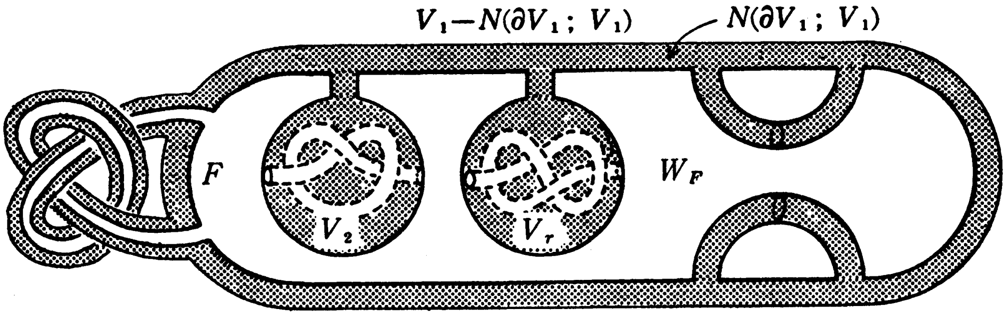

Figure 1

We have the following claim:

Claim 1. *There exists a meridian disk D for $W_F \cup U$ with $D \cap (V_2 \cup \cdots \cup V_r) = \emptyset$.*

Proof. Let $W = W_F \cup U$. We may show that

$$(*) \qquad W - {}^\circ(V_2 \cup \cdots \cup V_r) \cong r(D^2 \times S^1).$$

If $(*)$ is true, then we see that the homomorphism $\pi_1(\partial W) \to \pi_1(W - {}^\circ(V_2 \cup \cdots \cup V_r))$ of fundamental groups induced by the inclusion is not injective, and thus that there exists a simple essential loop α in ∂W such that α bounds a disk D in $W - {}^\circ(V_2 \cup \cdots \cup V_r)$ by the loop theorem, and D is a desired meridian disk.

We now show $(*)$ by induction on r. If $r=1$, then there is nothing to prove. So, we assume that $r \geqq 2$. We know that $W - {}^\circ(V_2 \cup \cdots \cup V_{r-1}) \cong (r-1)(D^2 \times S^1)$ by the induction hypothesis, and that V_r is contained in ${}^\circ(W - {}^\circ(V_2 \cup \cdots \cup V_{r-1}))$. Let us consider the following diagram of homomorphisms of fundamental groups induced by inclusions.

$$\begin{array}{ccccc} & & \pi_1(V_r) & & \\ & \overset{i_1}{\nearrow} & & \overset{i_3}{\searrow} & \\ \pi_1(\partial V_r) & & \xrightarrow{\quad i_0 \quad} & & \pi_1(W - {}^\circ(V_2 \cup \cdots \cup V_{r-1})) \\ & \underset{i_2}{\searrow} & & \underset{i_4}{\nearrow} & \\ & & \pi_1(W - {}^\circ(V_2 \cup \cdots \cup V_{r-1} \cup V_r)) & & \end{array}$$

The map i_1 is injective because V_r is an exterior of a nontrivial knot. If i_2 is injective, then both i_3 and i_4 are injective by Van Kampen's theorem, and then we have an injection from $Z\oplus Z$ to the free group of rank $r-1$, which is a contradiction. Therefore, i_2 is not injective. By the loop theorem, we have a simple essential loop β in ∂V_r such that β bounds a disk E in $W-{}^\circ(V_2\cup\cdots\cup V_{r-1}\cup V_r)$. Let $S=N(\partial V_r\cup E\,;\,W-{}^\circ(V_2\cup\cdots\cup V_{r-1}))$. Then, we can easily see that S is homeomorphic to a one-punctured solid torus and so $(W-{}^\circ(V_2\cup\cdots\cup V_{r-1}))-{}^\circ S$ is homeomorphic to a one-punctured $(r-1)$ $(D^2\times S^1)$. Now we can conclude that $W-{}^\circ(V_2\cup\cdots\cup V_r)\cong r(D^2\times S^1)$, and completing the proof of Claim 1. $\square$

By Claim 1 we can choose a meridian disk D for the solid torus $W_F\cup U$ such that $D\cap U$ consists of some disks and an annulus A. We may assume that $D\cap U$ has the minimum number of disks among all such meridian disks. It follows from the choice of D that the connected planar surface $P=D\cap W_F$ is incompressible in W_F.

The proof of Theorem is divided into two cases.

Case I. W_F is a handlebody (i.e. $B_i\cong D^2\times S^1$ for each i): In this case, we have a similar result to Haken's lemma (Casson and Gordon [CG], Lemma 1.1) for $D\subset W_F\cup U$. This result enables us to construct a 2-sphere in S^3 which gives a non-trivial decomposition for $(F\subset S^3)$.

We have the following claim:

Claim 2. *There exists a complete meridian-disk system $\boldsymbol{D}_W$ for W_F such that $P\cap\boldsymbol{D}_W=\emptyset$.*

Proof. Let $\boldsymbol{D}_W=D_1\cup\cdots\cup D_n$ be a complete meridian-disk system for W_F, and we assume that $P\cap\boldsymbol{D}_W$ has the minimum number of components among all such meridian-disk systems. By incompressibility of P and the standard innermost circle argument, we may assume that $P\cap\boldsymbol{D}_W$ consists of simple proper arcs in P.

We suppose that there exist arcs α_i in $P\cap\boldsymbol{D}_W$ which are inessential in P, and let ∇_i be the disks on P cut off by α_i. We choose an innermost arc, say α_1, so that ∇_1 does not contain any other α_i. We assume that $\alpha_1\subset P\cap D_1$, and α_1 divides D_1 into two subdisks, say d_1' and d_1''. Then, we have proper disks $D_1'=\nabla_1\cup d_1'$ and $D_1''=\nabla_1\cup d_1''$ in W_F. We can deform $D_1'\cup D_1''$ into general position in W_F, so that

$$P\cap(D_1'\cup D_1''\cup D_2\cup\cdots\cup D_n)=P\cap\boldsymbol{D}_W-\alpha_1\,,$$

and

$$(D_1'\cup D_1'')\cap\boldsymbol{D}_W=\emptyset\,.$$

(In fact, $D_1' \cup D_1''$ is obtained from D_1 by a modification ∇ along ∇_1 in the sense of [Sul, Def. 3.1]. See Figure 2.) Since $D_1' \cup D_1''$ is contained in the 3-ball $B^3 = cl(W_F - N(\boldsymbol{D}_W; W_F))$, both $\partial D_1'$ and $\partial D_1''$ bound disks on ∂B^3. It is easily checked that one of $\boldsymbol{D}' = D_1' \cup D_2 \cup \cdots \cup D_n$ and $\boldsymbol{D}'' = D_1'' \cup D_2 \cup \cdots \cup D_n$ is a complete meridian-disk system for W_F. This contradicts to the minimality of $P \cap \boldsymbol{D}_W$, and so $P \cap \boldsymbol{D}_W$ does not contain inessential arcs.

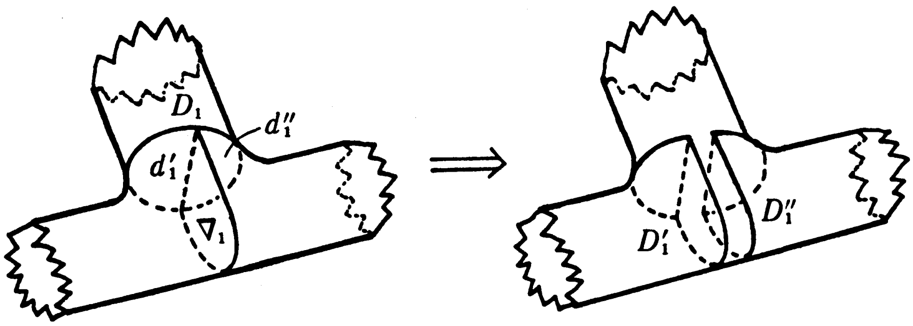

Figure 2

We now suppose that each component β_1 of $\boldsymbol{\beta} = P \cap \boldsymbol{D}_W$ is an essential arc in P. Let $D \cap F = C_1 \cup \cdots \cup C_m$ be simple loops. Let Q be the planar surface obtained from P by cutting along the arcs $\boldsymbol{\beta}$, that is, $Q = cl(P - N(P \cap \boldsymbol{D}_W; P))$, which is properly embedded in the 3-ball $B^3 = cl(W_F - N(\boldsymbol{D}_W; W_F))$. Since P is incompressible in W_F, Q is incompressible in B^3. Therefore we see that each component of Q is a disk and thus that $C_i \cap \boldsymbol{\beta} \neq \emptyset$ for each i.

Now we say that an arc β_i is of type I (resp. of type II) if the two points $\partial \beta_i$ contains a single component of $D \cap F$ (resp. two distinct components of $D \cap F$). Then, from the proof of Lemmas 1 and 2 of Ochiai [Oc], there exists a disk C_i such that each arc in $\boldsymbol{\beta}$ which meets C_i is of type II, and some sequence of isotopies of type A at these arcs (see Jaco [J] p. 24) has been to reduce the number of disks in $D \cap U$. This contradicts to the minimality of $D \cap U$ (and $P \cap \boldsymbol{D}_W$), and completing the proof of Claim 2. □

Since P is incompressible in W_F, by Claim 2, we conclude that P is a disk, and so $D \cap U$ consists of the annulus A. Now we have the following.

Claim 3. *There exists a complete disk system* $\boldsymbol{D}_V{}^*$ *for* U *with* $\boldsymbol{D}_V{}^* \cap A = \emptyset$.

Proof. We remember that $\boldsymbol{D}_V$ is a complete disk system for U, and let $\boldsymbol{D}_V = D_1 \cup \cdots \cup D_{n-1}$. If $\boldsymbol{D}_V \cap A = \emptyset$, then $\boldsymbol{D}_V$ is a required system for U. Thus, we may suppose that $\boldsymbol{D}_V \cap A \neq \emptyset$ and that each component of $\boldsymbol{D}_V \cap A$ is an arc since A is incompressible in U. It will be noticed that each arc in $\boldsymbol{D}_V \cap A$ is inessential in A, since its both endpoints are contained in one boundary component $\partial A \cap \partial_+ U = \partial A \cap F$ of A. Let α be an arc in $\boldsymbol{D}_V \cap A$ which is innermost on A, and let ∇ be the disk on A cut off by α. We assume that $\alpha\subset D_1\cap A$, and α divides D_1 into two subdisks, say d_1' and d_1''. Then, we have proper two disks $D_1'=\nabla\cup d_1'$ and $D_1''=\nabla\cup d_1''$ in U. We can deform $D_1'\cup D_1''$ into general position in U, so that

$$(D_1'\cup D_1''\cup D_2\cup\cdots\cup D_{n-1})\cap A=\boldsymbol{D}_V\cap A-\alpha\,,$$

and

$$(D_1'\cup D_1'')\cap\boldsymbol{D}_V=\emptyset\,.$$

We may assume that $D_1'\cup D_1''$ is contained in one of $N(\partial V_1;V_1)$, $V_2,\cdots,V_r$.

If $D_1'\cup D_1''\subset N(\partial V_1;V_1)$ (resp. $D_1'\cup D_1''\subset V_i$ for some i), then both D_1' and D_1'' bound disks on $\partial N(\partial V_1;V_1)$ (resp. ∂V_i) and cut off 3-balls from $N(\partial V_1;V_1)$ (resp. V_i), since both $N(\partial V_1;V_1)$ and V_i are ∂-irreducible and irreducible. By a similar way to the proof of Claim 2, it is easily checked that one of $\boldsymbol{D}_V'=D_1'\cup D_2\cup\cdots\cup D_{n-1}$ and $\boldsymbol{D}_V''=D_1''\cup D_2\cup\cdots\cup D_{n-1}$ is a complete disk system for U.

By the repetition of the procedure, we can get rid of all arcs in $\boldsymbol{D}_V\cap A$, and we have a required complete disk system $\boldsymbol{D}_V^*$. □

Now let $W^*=cl(U-N(\boldsymbol{D}_V^*;U))$, and let N^* be the component of W^* which corresponds to $F\times I$ in Definition 1.1 (2). Then, we can see that only one component of ∂N^* contains some disks in $cl(\partial N(\boldsymbol{D}_V^*;U)-\partial U)$. Now we denote this component by ∂_+N^*. Since only one component of ∂A is contained in ∂_+N^* and ∂A does not separate ∂_+N^*, we can take a simple loop γ in ∂_+N^* such that $\gamma\cap\partial A$ consists of one point and $\gamma\cap N(\boldsymbol{D}_V^*;U)=\emptyset$. Let

$$\Delta=cl(\partial N(P\cup\gamma;W_F)-F)\,,$$

where $P=D\cap W_F$ is a disk. Then, Δ is a proper disk in W_F, and $\partial\Delta$ bounds a disk in ∂_+N^*. Hence $\partial\Delta$ bounds a proper disk, say Δ^* in N^* and thus in $U=W^*\cup N(\boldsymbol{D}_V^*;U)\subset V_F$.

Let $\Sigma=\Delta\cup\Delta^*$. Then Σ is a 2-sphere which gives a decomposition for $(F\subset S^3)$ into a surface of genus 1 and a surface of genus $n-1$. This completes the proof of Case I. □

Case II. W_F is not a handlebody: In this case, we may assume that $B_i\not\cong D^2\times S^1$ for $i=1,\cdots,s$, $B_j\cong D^2\times S^1$ for $j=s+1,\cdots,n$ and $s\geqq 1$.

If $D\cap U$ has no disks, that is $D\cap U$ is an annulus, then we can construct a 2-sphere which gives a decomposition for $(F\subset S^3)$ as in Case I. Therefore, we may suppose that $D\cap U$ has some disks. We have the following claim by similar arguments to the proofs of Claim 2 and Claim 3.

Claim 4. *There exists a complete disk system $\boldsymbol{D}_W$ for W_F such that each component of $P\cap\boldsymbol{D}_W$ is an essential arc in P.* □

Let $W_1, \cdots, W_s$ be the components of $cl(W_F - N(\boldsymbol{D}_W; W_F))$ with $W_i \cong B_i$ for $i=1, \cdots, s$.

We suppose that $P \cap \boldsymbol{D}_W \neq \emptyset$, and let $d_1, \cdots, d_m$ be the disks of $D \cap U$ and let $C_i = \partial d_i$ for $i=1, \cdots, m$. Then we have the following claim.

Claim 5. *There exists C_i with $C_i \cap (P \cap \boldsymbol{D}_W) = \emptyset$.*

Proof. We suppose that Claim 5 is false. Then for every i, there exists an arc in $P \cap \boldsymbol{D}_W$ that meets C_i. By the technique of Ochiai [Oc], there exists C_j such that each arc in $P \cap \boldsymbol{D}_W$ that meets C_j is of type II, and some sequence of isotopies of type A reduces the number of disks in $D \cap U$, and contradicting minimality of $D \cap U$. □

Let C_1 be a loop with $C_1 \cap (P \cap \boldsymbol{D}_W) = \emptyset$, and let Q be the component of planar surface $cl(P - N(P \cap \boldsymbol{D}_W; P))$ with $Q \supset C_1$. It will be noticed that Q is a planar surface properly embedded in some W_j, and Q is incompressible in W_j since P is incompressible in W_F. Hence C_1 is essential in ∂W_j and bounds the disk d_1 in $U \subset V_F$. Since C_1 does not separate ∂W_j, we can take a simple loop, say γ, on ∂W_j such that $\gamma \cap C_1$ consists of one point and $\gamma \cap N(\boldsymbol{D}_W; W_F) = \emptyset$. Now let

$$\Delta = cl(\partial N(d_1 \cup \gamma; V_F) - \partial W_j).$$

Then Δ is a proper disk in V_F and $\partial\Delta$ bounds a disk in ∂W_j. Hence $\partial\Delta$ bounds a proper disk, say Δ', in $W_j \subset W_F$.

Then the 2-sphere $\Sigma = \Delta \cup \Delta'$ gives a decomposition for $(F \subset S^3)$ into a surface of genus 1 and a surface of genus $n-1$.

If $P \cap \boldsymbol{D}_W = \emptyset$, then we may assume that $P = D \cap W_F$ is contained in some W_j, and we have a 2-sphere which gives a decomposition for $(F \subset S^3)$ by the same argument as above (provided that P is substituted for Q).

This completes the proof of Case II, and we complete the proof of Theorem. □

References

[CG] A. J. Casson and C. McA. Gordon, *Reducing Heegaard splittings,* Topology and its Appl., **27** (1987), 275-283.

[F] R. H. Fox, *On the imbedding of polyhedra in 3-space,* Ann. of Math. (2), **49** (1948), 462-470.

[GL] C. McA. Gordon and J. Luecke, *Knots are determined by their complements,* J. Amer. Math. Soc., **2** (1989), 371-415.

[G] J. L. Gross, *A unique decomposition theorem for 3-manifolds with connected boundary,* Trans. Amer. Math. Soc., **142** (1969), 191-199.

[He] J. Hempel, 3-MANIFOLDS, Ann. of Math. Studies #86, Princeton Univ. Press, 1976.

[Ho] T. Homma, *On the existence of unknotted polygons on 2-manifolds in E^3*, Osaka Math. J., **6** (1954), 129-134.

[J] W. Jaco, LECTURES ON THREE-MANIFOLDS TOPOLOGY, CBMS Regional Conference Ser. in Math. #43, Amer. Math. Soc., 1980.

[M] M. Motto, *Maximal triads and prime decompositions of surfaces embedded in 3-manifolds*, Trans. Amer. Math. Soc., **331** (1992), 851-867.

[Oc] M. Ochiai, *On Haken's theorem and its extension*, Osaka J. Math. **20** (1983), 461-468.

[Su1] S. Suzuki, *On surfaces in 3-sphere, prime decompositions*, Hokkaido Math. J., **4** (1975), 179-195.

[Su2] ———, *On surfaces in 3-sphere, stable congruence*, Math. Japon., **20** (1975), 65-83.

[Su3] ———, *On surfaces of genus 3 in 3-sphere*, Math. Sem. Notes Kobe Univ., **9** (1981), 459-470.

[Sw] G. A. Swarup, *Some properties of 3-manifolds with boundary*, Quart. J. Math. Oxford (2), **21** (1970), 1-24.

[T1] Y. Tsukui, *On surfaces in 3-space*, Yokohama Math. J., **18** (1970), 93-104.

[T2] ———, *On a prime surface of genus 2 and homeomorphic splitting of 3-sphere*, Yokohama Math. J., **23** (1975), 63-75.

[W] F. Waldhausen, *Heegaard-Zerlegungen der 3-Sphäre*, Topology, **7** (1968), 196-203.

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