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ON THE INTEGRABLE G-INVARIANT METRICS

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Abstract. We consider Riemannian metrics, invariant with respect to a given smooth proper action. Then, we describe the structure of a manifold admitting an integrable invariant metric and prove that these metrics are "rare" among all invariants metrics, except of course some extreme cases.

1. Introduction

1.1. Let (G, M) be an effective, smooth proper action of the connected Lie group G on the (connected) paracompact C^{∞}-manifold M. Then, there is a Riemannian metric g on M invariant under the action of G i.e., G becomes a closed subgroup of the isometry group $I(M)=I(M, g)$ for this metric (cf. [5]). Suppose now that the orbits of the action are of the same dimension. Then, it is defined on M a distribution $\mathfrak{N}(g)$ consisting of the g-orthogonal complementary to the orbits subspaces.

We call the metric g *integrable* whenever $\mathfrak{N}(g)$ happens to be integrable. An integrable metric admits *local orthogonal cross sections,* provided that there is only one orbit type of the action (G, M) (cf. [2], [7]).

With the above notation our main result is the following:

1.2. Theorem. *Suppose that the proper action has only one orbit type and let g be a G-invariant metric on M. Then, $\mathfrak{N}(g)$ always has a prolongation to an integrable distribution $\mathfrak{D}(g)$ on M. Furthermore, if $\mathfrak{N}(g)$ is itself integrable then it gives rise to a flat connection in the principal bundle associated to (G, M).*

Then, we have two applications involving the orthogonal complementary distribution $\mathfrak{N}$.

1.3. When $\mathfrak{N}(g)$ is itself integrable, although (G, M) may be non principal, the representation of $\mathfrak{N}(g)$ given by the above stated Theorem has strong implications on the equivariant structure of M. In fact, we are able to obtain the obstruction morphism construncted in [2] in terms of holonomy morphisms

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and reductions. Thus, we rediscover in a different contex the structure theorem from [2], about actions admitting local orthogonal cross sections (cf. 4.1 below).

1.4. Concerning all G-invariant metrics, it is expected that among them the integrable ones should be rare. This fact was established by H. Abels (cf. [1]). As an application of our approach to G-spaces, we publish a proof of this in the following formulation.

1.5. Proposition. *Let G be a connected Lie group and (G, M) a smooth proper action. Then, under milde restrictions the G-invariant metrics which admit non-integrable orthogonal distribution form an open and dense subset in the set of all G-invariant metrics.*

The proof of our main result appears in Section 3, while in Section 2 we survey some properties of proper G-spaces and Section 4 is devoted to the applications.

2. Remarks about the geometry of certain proper G-spaces

Given the smooth proper action (G, M) we select some G-invariant metric g_0 and keep this fixed for the next of this section. Through every point $z \in M$ there exists a special slice S for the action, the "Koszul-slice" (cf. [6; 2.2.2]), which consists of geodesic arcs orthogonal to the orbit $G(z)$, at z. Furthermore, on the open-dense subset of the *principal orbits* we have

$$G(S) \underset{D}{=} G(z) \times S\,,$$

where "$\underset{D}{=}$" means equivariant diffeomorphism and S denotes Koszul slices (cf. [3]).

2.1. A satisfactory globalization of the above product structure is achived if the following additional condition is assumed (see also [2]).

(IGC) There is up to conjungancy, only one isotropy group.

Indded, let $x \in M$, $H = G_x$, $N(H)$ be the normalizer of H in G, $K = N(H)/H$ and

$$M^H = \{z \in M \mid H \subset G_z\} = \mathrm{Fix}(H, M)\,.$$

Then, M^H is a regular, totally geodesic submanifold of M [4; Vol. II p. 61], which also admits a differentiable, proper and free action of K. Meanwhile, K may be identified with Diffeo $^G(G(x))$, the group of G-diffeomorphisms of $G(x)$. Let p_H denotes the restriction of the natural map $p: M \to G\backslash M$ to M^H. The following nice discription for (G, M) is originally due to Borel.

2.2. Proposition [3; Ch. II, 5.11]. *Suppose that the proper smooth action* (G, M) *satisfies* (IGC). *Then, the map* p *is the projection in a fiber bundle with associated principal bundle the* K*-space* M^H *and projection* p_H. *In particular*

$$M \underset{D}{=} G(x) \times_K M^H \ \textit{(twisted product)}.$$

If, following [2], for g_0 we further assume

(**NI**) The distribution $\mathfrak{N}(g_0)$ is integrable,

then, another type of fibering of M over $G\backslash M$ is also possible. Indeed, let $N(x)$ denote the maximal integral manifold of $\mathfrak{N}$ through x. Then $p|N(x)$: $N(x) \to G\backslash M$ is a regular covering map and its group of deck transformations $H(x)$ can be realized as a subgroup of K. The obstruction morphism for (G, M) is obtained as follows:

Let c: $[0, 1] \to M$, $c(0)=c(1)=x$ be a closed path and $\tilde{c}$ denote the lifting of $p \circ c$ with respect $p|N(x)$, starting at x. The correspondence $[c] \to \tilde{c}(1)$ defines a morphism

$$\phi_x(:=\phi):\ \pi_1(M, x) \longrightarrow K,$$

with $\operatorname{Im} \phi = H(x)$.

2.3. Proposition [2; 1.14]. *Suppose that* (G, M) *admits local orthogonal cross sections. Then* p: $M \to G\backslash M$ *is the* (*locally trivial*) *fiber bundle with fibre* G/G_x, *structure group* $H(x)$ *and associated principal fiber bundle the covering space* $p|N(x)$: $N(x) \to G\backslash M$. *Furthermore,*

$$M \underset{D}{=} G(x) \times_{\operatorname{Im}\phi} N(x).$$

In Section 4 we shall indicate how this fibration can be obtained from Proposition 2.2 and our prolongation of $\mathfrak{N}$.

3. Proof of Theorem 1.2.

In this section we deal with the proper smooth action (G, M) satisfying (IGC) and fixe some G-invariant metric g_0. With the notation of Section 2 still in force, we first examine some differential-geometric aspects concerning (G, M).

3.1. For $h \in H$, the differential $(dh)_x$ maps T_xM onto itself and preserves the orthogonal splitting

$$T_xM = T_xG(x) \oplus T_xS.$$

Let

$$D_x = \{\vartheta \in T_xM:\ (dh)_x(\vartheta) = \vartheta,\ \forall h \in G_x\}.$$

Then, beacause of (IGC), $N_x := T_xS$ is contained in D_x and

$$D_x = F_x \oplus N_x, \quad \text{while} \quad F_x = D_x \cap T_xG(x).$$

The correspondence $x \to N_x$ defines a G-invariant distribution $\mathfrak{N}$, complementary to the G-orbits. If $y=gx$, then

$$G_y=g^{-1}G_x g$$

and, as an easy calculation shows,

$$F_y=(dg)_x(F_x).$$

Finally the correspondence $x \to F_x$ defines a G-invariant distribution $\mathfrak{F}$ tangent to the orbits. We set

$$\mathfrak{D}=\mathfrak{F}\oplus\mathfrak{N}.$$

Looking at a normal neighbourhood of $x \in M$, we obtain

$$D_x=T_x M^H,$$

which implies that $\mathfrak{D}$ is integrable. Let $y \in M$ and $D(x)$ be the maximal integral manifold of $\mathfrak{D}$, through x. If we consider a path γ in $G\backslash M$ from $p(x)$ to $p(y)$, then γ has some lifting $\hat{\gamma}$ in M starting at x. Beacause of the trivializations

$$G(S_z) \underset{D}{=} G(z) \times S,$$

$\hat{\gamma}$ lies entirely on slices and (IGC) implies that $G(y) \cap D(x) \neq \emptyset$. We summarize the above discussion in the following.

3.2. Proposition. *If* (IGC) *holds, then* $\mathfrak{D}$ *is integrable with maximal integral manifold* $D(x)$, *through* x, *the connected component of* M^H *which contains* x. *Furthermore,* $D(x)$ *intersects every* G*-orbit.*

3.3. It is now clear that $\mathfrak{F}$ is also integrable and in the G-space

$$G(x) \underset{D}{=} G/H, \qquad \mathrm{Fix}(H, M) \cap G(x) \cong N(H)/H.$$

Contrary to this, $\mathfrak{N}$ is not always integrable. The integrability of $\mathfrak{N}$ yields another slise S^* at x, on the integral manifold $N(x)$ [6; 2.2 Lemma]. The next result shows that some "Koszul-slice" S and S^* are in fact the same.

3.4. Lemma. *If* $\mathfrak{N}$ *is integrable then it is geodesible, that is every integral manifold of* $\mathfrak{N}$ *is a totally geodesic, immersed submanifold.*

Proof. Let X^* be a fundamental (: Killing) vector field of the action and U, V vector fields of M such that $U(y)$, $V(y) \in N_y$ for all $y \in M$. Using the easily obtained relation

$$\langle X^*, [U, V]\rangle = -2\langle X^*, \nabla_V U\rangle,$$

the lemma follows.

3.5. As **3.2.** suggests, $\mathfrak{N}$ has always a prolongation to an integrable distribution, namely $\mathfrak{D}$. On the other hand, the integrability of $\mathfrak{N}$ may be a consequence either of some special structure of G/H, or of the "rigidity" of the orbits.

3.6. Corollary. *Each one of the following conditions implies that* $\mathfrak{N}$ *is integrable.*

(a) *The linear isotropy representation does not fix any non zero vector. Equivalently stated,* $\dim F_x=0$.

(b) *The local isometries for some orbit, in the Riemannian structure induced from M, are reduced to the identity only.*

3.7. We complete now the proof of our main result. Because of the existence of local orthogonal cross sections, the distribution $\mathfrak{N}$ is integrable. This, in view of **3.4,** implies that the "Koszul-slices" are in fact local sections of p: $M\to G\backslash M$, which are everywhere orthogonal to the orbits. Because of **3.1** and **3.2,** $\mathfrak{N}$ defines also an integrable distribution in M^H. Thus, $\mathfrak{N}$ defines a flat, K-invariant connection in the principal K-bundle $p_H: M^H\to G\backslash M$.

4. Applications

A. The obstruction morphism for actions admitting local orhogonal cross sections.

4.1. If for the action (G, M) and the G-invariant metric g_0 the conditions (IGC) and (NI) from Section 2 are satisfied, using Theorem 1.2, we have a flat connection $\mathfrak{N}$ in the associated principal K-bundle $p_H: M^H\to G\backslash M$. Using folk results, the holonomy group $\Phi(x)$ of this connection must be a discrete subgroup of K and its holonomy bundle through x is obviously $N(x)$. Also, the group $\Phi(x)$ may be identified with the group of deck transformations of the regular covering $p_H|N(x): N(x)\to G\backslash M$. Finally,

$$M^H \underset{D}{=} K\times_{\Phi(x)} N(x)$$

(cf. [4; Vol. I; Ch. II, 9]).

4.2. Proposition [2; 1.12]. *Suppose that the smooth effective proper action* (G, M) *admits local orthogonal cross sections. Then there is a homomorphism*

$$\phi:\ \pi_1(M, x)\longrightarrow \mathrm{Diffeo}^G(G(x))$$

and

$$M \underset{D}{=} G(x)\times_{\mathrm{Im}\phi} N(x).$$

In fact, $\mathrm{Im}\,\phi\subset N(H)/H$.

Proof. The orbit map $p: M \to G\backslash M$ induces an epimorphism

$$(p_x)_\#: \pi_1(M, x) \longrightarrow \pi_1(G\backslash M, p(x)).$$

Let $\phi = \Phi_x \circ (p_x)_\#$, where

$$\Phi_x: \pi_1(G\backslash M, p(x)) \longrightarrow K$$

is the holonomy homomorphism of $\mathfrak{N}$. Since $K \subset \mathrm{Diffeo}^G(G(X))$, $\mathrm{Im}\phi$ acts G-equivariantly on $G(x)$ and we have

$$M \underset{D}{=} G(x) \times_K M^H \underset{D}{=} G(x) \times_K (K \times_{\phi(x)} N(x)) \underset{D}{=} G(x) \times_{\mathrm{Im}\phi} N(x),$$

where some standard results about twisted products involving group actions were used (cf. [3; Ch. II]).

4.3. For smooth proper actions another kind of *sections* was considered in [7; 3.1], where further supposed that the integrable $\mathfrak{N}$ has closed integral manifolds. This is not necessarily the case for an action admitting local orthogonal cross sections. Of course results of "local" nature are true in both cases (cf. [2; 1.9] in connection to [7; 4] and our 3.6(a) compared to [7; 5.3]). It is the condition (IGC) which leads to global results like **4.2.**

Example. Consider the torus $\boldsymbol{T}^2$ as the orbit space of $\boldsymbol{R}^2$ under the usual action of the integer lattice. On $\boldsymbol{R}^2$ we define the Riemannian structure assuming that for $\tan\phi \in (\boldsymbol{R}-\boldsymbol{Q})$, the vectors $(\cos\phi, \sin\phi)$ and $(0, 1)$ form an orthonormal framming and project this structure on $\boldsymbol{T}^2$. Let $(\boldsymbol{R}, \boldsymbol{R}^2)$ be now the proper smooth action with $(t, (x, y)) \to (x, t+y)$. Then, it preserves the above defined metric of $\boldsymbol{R}^2$ and projects to an isometric S^1-action on $\boldsymbol{T}^2$. This action admits local orthogonal cross sections and the maximal integral manifolds of $\mathfrak{N}$ are the projections of the lines $y = (\tan\phi)x + c$. As $\tan\phi \in (\boldsymbol{R}-\boldsymbol{Q})$, these are dense subsets of $\boldsymbol{T}^2$ and $(S^1, \boldsymbol{T}^2)$ has not sections.

B. Non-integrable G-invariant metrics.

If the action (G, M) has orbits of codim -1, then every G-invariant metric is (trivially) integrable. If (IGC) is satisfied, this is also true, provided that $\dim F_x = 0$ (cf. 3.6 (a)). The next result shows that, except of these two cases, there are many non integrable metrics.

4.4. Proposition. *Let (G, M) be an effective smooth proper action with only one orbit type. Suppose further that the orbits of the action are of codimension greater than one and that the linear isotropy representation fixes some non zero vector. Then, the G-invariant metrics which are not integrable form an open and dense subset in the space of all G-invariant metrics (equippied with the C^∞-*

topology).

Proof. We first prove that the non integrable metrics form an open set. Let g^* be a non integrable G-invariant metric. Then there are vector fields X and Y g^*-orthogonal to the orbits of the action and a point $x \in M$ such that $g^*(\vartheta, [X, Y](x)) \neq 0$, for some $\vartheta \in T_x G(x)$. Given another G-invariant metric g, the projections of X, Y in $\mathfrak{N}(g)$ yield vector fields $\bar{X}$, $\bar{Y}$ g-orthogonal to the orbits. The correspondence $g \to g(\vartheta, [\bar{X}, \bar{Y}](x))$ involves, in local coordinates, only the local coordinates (g_{ij}) of g and their derivatives. Hence, it is continous and if g is close enough to g^*, then $g(\vartheta, [\bar{X}, \bar{Y}](x)) \neq 0$.

We now show that, an appropriate small pertubation of the G-invariant integrable metric g_0 gives a new G-invariant metric $\tilde{g}$ with $\mathfrak{N}(\tilde{g})$ non integrable. Indeed, for $x \in M$, let v be some non zero vector of F_x and S a "Koszul-slice" for g_0, at x. Because of **2.1,** the correspondence

$$hx \longrightarrow (dh)_x(v) \in T_{hx} G(x)$$

defines a vector field on $G(x)$. Using the trivialization

$$G(S) \underset{D}{=} G(x) \times S$$

we can extend this field to a G-invariant vector field $\hat{V}$ on the open set $G(S)$. Let the vector fields $X_1, X_2, \cdots, X_k$, $k = \operatorname{codim} T_x G(x)$ be selected in order that $\{X_1(z), X_2(z), \cdots, X_k(z)\}$ is a g_0-orthonormal basis of $T_z S$, for all $z \in S$. Then the vector fields $\tilde{X}_i = (dh)(X_i)$, $i = 1, 2, \cdots, k$, $h \in G$ are G-invariant and define an orthonormal basis in $T_{hz} S$. Furthermore, let $\nu : S \to \boldsymbol{R}$ be a non constant, smooth function with compact support and $\tilde{\nu} : G(S) \to \boldsymbol{R}$ defined by $\tilde{\nu}(hz) = \nu(z)$, $z \in S$. We define the new metric $\tilde{g}$ on M setting

$$\tilde{g} \,|\, M - G(S) = g_0 \,|\, M - G(S), \qquad \tilde{g} \,|\, T_{hx} G(x) = g_0 \,|\, T_{hx} G(x)$$

and assuming that, on $G(S)$ the vector fields $\{\tilde{X}_1 + \tilde{\nu}\hat{V}, \tilde{X}_2, \cdots, \tilde{X}_k\}$ define the $\tilde{g}$-orthogonal bases on the complementary to the orbits subspaces. We claim that $\mathfrak{N}(\tilde{g})$ is non integrable. Indeed, we have the relation

$$[\tilde{X}_1 + \tilde{\nu}\hat{V}, \tilde{X}_2] = [\tilde{X}_1, \tilde{X}_2] + \tilde{\nu}[\hat{V}, \tilde{X}_2] + \tilde{X}_2(\tilde{\nu})\hat{V},$$

where we can suppose that $\nu(x) = 0$ and $X_2(\nu)(x) \neq 0$. Therefore, the summand $\tilde{X}_2(\tilde{\nu})\hat{V}$ is not indentically zero and it is tangent to the orbit $G(x)$ at x. Also note that

$$\tilde{\nu}(x)[\hat{V}, \tilde{X}_2](x) = 0 \quad \text{and} \quad [\tilde{X}_1, \tilde{X}_2](x) \in \mathfrak{N}(\tilde{g}) = \mathfrak{N}(g_0)(x).$$

Hence, for the2vector fields $\tilde{X}_1 + \tilde{\nu}\hat{V}$ and $\tilde{X}_2$, the Frobenious integrability condition fails at x.

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