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ASYMPTOTIC BEHAVIOR OF FLUCTUATION AND DEVIATION FROM LIMIT SYSTEM IN THE SMOLUCHOWSKI-KRAMERS APPROXIMATION FOR SDE

By

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Abstract. Two kinds of the stochastic differential equations of the McKean type are considered. The one contains a large parameter $\alpha>0$ and describes the state of the particle in two dimension by its position and velocity variables, corresponding to the Fokker-Planck equation known as the Kramers equation. Here the phase variables split into the slow position and the fast velocity. The other describes the limit system of the position variable in one dimension as $\alpha\rightarrow\infty$, corresponding to the Fokker-Planck equation known as the Smoluchowski equation. For the position variable, the limit distributions of the fluctuation and the deviation from the limit system are obtained, with the help of estimates for the rate of decay of the remainder term. For the velocity variable, the limit distributions of the rescaled processes and the stability over an infinite time interval are obtained.

1. Introduction and motivation

Let $(\Omega, \boldsymbol{F}, P)$ be a probability space with an increasing family $\{\boldsymbol{F}_t ; t\geqq 0\}$ of sub-σ-algebras of $\boldsymbol{F}$ and let $w(t)$ be a one-dimensional Brownian motion process adapted to $\boldsymbol{F}_t$. Let α be a large parameter such that $\alpha\gg 1$. Then our goal of this paper is to study a rigorous detail of the limit behaviors as $\alpha\rightarrow\infty$ for the solution $(x^{\alpha}(t), y^{\alpha}(t))$ of the following two-dimensional stochastic differential equation with mean-field of the McKean type:

$$
(1.1)\qquad
\begin{aligned}
&dx^{\alpha}(t)=y^{\alpha}(t)dt\,,\\
&dy^{\alpha}(t)=[-\alpha\kappa y^{\alpha}(t)-\alpha g(x^{\alpha}(t))-\alpha\gamma\{y^{\alpha}(t)-E[y^{\alpha}(t)]\}]dt+\alpha\delta dw(t)\,,\\
&(x^{\alpha}(0),\ y^{\alpha}(0))=(\xi,\ \eta)=\phi\,.
\end{aligned}
$$

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Here and hereafter $\{\gamma, \delta, \kappa\}$ is a family of positive constants, $g(x)$ is a scalar function on $R^1=(-\infty, \infty)$ and $E[\ ]$ denotes the mathematical expectation, and also $\phi=(\xi, \eta)$ is a two-dimensional random vector being independent of the two-dimensional Brownian motion process.

The equation (1.1) arises from the equation (1.1)′ with a large parameter $\beta \gg 1$ written below, under consideration that

$$\alpha=\beta^2, \quad x^\alpha(t)=x(\beta t) \quad \text{and} \quad y^\alpha(t)=\beta y(\beta t),$$

$$\begin{aligned} &dx(t)=y(t)dt\,, \\ &dy(t)=[-\beta\kappa y(t)-g(x(t))-\beta\gamma\{y(t)-E[y(t)]\}]dt+\sqrt{\beta}\,\delta db(t), \end{aligned} \tag{1.1$'$}$$

where $b(t)$ is a one-dimensional Brownian motion process. The solution $(x(t), y(t))$ of (1.1)′ is an example of a formulation of the response $(x(t), \dot{x}(t))$ of the oscillator

$$\ddot{x}+\beta\kappa\dot{x}+g(x)+\beta\gamma\{\dot{x}-E[\dot{x}]\}=\sqrt{\beta}\,\delta\dot{b}$$

to the formal white noise $\dot{b}$. Here the dotted notation indicates the symbolic derivative d/dt.

The *reduced equation* for $x^\alpha(t)$ as $\alpha\to\infty$ can be derived by (1.1) with the following result:

$$\begin{aligned} &dx(t)=\left[-\frac{1}{\kappa+\gamma}g(x(t))-\frac{\gamma}{\kappa(\kappa+\gamma)}E[g(x(t))]\right]dt+\frac{\delta}{\kappa+\gamma}dw(t), \\ &x(0)=\xi\,, \end{aligned} \tag{1.2}$$

where ξ is the first component of the initial state ϕ in (1.1).

Assumption 1.1. There exists a constant $l>0$ satisfying

$$|g(x)-g(x')|\leqq l|x-x'|$$

for all $x\in R^1$ and $x'\in R^1$.

Assumption 1.2. The random vector $\phi=(\xi, \eta)$ is independent of the two-dimensional Brownian motion process, satisfying

$$E[|\phi|^{2m}]<\infty \qquad \text{for an integer } m\geqq 2.$$

Theorem 1.1 ([5], [6]). *Suppose that Assumptions* 1.1 *and* 1.2 *hold. Then the following results hold with the same exponent m as in Assumption* 1.2:

(i) $$\sup_{\alpha>0} E\left[\sup_{0\leq t\leq T}|x^\alpha(t)|^{2m}\right]\leqq M\exp[NT^{2m}]$$

for every $T<\infty$, *where* M *and* N *are positive constants being independent of* α *(*M *depends on* T *and* N *does not depend on* T*).*

(ii) $$E\left[\sup_{0\le t\le T}|x^{\alpha}(t)-x(t)|^{2m}\right]\longrightarrow 0 \qquad as\ \alpha\to\infty$$

for every $T<\infty$, *where* $x(t)$ *is the solution of* (1.2).

(iii) *For the solution* $y^{\alpha}(t)$ *of* (1.1), *define* $\tilde{y}^{\alpha}(t)$ *by*

$$\tilde{y}^{\alpha}(t)=(1/\sqrt{\alpha})y^{\alpha}(t/\alpha).$$

Then

$$E\left[\sup_{0\le t\le T}|\tilde{y}^{\alpha}(t)-\tilde{y}(t)|^{2m}\right]\longrightarrow 0 \qquad as\ \alpha\to 0$$

for every $T<\infty$. *Here* $\tilde{y}(t)$ *is the Ornstein-Uhlenbeck process satisfying the following Langevin equation*:

$$d\tilde{y}(t)=-(\kappa+\gamma)\tilde{y}(t)dt+\delta d\tilde{w}(t), \qquad \tilde{y}(0)=0$$

with the one-dimensional Brownian motion process $\tilde{w}(t)$ *given by*

$$\tilde{w}(t)=\sqrt{\alpha}\,w(t/\alpha).$$

The above result (ii) of Theorem 1.1 corresponds to the so-called *Smoluchowski-Kramers approximation.* In case $\gamma=0$, Schuss ([7], Ch. 6) gives several derivations for the limit of $x^{\alpha}(t)$ as $\alpha\to\infty$, Karatzas and Shreve ([4], p. 299) treats a model with a bounded drift coefficient, and besides Gardiner ([3], p. 196, l. l. 11-16) suggests necessity of successful development of such a scheme of systematic approximation on the stochastic differential equation. Being inspired by them, we seek details of various convergences to the limit system of as $\alpha\to\infty$ with rigour.

Notation 1.1. We shall use the following notations:

$$I_0^{\alpha}(t)=\exp[-\alpha(\kappa+\gamma)t]\int_0^t \exp[\alpha(\kappa+\gamma)u]\eta\,du$$

$$=\eta\frac{1}{\alpha(\kappa+\gamma)}\Big(1-\exp[-\alpha(\kappa+\gamma)t]\Big), \qquad \text{where } \eta=y^{\alpha}(0),$$

$$I_1^{\alpha}(t)=\frac{1}{\kappa+\gamma}\exp[-\alpha(\kappa+\gamma)t]\int_0^t \exp[\alpha(\kappa+\gamma)u]g(x^{\alpha}(u))du,$$

$$I_2^{\alpha}(t)=\frac{\gamma}{\kappa+\gamma}\exp[-\alpha(\kappa+\gamma)t]\int_0^t \exp[\alpha(\kappa+\gamma)u]n^{\alpha}(u)du,$$

$$\text{where } n^{\alpha}(u)=E[y^{\alpha}(u)],$$

$$I_3^{\alpha}(t)=\frac{\delta}{\kappa+\gamma}\exp[-\alpha(\kappa+\gamma)t]\int_0^t \exp[\alpha(\kappa+\gamma)u]dw(u),$$

$$I_4^\alpha(t)=\frac{\gamma}{\kappa+\gamma}\exp[-\alpha\kappa t]\int_0^t \exp[\alpha\kappa u]\Big(E[\eta]+\frac{1}{\kappa}E[g(x^\alpha(u))]\Big)du$$
$$=\frac{\gamma}{\kappa+\gamma}\Big[E[\eta]\frac{1}{\alpha\kappa}\Big(1-\exp[-\alpha\kappa t]\Big)$$
$$+\frac{1}{\kappa}\exp[-\alpha\kappa t]\int_0^t \exp[\alpha\kappa u]E[g(x^\alpha(u))]\,du\Big].$$

Remainder term

$$R^\alpha(t)=I_0^\alpha(t)+I_1^\alpha(t)-I_2^\alpha(t)-I_3^\alpha(t)+I_4^\alpha(t),$$
$$r^\alpha(t)=I_0^\alpha(t)+I_1^\alpha(t)-I_2^\alpha(t)+I_4^\alpha(t).$$

Deviation process

$$\Delta^\alpha(t)=x^\alpha(t)-x(t),$$

where $x^\alpha(t)$ and $x(t)$ are the solutions of (1.1) and (1.2), respectively.

According to the decomposition formula (6.5) and (1.2), the deviation process $\Delta^\alpha(t)$ satisfies

$$\Delta^\alpha(t)=-\frac{1}{\kappa+\gamma}\int_0^t[g(x^\alpha(u))-g(x(u))]du$$
$$-\frac{\gamma}{\gamma(\kappa+\kappa)}\int_0^t E[g(x^\alpha(u))-g(x(u))]du+R^\alpha(t). \tag{1.3}$$

In particular, if $g(x)\equiv$constant, then $\Delta^\alpha(t)=R^\alpha(t)$.

Notation 1.2. We say a sequence $\{X^\alpha\}$ of random elements *converges in distribution* to the random element X, and write

$$X^\alpha \Longrightarrow X \tag{1.4}$$

if the distributions μ^α of X^α converge weakly to the distribution μ of X: $E[f(X^\alpha)]\to E[f(X)]$ for each bounded and continuous function f on R^1. When (1.4) holds, we will frequently abuse terminology by saying that X^α *converges weakly* to X.

The contents of this paper are as follows:

Each proof of the theorems in Section 2 is given in Sections 3, 4 and 5, with the help of auxiliary estimates in Sections 6 and 7. We will refer to Sections

6 and 7 for the exact proof of the auxiliary estimates.

Result 1.1 (Fluctuation of position process). As the estimate (7.11) shows, $\Delta^\alpha(t)$ satisfies

$$E[|\Delta^\alpha(t)|^2]=O\left(\frac{1}{\alpha^2}+\frac{1}{\alpha}\right) \quad \text{for } 0\leqq t\leqq T, \tag{1.5}$$

where a constant in O may depend on T. This implies that the variable $\sqrt{\alpha}\,\Delta^\alpha(t)$ can have a limit distribution as $\alpha\to\infty$. Further the relation (1.5) gives us that

$$E\left[\left|\sqrt{\alpha}\,\Delta^\alpha\left(\frac{t}{\alpha}\right)\right|^2\right]=O\left(\frac{1}{\alpha}+1\right) \quad \text{for } 0\leqq t\leqq T.$$

Therefore, it might be expected that the process $\sqrt{\alpha}\,\Delta^\alpha(t/\alpha)$ also has a limit distribution as $\alpha\to\infty$.

Theorem 2.1 shows that

$$\sqrt{\alpha}\,\Delta^\alpha(t)\Longrightarrow \overline{W} \quad \text{as } \alpha\to\infty \quad \text{for each } t>0, \tag{1.6}$$

where $\overline{W}$ is a Gaussian random variable with mean 0 and variance

$$\sigma^2=\left(\frac{\delta}{\kappa+\gamma}\right)^2\frac{1}{2(\kappa+\gamma)}.$$

Theorem 2.2 shows that

$$\sqrt{\alpha}\,\Delta^\alpha\left(\frac{t}{\alpha}\right)\Longrightarrow U(t) \quad \text{as } \alpha\to\infty \quad \text{for all } t\geqq 0,$$

where $U(t)$ is the Ornstein-Uhlenbeck process governed by (2.2), since $-U(t)$ and $U(t)$ have the same probability distribution.

Result 1.2 (Rescaled velocity process). As the second equation of (1.1) suggests, the velocity process $y^\alpha(t)$ is *wide band noise* process which *blows* up to white noise as $\alpha\to\infty$. How to transform the space-time parameter in order to get a nontrivial limit of $y^\alpha(t)$ as $\alpha\to\infty$? For this question, the convergence (iii) of Theorem 1.1 is one result.

Theorem 2.3 shows that

$$\frac{1}{\sqrt{\alpha}}\,y^\alpha(t)\Longrightarrow \overline{W}_1 \quad \text{as } \alpha\to\infty \quad \text{for each } t>0,$$

where $\overline{W}_1$ is a Gaussian random variable with mean 0 and variance

$$\sigma_1^2=\frac{\delta^2}{2(\kappa+\gamma)}.$$

Theorem 2.4 shows that

$$y^{\alpha}\left(\frac{t}{\alpha^2}\right)-\eta \exp\left(-\frac{(\kappa+\gamma)t}{\alpha}\right) \Longrightarrow \delta \hat{W}(t) \qquad \text{as } \alpha\to\infty$$

for all $t\geqq 0$ with a one-dimensional Brownian motion process $\hat{W}(t)$.

Of special interest is the behavior of $y^{\alpha}(t)$ over an infinite time interval $0\leqq t<\infty$. Then, Theorem 2.5 shows a stability such that $(1/\sqrt{\alpha})y^{\alpha}(t)$ is uniformly bounded in mean square.

Result 1.3 (Behavior of remainder term). As follows from the estimate (7.5), $R^{\alpha}(t)$ satisfies

$$E[|R^{\alpha}(t)|^2]=O\left(\frac{1}{\alpha^2}+\frac{1}{\alpha}\right) \qquad \text{for } 0\leqq t\leqq T,$$

where a constant in O may depend on T. So, for any $c>0$

$$P(|R^{\alpha}(t)|>c) \longrightarrow 0 \qquad \text{as } \alpha\to\infty .$$

On the other hand, Notation 1.1 implies

$$R^{\alpha}(t)=r^{\alpha}(t)-I_3^{\alpha}(t) .$$

If the function $g(x)$ and the initial vector $\phi=(\xi, \eta)$ are bounded, then it can be proved from Notation 1.1 that $|r^{\alpha}(t)|=O(1/\alpha)$ uniformly in $t\geqq 0$ with probability 1. Now, $I_3^{\alpha}(t)$ is the pathwise unique solution of the Langevin equation:

$$dI_3^{\alpha}(t)=-\alpha(\kappa+\gamma)I_3^{\alpha}(t)dt+\frac{\delta}{\kappa+\gamma}dw(t) \qquad \text{with } I_3^{\alpha}(0)=0 . \tag{1.7}$$

Here, it can be proved from Lemma 5.1 that for each $t>0$, $P(|I_3^{\alpha}(t)|>c)$ goes to zero as $\alpha\to\infty$ exponentially fast. Therefore, it is plausible that this exponential decay can be transferred to the remainder term $R^{\alpha}(t)$.

Theorem 2.6 and Theorem 2.7 show that the large deviation principle holds for $R^{\alpha}(t)$ and $R^{\alpha}(t/\alpha)$.

Remark 1.1. For $\alpha\gg 1$, put $\varepsilon=1/\alpha$. Then the equation (1.1) is equivalent to the following equation with a small parameter ε:

$$dx^{\varepsilon}(t)=y^{\varepsilon}(t)dt ,$$

$$\varepsilon dy^{\varepsilon}(t)=[-\kappa y^{\varepsilon}(t)-g(x^{\varepsilon}(t))-\gamma\{y^{\varepsilon}(t)-E[y^{\varepsilon}(t)]\}]dt+\delta dw(t) ,$$

$$(x^{\varepsilon}(0),\ y^{\varepsilon}(0))=(\xi, \eta)=\phi ,$$

or equivalently,

$$\varepsilon\ddot{x}+\kappa\dot{x}+g(x)+\gamma\{\dot{x}-E[\dot{x}]\}=\delta\dot{w} .$$

In the above system, $x^{\varepsilon}(t)$ changes at a *normal rate* and $y^{\varepsilon}(t)$ changes at a much *faster rate*. Thus, our investigation is related with *singular perturbations* of stochastic differential equations of the McKean type.

2. Theorems

Theorem 2.1. *Suppose that $g(x)$ is a thrice continuously differentiable function satisfying*

$$|g'(x)|+|g''(x)|+|g'''(x)|\leqq A \qquad \text{for all } x\in R^1$$

with a constant $A>0$. Let $\phi=(\xi, \eta)$ be the same random vector as in Assumption 1.2. Then $\sqrt{\alpha}\,\Delta^{\alpha}(t)$ converges weakly to $\overline{W}$ as $\alpha\to\infty$ for each $t>0$, where $\overline{W}$ is a Gaussian random variable with mean 0 and variance

$$\sigma^2=\Big(\frac{\delta}{\kappa+\gamma}\Big)^2\frac{1}{2(\kappa+\gamma)}$$

and independent of $t>0$.

Theorem 2.2. *Suppose that $g(x)$ is a differentiable function satisfying*

$$|g'(x)|\leqq B \qquad \text{for all } x\in R^1$$

with a constant $B>0$. Let $\phi=(\xi, \eta)$ be the same random vector as in Assumption 1.2. Let $T<\infty$ be arbitrary and fixed. Then

$$E\Big[\Big|\sqrt{\alpha}\,\Delta^{\alpha}\Big(\frac{t}{\alpha}\Big)-(-U(t))\Big|^2\Big]\leqq\overline{K}(T)\Big(\frac{1}{\alpha^3}+\frac{1}{\alpha^2}+\frac{1}{\alpha}\Big) \tag{2.1}$$

for $0\leqq t\leqq T$, where $\overline{K}(T)$ is a positive constant depending on T and being independent of α. Here $U(t)$ is the Ornstein-Uhlenbeck process satisfying the following Langevin equation:

$$dU(t)=-(\kappa+\gamma)U(t)dt+\frac{\delta}{\kappa+\gamma}d\tilde{w}(t), \qquad U(0)=0 \tag{2.2}$$

with the one-dimensional Brownian motion process $\tilde{w}(t)=\sqrt{\alpha}\,w(t/\alpha)$.

Theorem 2.3. *Suppose that the same assumptions as in Theorem 2.2 hold. Then $(1/\sqrt{\alpha})y^{\alpha}(t)$ converges weakly to $(\kappa+\gamma)\overline{W}$ as $\alpha\to\infty$ for each $t>0$, where $\overline{W}$ is the same Gaussian random variable as in Theorem 2.1.*

Theorem 2.4. *Suppose that the same assumptions as in Theorem 2.2 hold. Define $Y^{\alpha}(t)$ by*

$$Y^{\alpha}(t)=y^{\alpha}\Big(\frac{t}{\alpha^2}\Big)-\eta\exp\Big(-\frac{(\kappa+\gamma)t}{\alpha}\Big), \qquad \text{where } \eta=y^{\alpha}(0).$$

Let $T<\infty$ be arbitrary and fixed. Then

$$\begin{aligned} &E[|Y^{\alpha}(t)-\delta\hat{W}(t)|^2] \\ &\leqq\hat{K}(T)\Big[\Big\{1-\exp\Big(-\frac{(\kappa+\gamma)t}{\alpha}\Big)\Big\}^2+\frac{1}{\alpha}\Big\{1-\exp\Big(-\frac{2(\kappa+\gamma)t}{\alpha}\Big)\Big\}\Big] \end{aligned} \tag{2.3}$$

for $0 \leqq t \leqq T$, *where* $\hat{K}(T)$ *is a positive constant depending on* T *and being independent of* α, *and besides* $\hat{W}(t)$ *is the one-dimensional Brownian motion process given by* $\hat{W}(t)=\alpha w(t/\alpha^2)$.

Theorem 2.5. *Suppose that* $g(x)$ *is a bounded function satisfying Assumption* 1.1, *such that* $|g(x)| \leqq C$ *for all* $x \in R^1$ *with a constant* $C>0$. *Let* $\phi=(\xi, \eta)$ *be the same random vector as in Assumption* 1.2. *Then*

$$\sup_{t\geq 0} E\Big[\Big|\frac{1}{\sqrt{\alpha}}\, y^{\alpha}(t)\Big|^2\Big] \leqq \bar{M}\Big(\frac{1}{\alpha}+1\Big), \tag{2.4}$$

where $\bar{M}$ *is a positive constant being independent of* α. *Namely,*

$$\sup_{\alpha\geq 1}\Big\{\sup_{t\geq 0} E\Big[\Big|\frac{1}{\sqrt{\alpha}}\, y^{\alpha}(t)\Big|^2\Big]\Big\} \equiv \bar{N} < \infty .$$

Now, according to [2], Ch. 5, we will give the definition of large deviations. Let $\{\mu^{\alpha}\}_{\alpha\geq 1}$ be a family of probability measures in R^1, and put

$$H^{\alpha}(z)=\log \int_{R^1} \exp[zx]\mu^{\alpha}(dx) .$$

Let $\lambda(\alpha)$ be a numerical-valued function converging to $+\infty$ as $\alpha \to \infty$. We assume that the limit

$$\lim_{\alpha\to\infty} \frac{1}{\lambda(\alpha)} H^{\alpha}(\lambda(\alpha)z)=H(z)$$

exists for all $z \in R^1$. We will denote by $L(x)$, the Legendre transformation of $H(z)$:

$$L(x)=\sup_{z}\{zx-H(z)\} .$$

Definition 2.1. The family $\{\mu^{\alpha}\}_{\alpha\geq 1}$ is said to satisfy the large deviation principle with the *action functional* $\lambda(\alpha)L(x)$ if it satisfies the following Conditions (0), (I) and (II):

Condition (0). The set $\Phi(s)=\{x\,;\, L(x)\leqq s\}$ is compact for $s\geqq 0$.

Condition (I). For any $\varepsilon>0$, any $\theta>0$ and any $x\in R^1$, there exists $\alpha_0>0$ such that

$$\mu^{\alpha}\{y\,;\,\rho(y, x)<\varepsilon\} \geqq \exp\{-\lambda(\alpha)[L(x)+\theta]\} \qquad \text{for all } \alpha \geqq \alpha_0 .$$

Here and hereafter, for $y\in R^1$ and $x\in R^1$, $\rho(y, x)$ is defined by

$$\rho(y, x)=|y-x| .$$

Condition (II). For any $\varepsilon>0$, any $\theta>0$ and any $s>0$, there exists $\alpha_0>0$ such that

$$\mu^{\alpha}\{y\,;\,\rho(y, \Phi(s))\geqq\varepsilon\} \leqq \exp\{-\lambda(\alpha)(s-\theta)\} \qquad \text{for all } \alpha\geqq\alpha_0 ,$$

where

$$\rho(y, \Phi(s)) = \inf\{\rho(y, z)\,;\, z \in \Phi(s)\}.$$

Theorem 2.6. *Suppose that $g(x)$ is a bounded and differentiable function satisfying*

$$|g(x)| + |g'(x)| \leqq D \qquad \text{for all } x \in R^1$$

with a constant $D>0$. Let $\phi=(\xi, \eta)$ be any two-dimensional random vector independent of the two-dimensional Brownian motion process, satisfying $|\phi| \leqq a$ with probability 1 with a constant $a>0$. Let ν_t^α be the probability distribution in R^1 of the remainder term $R^\alpha(t)$. Then, for each $t>0$, the family $\{\nu_t^\alpha\}_{\alpha \geqq 1}$ satisfies the large deviation principle with the action functional $\lambda(\alpha)L(x)$, where

$$\lambda(\alpha) = \alpha, \quad L(x) = \frac{1}{2\sigma^2}x^2 \quad \text{and} \quad \sigma^2 = \left(\frac{\delta}{\kappa+\gamma}\right)^2 \frac{1}{2(\kappa+\gamma)}.$$

In the following, we consider the rescaled process $R^\alpha(t/\alpha)$ for the remainder term $R^\alpha(t)$ of Notation 1.1. Put

$$A^\alpha(t) = \frac{\gamma}{\kappa}\left\{E[\eta] - n^\alpha(t)\right\} + \left\{\eta + \frac{1}{\kappa+\gamma} g(x^\alpha(t))\right\} + \frac{\gamma}{\kappa(\kappa+\gamma)} E[g(x^\alpha(t))], \tag{2.5}$$

where $\eta = y^\alpha(0)$ and $n^\alpha(t) = E[y^\alpha(t)]$. For $R^\alpha(t)$ and $A^\alpha(t)$, define $\tilde{R}^\alpha(t)$ and $\tilde{A}^\alpha(t)$ by

$$\tilde{R}^\alpha(t) = R^\alpha\left(\frac{t}{\alpha}\right) \qquad \text{and} \qquad \tilde{A}^\alpha(t) = A^\alpha\left(\frac{t}{\alpha}\right).$$

Then, according to (6.8), $\tilde{R}^\alpha(t)$ satisfies the following equation:

$$d\tilde{R}^\alpha(t) = -(\kappa+\gamma)\tilde{R}^\alpha(t)dt + \frac{1}{\alpha}\tilde{A}^\alpha(t)dt + \frac{1}{\sqrt{\alpha}}\left(-\frac{\delta}{\kappa+\gamma}\right)d\tilde{w}(t), \quad \tilde{R}^\alpha(0) = 0, \tag{2.6}$$

where $\tilde{w}(t)$ is given by $\tilde{w}(t) = \sqrt{\alpha}\, w(t/\alpha)$.

Now, for $\alpha \gg 1$, set $\varepsilon = 1/\alpha$. Then, emphasizing the dependence on the small parameter $\varepsilon \ll 1$, we set

$$R^\varepsilon(t) = \tilde{R}^\alpha(t) \qquad \text{and} \qquad A^\varepsilon(t) = \tilde{A}^\alpha(t).$$

Then (2.6) can be rewritten as follows:

$$dR^\varepsilon(t) = [-(\kappa+\gamma)R^\varepsilon(t) + \varepsilon A^\varepsilon(t)]dt + \sqrt{\varepsilon}\left(-\frac{\delta}{\kappa+\gamma}\right)d\tilde{w}(t), \qquad R^\varepsilon(0) = 0. \tag{2.7}$$

By $C([0, \infty)\,;\, R^1)$ (resp. $C([0, T]\,;\, R^1)$) we denote the space of all continuous functions $\varphi(t)$, $0 \leqq t < \infty$ (resp. $0 \leqq t \leqq T$), with range R^1.

Theorem 2.7. *Suppose that the same assumptions as in Theorem 2.6 hold. Let $\{P^\varepsilon\}_{0<\varepsilon<1}$ be the family of probability measures induced by $R^\varepsilon(\cdot) = \{R^\varepsilon(t)\}_{0<\varepsilon<1}$*

on $C([0, \infty); R^1)$. *Define* $S(\varphi)$ *on* $C([0, T]; R^1)$ *by*

$$(2.8)\qquad S(\varphi)=\begin{cases} \dfrac{1}{2}\displaystyle\int_0^T\Big(\frac{\delta}{\kappa+\gamma}\Big)^{-2}\Big\{\frac{d}{dt}\varphi(t)+(\kappa+\gamma)\varphi(t)\Big\}^2 dt \\ \qquad\qquad \textit{if } \varphi(0)=0 \textit{ and } \varphi(t) \textit{ is absolutely continuous on } [0, T], \\ \infty \qquad \textit{otherwise.} \end{cases}$$

Then $\{P^\varepsilon\}_{0<\varepsilon<1}$ *satisfies the large deviation principle with the action functional* $(1/\varepsilon)S(\varphi)$.

In the above theorem, we say that the family $\{P^\varepsilon\}_{0<\varepsilon<1}$ on $C([0, \infty); R^1)$ satisfies the large deviation principle with the action functional $(1/\varepsilon)S(\varphi)$ if it satisfies the conditions (0), (I) and (II) in [2] (p. 80) with $\lambda(\varepsilon)=1/\varepsilon$, which is an analogue of the above Definition 2.1 with metric space $(R^1, \rho(y, x)=|y-x|)$ replaced by $(C([0, T); R^1), \rho(\varphi, \phi)=\sup_{0\le t\le T}|\varphi(t)-\phi(t)|)$.

3. Limit distribution of fluctuation of position process

In order to prove Theorems 2.1 and 2.2 we prepare several lemmas.

Lemma 3.1. *For the process* $I_3^\alpha(t)$ *given by Notation* 1.1, *define* $W^\alpha(t)$ *by* $W^\alpha(t)=\sqrt{\alpha}I_3^\alpha(t)$. *Then* $W^\alpha(t)$ *satisfies the following Langevin equation*:

$$(3.1)\qquad dW^\alpha(t)=-\alpha(\kappa+\gamma)W^\alpha(t)dt+\sqrt{\alpha}\Big(\frac{\delta}{\kappa+\delta}\Big)dw(t), \qquad W^\alpha(0)=0.$$

Moreover, for each $t>0$, $W^\alpha(t)$ *converges weakly to* $\overline{W}$ *as* $\alpha\to\infty$, *where* $\overline{W}$ *is a Gaussian random variable with mean* 0 *and variance*

$$\sigma^2=\Big(\frac{\delta}{\kappa+\gamma}\Big)^2\frac{1}{2(\kappa+\gamma)}$$

and independent of $t>0$.

Proof. By Remark 6.1, since $I_3^\alpha(t)$ satisfies the Langevin equation (1.7), $W^\alpha(t)$ automatically satisfies (3.1). Evidently, $W^\alpha(t)$ is a Gaussian stochastic process with mean $m^\alpha(t)=\sqrt{\alpha}E[I_3^\alpha(t)]=0$ and variance $v^\alpha(t)=\alpha E[|I_3^\alpha(t)|^2]$. The estimate (7.2) applies to the moment of $I_3^\alpha(t)$, and so

$$v^\alpha(t)=\sigma^2[1-\exp\{-2\alpha(\kappa+\gamma)t\}], \quad \text{where} \quad \sigma^2=\Big(\frac{\delta}{\kappa+\gamma}\Big)^2\frac{1}{2(\kappa+\gamma)}.$$

Thus, $m^\alpha(t)=0$ for all $t\ge 0$, and moreover

$$v^\alpha(t)\longrightarrow\sigma^2 \quad \text{as } \alpha\to\infty \text{ for each } t>0.$$

Accordingly, for each $t>0$, the limit distribution of $W^\alpha(t)$ as $\alpha\to\infty$ is the normal distribution with mean 0 and variance σ^2, which follows from [1] (p. 303) by consideration of the characteristic functions. Hence the proof is complete.

Lemma 3.2. *Suppose that the function $g(x)$ satisfies the global Lipschitz condition in $x \in R^1$. Let ξ be any random variable independent of the one-dimensional Brownian motion process $w(t)$, such that $E[|\xi|^{2n}]<\infty$ for an integer $n \geqq 1$. Let $x(t)$ be the solution of* (1.2) *with the initial state $x(0)=\xi$. Then*

$$E[|x(t)|^{2n}] \leqq (1+E[|\xi|^{2n}]) \exp [\bar{K}t]-1 \qquad \textit{for all } t \geqq 0 \tag{3.2}$$

with a constant $\bar{K}>0$ depending only on the family $\{n, l, \kappa, \gamma, \delta\}$ of constants, where l is the Lipschitz constant for $g(x)$.

Proof. Under the assumptions, (1.2) has the pathwise unique solution $x(t)$. Ito's formula applies to $x(t)^{2n}$, and so

$$dx(t)^{2n}=A(t)dt+\frac{\delta}{\kappa+\gamma}(2n)x(t)^{2n-1}dw(t)\,, \tag{3.3}$$

where

$$A(t)=\left\{-\frac{1}{\kappa+\gamma}g(x(t))-\frac{\gamma}{\kappa(\kappa+\gamma)}E[g(x(t))]\right\}(2n)x(t)^{2n-1}$$
$$+\frac{1}{2}\left(\frac{\delta}{\kappa+\gamma}\right)^2(2n)(2n-1)x(t)^{2n-2}\,.$$

Set $c=\max\{l, |g(0)|\}$. Then, since

$$|g(x)| \leqq c(1+|x|) \qquad \text{for all } x \in R^1\,,$$

we have

$$|g(x)x^{2n-1}|\, c(|x|^{2n-1}+|x|^{2n}) \leqq 2c(1+|x|^{2n})$$

for all $x \in R^1$, and so

$$|g(x(t))x(t)^{2n-1}| \leqq 2c(1+|x(t)|^{2n}) \qquad \text{for all } t \geqq 0. \tag{3.4}$$

Use the Young inequality:

If $a>0$, $b>0$, $p>1$ and $1/p+1/q=1$, then $ab \leqq a^p/p+b^q/q$.

Here, put

$$a=|E[g(x(t))]|, \quad b=|x(t)|^{2n-1}, \quad p=2n \quad \text{and} \quad q=\frac{2n}{2n-1},$$

and note that

$$|g(x)|^{2n} \leqq c^{2n}(1+|x|)^{2n} \leqq c^{2n}2^{2n-1}(1+|x|^{2n})$$

for all $x \in R^1$. Then we see

$$
\begin{aligned}
&|E[g(x(t))]|\cdot|x(t)|^{2n-1}\\
(3.5)\qquad &\leqq \frac{1}{2n}E[|g(x(t))|^{2n}]+\frac{2n-1}{2n}|x(t)|^{2n}\\
&\leqq \frac{1}{2n}c^{2n}2^{2n-1}(1+E[|x(t)|^{2n}])+\frac{2n-1}{2n}|x(t)|^{2n}
\end{aligned}
$$

for all $t\geqq 0$. Moreover

$$
(3.6)\qquad |x(t)|^{2n-2}\leqq 1+|x(t)|^{2n} \qquad \text{for all } t\geqq 0.
$$

Therefore, by (3.4), (3.5) and (3.6), the drift coefficient $A(t)$ in (3.3) satisfies

$$
\begin{aligned}
|A(t)| &\leqq \frac{1}{\kappa+\gamma}(2n)(2c)\{1+|x(t)|^{2n}\}\\
(3.7)\qquad &+\frac{\gamma}{\kappa(\kappa+\gamma)}\{c^{2n}2^{2n-1}(1+E[|x(t)|^{2n}])+(2n-1)|x(t)|^{2n}\}\\
&+\left(\frac{\delta}{\kappa+\gamma}\right)^2 n(2n-1)\{1+|x(t)|^{2n}\}
\end{aligned}
$$

for all $t\geqq 0$. Take expectations on (3.3). Then, by (3.7) we get

$$
E[|x(t)|^{2n}]\leqq E[|\xi|^{2n}]+\bar{K}\int_0^t\{1+E[|x(u)|^{2n}]\}\,du
$$

for all $t\geqq 0$, where $\bar{K}$ is a positive constant depending on the family $\{n, c, \kappa, \gamma, \delta\}$ of constants. The estimate (3.2) follows from the specic Gronwall-Bellman inequality: If $\varphi(t)$ is a nonnegative and continuous function satisfying

$$
\varphi(t)\leqq \delta_2 t+\delta_1\int_0^t\varphi(s)ds+\delta_3 \qquad \text{for all } t\geqq 0
$$

with contants $\delta_1>0$, $\delta_2\geqq 0$ and $\delta_3\geqq 0$, then

$$
\varphi(t)\leqq\left(\frac{\delta_2}{\delta_1}+\delta_3\right)\exp[\delta_1 t]-\frac{\delta_2}{\delta_1} \qquad \text{for all } t\geqq 0.
$$

Hence the proof is complete.

Next we proceed to a key lemma for the proof of Theorem 2.1.

Lemma 3.3. *Suppose that the function $g(x)$ and the random vector $\phi=(\xi, \eta)$ satisfy Assumption 1.1 and Assumption 1.2, respectively, and suppose that $f(x)$ is a twice continuously differentiable function satisfying*

$$
|f(x)|+|f'(x)|+|f''(x)|\leqq D \qquad \textit{for all } x\in R^1
$$

with a constant $D>0$. For the process $I_3^\alpha(t)$ given by Notation 1.1, set $W^\alpha(t)=\sqrt{\alpha}\,I_3^\alpha(t)$, and let $x(t)$ be the solution of (1.2) with the initial state $x(0)=\xi$. Let

$T<\infty$ be arbitrary and fixed. Then

$$E\Big[\Big(\int_0^t W^\alpha(u)f(x(u))du\Big)^2\Big]\leqq \bar{K}(T)\Big(\frac{1}{\alpha^2}+\frac{1}{\alpha}\Big) \qquad \text{for } 0\leqq t\leqq T, \tag{3.8}$$

where $\bar{K}(T)$ is a positive constant depending on T and being independent of α.

Proof. For simplicity, set

$$x_1(t)=W^\alpha(t) \quad \text{and} \quad x_2(t)=f(x(t)).$$

Then (3.1) and Ito's formula imply the following equations:

$$dx_1(t)=a_1(t)dt+b_1(t)dw(t),$$

$$a_1(t)=-\alpha(\kappa+\gamma)W^\alpha(t), \qquad b_1(t)=\sqrt{\alpha}\Big(\frac{\delta}{\kappa+\gamma}\Big).$$

$$dx_2(t)=a_2(t)dt+b_2(t)dw(t),$$

$$\begin{aligned} a_2(t)=&\Big\{-\frac{1}{\kappa+\gamma}g(x(t))-\frac{\gamma}{\kappa(\kappa+\gamma)}E[g(x(t))]\Big\}f'(x(t)) \\ &+\frac{1}{2}\Big(\frac{\delta}{\kappa+\gamma}\Big)^2 f''(x(t)), \end{aligned} \tag{3.9}$$

$$b_2(t)=\frac{\delta}{\kappa+\gamma}f'(x(t)).$$

$$d[x_1(t)x_2(t)]=x_1(t)dx_2(t)+x_2(t)dx_1(t)+b_1(t)b_2(t)dt,$$

namely,

$$\begin{aligned} &d[W^\alpha(t)f(x(t))] \\ &=W^\alpha(t)\{a_2(t)dt+b_2(t)dw(t)\}-\alpha(\kappa+\gamma)\{W^\alpha(t)f(x(t))\}dt \\ &\quad+\sqrt{\alpha}\Big(\frac{\delta}{\kappa+\gamma}\Big)f(x(t))dw(t)+\sqrt{\alpha}\Big(\frac{\delta}{\kappa+\gamma}\Big)^2 f'(x(t))dt. \end{aligned}$$

Taking notice of the term $\{W^\alpha(t)f(x(t))\}dt$, we get

$$\int_0^t W^\alpha(u)f(x(u))du=\frac{1}{\alpha(\kappa+\gamma)}\Big[F_1(t)+F_2(t)+F_3(t)+\sqrt{\alpha}F_4(t)+\sqrt{\alpha}F_5(t)\Big], \tag{3.10}$$

where

$$F_1(t)=W^\alpha(0)f(x(0))-W^\alpha(t)f(x(t))=-W^\alpha(t)f(x(t)),$$

$$F_2(t)=\int_0^t W^\alpha(u)a_2(u)du, \quad F_3(t)=\int_0^t W^\alpha(u)b_2(u)dw(u),$$

$$F_4(t)=\frac{\delta}{\kappa+\gamma}\int_0^t f(x(u))dw(u) \quad \text{and} \quad F_5(t)=\Big(\frac{\delta}{\kappa+\gamma}\Big)^2\int_0^t f'(x(u))du.$$

In order to prove (3.8) we must evaluate each moment of $F_i(t)$ for $1\leqq i\leqq 5$. Under Assumption 1.1, we note that

$$|g(x)| \leqq c(1+|x|) \qquad \text{for all } x \in R^1,$$

where $c=\max\{l, |g(0)|\}$ with the Lipschitz constant $l>0$ for $g(x)$. In the following, we shall denote various positive constants being independent of α by the same symbol K (K may depend on $\{D, c, \kappa, \gamma, \delta\}$ of constants).

Step 1. Since $f(x)$ is bounded, as follows from the estimate (7.2), $F_1(t)$ satisfies

$$E[|F_1(t)|^2] \leqq KE[|W^\alpha(t)|^2] = K\alpha E[|I_3^\alpha(t)|^2] \leqq K\alpha(1/\alpha) \leqq K \tag{3.11}$$

for all $t \geqq 0$.

Step 2. The Schwarz inequality, together with the inequality such that $|xy| \leqq (|x|^2+|y|^2)/2$ for $x \in R^1$ and $y \in R^1$, implies

$$\begin{aligned}|F_2(t)|^2 &\leqq t\int_0^t |W^\alpha(u)|^2 |a_2(u)|^2 du \\ &\leqq t\int_0^t \frac{1}{2}\{|W^\alpha(u)|^4 + |a_2(u)|^4\} du .\end{aligned}$$

Now, the estimate (7.3) yields

$$E[|W^\alpha(u)|^4] = \alpha^2 E[|I_3^\alpha(u)|^4] \leqq \alpha^2 K(1/\alpha^2) \leqq K \qquad \text{for all } u \geqq 0 .$$

On the other hand, since $|g(x)| \leqq c(1+|x|)$ for all $x \in R^1$, and since $|f'(x)| + |f''(x)| \leqq D$ for all $x \in R^1$, the drift coefficient $a_2(u)$ in (3.9) satisfies

$$|a_2(u)| \leqq K(1+|x(u)|+E[|x(u)|]) \qquad \text{for all } u \geqq 0 .$$

This yields

$$E[|a_2(u)|^4] \leqq K(1+E[|x(u)|^4]) \qquad \text{for all } u \geqq 0 .$$

Accordingly, the estimate (3.2) assures us of the following result:

$$\begin{aligned}E[|F_2(t)|^2] &\leqq tK\int_0^t (1+E[|x(u)|^4]) du \\ &\leqq tK\int_0^t (1+E[|\xi|^4]) \exp[\bar{K}u] du \\ &\leqq KT^2(1+E[|\xi|^4]) \exp[\bar{K}T] \qquad \text{for } 0 \leqq t \leqq T .\end{aligned} \tag{3.12}$$

Step 3. Since $f'(x)$ is bounded, and by Lemma 7.2 since

$$E[|W^\alpha(u)|^2] = \alpha E[|I_3^\alpha(u)|^2] \leqq K \qquad \text{for all } u \geqq 0 ,$$

$F_3(t)$ satisfies

$$E[|F_3(t)|^2] \leqq K\int_0^t E[|W^\alpha(u)|^2] du \leqq Kt \leqq KT \qquad \text{for } 0 \leqq t \leqq T. \tag{3.13}$$

Step 4. Since $f(x)$ is bounded, $F_4(t)$ satisfies

(3.14) $$E[|F_4(t)|^2]=\left(\frac{\delta}{\kappa+\gamma}\right)^2 E\left[\left(\int_0^t f(x(u))dw(u)\right)^2\right]\leqq Kt\leqq KT$$

for $0\leqq t\leqq T$.

Step. 5. Since $f'(x)$ is bounded, $F_5(t)$ satisfies

(3.15) $$E[|F_5(t)|^2]\leqq Kt^2\leqq KT^2 \qquad \text{for } 0\leqq t\leqq T\,.$$

Lastly, take expectations on the square of (3.10) and use the inequality such that $(x_1+x_2+\cdots+x_5)^2\leqq 5(x_1^2+x_2^2+\cdots+x_5^2)$. Then, by (3.11)~(3.15) we obtain

$$\begin{aligned}E&\left[\left(\int_0^t W^\alpha(u)f(x(u))du\right)^2\right]\\ &\leqq\left(\frac{1}{\alpha(\kappa+\gamma)}\right)^2 5\Big(E[|F_1(t)|^2+E[|F_2(t)|^2]+E[|F_3(t)|^2]\\ &\qquad\qquad +\alpha E[|F_4(t)|^2]+\alpha E[|F_5(t)|^2]\Big)\\ &\leqq\bar{K}(T)\left(\frac{1}{\alpha^2}+\frac{1}{\alpha}\right)\end{aligned}$$

for $0\leqq t\leqq T$ with a constant $\bar{K}(T)>0$ depending on T and being independent of α. Hence the proof is complete.

Next we will show that

$$\sqrt{\alpha}(\Delta^\alpha(t)-R^\alpha(t))\longrightarrow 0 \qquad \text{in mean square as } \alpha\to\infty\,.$$

Lemma 3.4. *Suppose that $g(x)$ is a thrice continuously differentiable function satisfying*

$$|g'(x)|+|g''(x)|+|g'''(x)|\leqq A \qquad \textit{for all } x\in R^1$$

with a constant $A>0$. Let $\phi=(\xi,\eta)$ be the same random vector as in Assumption 1.2. For the deviation process $\Delta^\alpha(t)=x^\alpha(t)-x(t)$ and the remainder term $R^\alpha(t)$ given by Notation 1.1, set

$$X^\alpha(t)=\sqrt{\alpha}(\Delta^\alpha(t)-R^\alpha(t))\,.$$

Let $T<\infty$ be arbitrary and fixed. Then

(3.16) $$E[|X^\alpha(t)|^2]\leqq\bar{M}(T)\left(\frac{1}{\alpha^3}+\frac{1}{\alpha^2}+\frac{1}{\alpha}\right)\exp[\bar{M}(T)t] \qquad \textit{for } 0\leqq t\leqq T\,,$$

where $\bar{M}(T)$ is a positive constant depending on T and being independent of α.

Proof. According to (1.3) and (6.6), $\sqrt{\alpha}\Delta(t)$ satisfies

$$\begin{aligned}\sqrt{\alpha}\Delta^\alpha(t)=&-\frac{1}{\kappa+\gamma}\int_0^t\{\sqrt{\alpha}\Delta^\alpha(u)\}g'(\zeta^\alpha(u))du\\ &-\frac{\gamma}{\kappa(\kappa+\gamma)}\int_0^t E[\{\sqrt{\alpha}\Delta^\alpha(u)\}g'(\zeta^\alpha(u))]du+\sqrt{\alpha}R^\alpha(t)\,,\end{aligned}$$

where

$$\zeta^\alpha(u)=x(u)+\theta^\alpha(u)\Delta^\alpha(u) \quad \text{and} \quad 0<\theta^\alpha(u)<1 .$$

Since $\sqrt{\alpha}\,\Delta^\alpha(u)=X^\alpha(u)+\sqrt{\alpha}\,R^\alpha(u)$, we have

$$\begin{aligned} X^\alpha(t)=&-\frac{1}{\kappa+\gamma}\int_0^t \{X^\alpha(u)+\sqrt{\alpha}\,R^\alpha(u)\}\,g'(\zeta^\alpha(u))du \\ &-\frac{\gamma}{\kappa(\kappa+\gamma)}\int_0^t E[\{X^\alpha(u)+\sqrt{\alpha}\,R^\alpha(u)\}\,g'(\zeta^\alpha(u))]du. \end{aligned} \tag{3.17}$$

First, the mean value theorem is applicable to $g'(x)$, and so

$$g'(x+h)=g'(x)+hg''(\rho), \quad \rho=x+\theta_1 h, \quad 0<\theta_1<1.$$

This implies

$$g'(\zeta^\alpha(u))=g'(x(u))+h^\alpha(u)g''(\rho^\alpha(u)) ,$$

where

$$h^\alpha(u)=\theta^\alpha(u)\Delta^\alpha(u), \quad \rho^\alpha(u)=x(u)+\theta_1^\alpha(u)h^\alpha(u) \quad \text{and} \quad 0<\theta_1^\alpha(u)<1 .$$

Secondly, Notation 1.1 implies

$$R^\alpha(u)=r^\alpha(u)-I_3^\alpha(u) \quad \text{with} \quad r^\alpha(u)=I_0^\alpha(u)+I_1^\alpha(u)-I_2^\alpha(u)+I_4^\alpha(u) .$$

So, the integrand of the right-hand side of (3.17) is rewritten as follows:

$$\begin{aligned} &\{X^\alpha(u)+\sqrt{\alpha}\,R^\alpha(u)\}\,g'(\zeta^\alpha(u)) \\ &=X^\alpha(u)g'(\zeta^\alpha(u))+\sqrt{\alpha}\,R^\alpha(u)\{g'(x(u))+h^\alpha(u)g''(\rho^\alpha(u))\} \\ &=X^\alpha(u)g'(\zeta^\alpha(u))+\sqrt{\alpha}\,\{r^\alpha(u)-I_3^\alpha(u)\}\,g'(x(u))+\sqrt{\alpha}\,R^\alpha(u)h^\alpha(u)g''(\rho^\alpha(u)) \\ &=X^\alpha(u)g'(\zeta^\alpha(u))+\sqrt{\alpha}\,r^\alpha(u)g'(x(u))-W^\alpha(u)g'(x(u))+\sqrt{\alpha}\,R^\alpha(u)h^\alpha(u)g''(\rho^\alpha(u)), \end{aligned}$$

where $W^\alpha(u)=\sqrt{\alpha}\,I_3^\alpha(u)$. Substituting the above expression into the right-hand side of (3.17), we get the following relation:

$$\begin{aligned} &X^\alpha(t)=-\frac{1}{\kappa+\gamma}F^\alpha(t)-\frac{\gamma}{\kappa(\kappa+\gamma)}E[F^\alpha(t)] , \\ &F^\alpha(t)=X_1^\alpha(t)+\sqrt{\alpha}\,X_2^\alpha(t)-X_3^\alpha(t)+\sqrt{\alpha}\,X_4^\alpha(t) , \\ &X_1^\alpha(t)=\int_0^t X^\alpha(u)g'(\zeta^\alpha(u))du, \quad X_2^\alpha(t)=\int_0^t r^\alpha(u)g'(x(u))du , \\ &X_3^\alpha(t)=\int_0^t W^\alpha(u)g'(x(u))du, \quad X_4^\alpha(t)=\int_0^t R^\alpha(u)h^\alpha(u)g''(\rho^\alpha(u))du . \end{aligned} \tag{3.18}$$

In order to prove (3.16) we must evaluate each $E[|X_i^\alpha(t)|^2]$ for $i=1$, 2, 3 and 4. In the following we shall use the Schwarz inequality such that

$$\left(\int_0^t k(u)q(u)du\right)^2 \leqq A^2 t\int_0^t |k(u)|^2 du$$

if $|q(u)|\leqq A$ for all $u\geqq 0$ with a constant $A>0$.

Step 1. Since $|g'(x)|\leqq A$ for all $x\in R^1$, $X_1^\alpha(t)$ satisfies

$$E[|X_1^\alpha(t)|^2]\leqq A^2t\int_0^t E[|X^\alpha(u)|^2]du\leqq A^2T\int_0^t E[|X^\alpha(u)|^2]du \quad \text{for } 0\leqq t\leqq T.$$

Step 2. By (7.7), since

$$E[|r^\alpha(u)|^2]\leqq K(T)\Big(\frac{1}{\alpha^2}\Big) \qquad \text{for } 0\leqq t\leqq T,$$

where $K(T)$ is a constant depending on T and being independent of α, $X_2^\alpha(t)$ satisfies

$$E[|X_2^\alpha(t)|^2]\leqq A^2t\int_0^t E[|r^\alpha(u)|^2]du\leqq A^2T^2K(T)\Big(\frac{1}{\alpha^2}\Big) \qquad \text{for } 0\leqq t\leqq T.$$

Step 3. In Lemma 3.3, set $f(x)=g'(x)$. Then (3.8) yields

$$E[|X_3^\alpha(t)|^2]=E\Big[\Big(\int_0^t W^\alpha(u)g'(x(u))du\Big)^2\Big]\leqq \bar{K}(T)\Big(\frac{1}{\alpha^2}+\frac{1}{\alpha}\Big) \quad \text{for } 0\leqq t\leqq T,$$

where $\bar{K}(T)$ is a positive constant depending on T and being independen of α.

Step 4. Since $|g''(x)|\leqq A$ for all $x\in R^1$, and since

$$|h^\alpha(u)|=|\theta^\alpha(u)\Delta^\alpha(u)|<|\Delta^\alpha(u)| \qquad \text{for all } u\geqq 0,$$

$X_4^\alpha(t)$ satisfies

$$\begin{aligned}|X_4^\alpha(t)|^2&\leqq A^2t\int_0^t|R^\alpha(u)|^2|\Delta^\alpha(u)|^2du\\ &\leqq A^2t\int_0^t\frac{1}{2}\{|R^\alpha(u)|^4+|\Delta^\alpha(u)|^4\}du,\end{aligned}$$

where the inequality, such that $2|xy|\leqq|x|^2+|y|^2$ for $x\in R^1$ and $y\in R^1$, is used. Now, as follows from (7.6) and (7.12), both $R^\alpha(u)$ and $\Delta^\alpha(u)$ satisfy that

$$E[|R^\alpha(u)|^4]\leqq K(T)\Big(\frac{1}{\alpha^4}+\frac{1}{\alpha^2}\Big) \qquad \text{for } 0\leqq u\leqq T$$

and

$$E[|\Delta^\alpha(u)|^4]\leqq \tilde{K}(T)\Big(\frac{1}{\alpha^4}+\frac{1}{\alpha^2}\Big) \qquad \text{for } 0\leqq u\leqq T,$$

where $K(T)$ and $\tilde{K}(T)$ are positive constants. Accordingly, $X_4^\alpha(t)$ satisfies

$$E[|X_4^\alpha(t)|^2]\leqq A^2T^2\frac{1}{2}\{K(T)+\tilde{K}(T)\}\Big(\frac{1}{\alpha^4}+\frac{1}{\alpha^2}\Big) \qquad \text{for } 0\leqq t\leqq T.$$

Lastly, consider the square of (3.18) and use the inequality such that

$$(a_1+a_2+\cdots+a_8)^2 \leqq 8(a_1^2+a_2^2+\cdots+a_8^2) \qquad \text{for } a_1, a_2, \cdots, a_8 \in R^1.$$

Then, by Steps 1, 2, 3 and 4 we see

$$\begin{aligned} E[|X^\alpha(t)|^2] &\leqq 8\Big\{\Big(\frac{1}{\kappa+\gamma}\Big)^2+\Big(\frac{\gamma}{\kappa(\kappa+\gamma)}\Big)^2\Big\} \\ &\qquad \times \{E[|X_1^\alpha(t)|^2]+\alpha E[|X_2^\alpha(t)|^2]+E[|X_3^\alpha(t)|^2]+\alpha E[|X_4^\alpha(t)|^2]\} \\ &\leqq \overline{M}(T)\Big\{\int_0^t E[|X^\alpha(u)|^2]du+\Big(\frac{1}{\alpha^3}+\frac{1}{\alpha^2}+\frac{1}{\alpha}\Big)\Big\} \qquad \text{for } 0\leqq t\leqq T \end{aligned}$$

with a constant $\overline{M}(T)>0$ depending on T and being independent of α. Hence by the Gronwall-Bellman inequality we get (3.16), showing the proof.

Proof of Theorem 2.1. For the deviation process $\Delta^\alpha(t)$ and the remainder term $R^\alpha(t)$, set $X^\alpha(t)=\sqrt{\alpha}\,\Delta^\alpha(t)-\sqrt{\alpha}\,R^\alpha(t)$. Further, put $W^\alpha(t)=\sqrt{\alpha}\,I_3^\alpha(t)$. Then, by Notation 1.1, since $R^\alpha(t)=r^\alpha(t)-I_3^\alpha(t)$, we have

$$(-W^\alpha(t))-\sqrt{\alpha}\,\Delta^\alpha(t)=-(X^\alpha(t)+\sqrt{\alpha}\,r^\alpha(t)). \tag{3.19}$$

First, it follows from Lemma 3.1 that

$$W^\alpha(t) \Longrightarrow \overline{W} \qquad \text{as } \alpha\to\infty \quad \text{for each } t>0,$$

where $\overline{W}$ is a Gaussian random variable with mean 0 and variance

$$\sigma^2=\Big(\frac{\delta}{\kappa+\gamma}\Big)^2\frac{1}{2(\kappa+\gamma)}.$$

This convergence with Example 25.8 in [1] (p. 288) implies that

$$(-W^\alpha(t)) \Longrightarrow (-\overline{W}) \qquad \text{as } \alpha\to\infty \quad \text{for each } t>0.$$

Therefore

$$(-W^\alpha(t)) \Longrightarrow \overline{W} \qquad \text{as } \alpha\to\infty \quad \text{for each } t>0, \tag{3.20}$$

since $-\overline{W}$ and $\overline{W}$ have the same normal distribution with mean 0 and variance σ^2. Secondly, it follows from (3.16) and (7.7) that

$$\begin{aligned} E[|X^\alpha(t)+\sqrt{\alpha}\,r^\alpha(t)|^2] &\leqq 2\{E[|X^\alpha(t)|^2+\alpha E[|r^\alpha(t)|^2]\} \\ &\leqq 2\Big\{N(T)\Big(\frac{1}{\alpha^3}+\frac{1}{\alpha^2}+\frac{1}{\alpha}\Big)+\alpha K(T)\Big(\frac{1}{\alpha^2}\Big)\Big\} \end{aligned}$$

for $0\leqq t\leqq T$ with constants $N(T)>0$ and $K(T)>0$ depending on T. This implies that

$$X^\alpha(t)+\sqrt{\alpha}\,r^\alpha(t) \longrightarrow 0 \qquad \text{in mean square as } \alpha\to\infty \quad \text{for } 0\leqq t\leqq T.$$

Thus, the relation (3.19) yields

$$(-W^{\alpha}(t))-\sqrt{\alpha}\,\Delta^{\alpha}(t)\Longrightarrow 0 \quad \text{as } \alpha\to\infty \quad \text{for } 0\leqq t\leqq T\,. \tag{3.21}$$

Therefore, appealing to Theorem 25.4 in [1] (p. 285), by (3.20) and (3.21) we can conclude that

$$\sqrt{\alpha}\,\Delta^{\alpha}(t)\Longrightarrow \overline{W} \quad \text{as } \alpha\to\infty \quad \text{for each } t>0\,.$$

Hence the proof is complete.

Proof of Theorem 2.2. Set $D^{\alpha}(t)=\sqrt{\alpha}\,\Delta^{\alpha}(t/\alpha)$. Then, by the equations (1.3) and (6.6), $D^{\alpha}(t)$ satisfies

$$\begin{aligned} D^{\alpha}(t)=&-\left(\frac{1}{\alpha}\right)\frac{1}{\kappa+\gamma}\int_0^t D^{\alpha}(s)g'\left(\zeta^{\alpha}\left(\frac{s}{\alpha}\right)\right)ds \\ &-\left(\frac{1}{\alpha}\right)\frac{\gamma}{\kappa(\kappa+\gamma)}\int_0^t E\left[D^{\alpha}(s)g'\left(\zeta^{\alpha}\left(\frac{s}{\alpha}\right)\right)\right]ds+\sqrt{\alpha}\,R^{\alpha}\left(\frac{t}{\alpha}\right), \end{aligned}$$

where

$$\zeta^{\alpha}(s)=x(s)+\theta^{\alpha}(s)\Delta^{\alpha}(s) \quad \text{and} \quad 0<\theta^{\alpha}(s)<1.$$

Set $U^{\alpha}(t)=\sqrt{\alpha}\,I_3^{\alpha}(t/\alpha)$. Then, by Notation 1.1, since $R^{\alpha}(t)=r^{\alpha}(t)-I_3^{\alpha}(t)$, we see

$$\sqrt{\alpha}\,R^{\alpha}\left(\frac{t}{\alpha}\right)=\sqrt{\alpha}\,r^{\alpha}\left(\frac{t}{\alpha}\right)-U^{\alpha}(t)\,.$$

Therefore, $D^{\alpha}(t)$ satisfies

$$\begin{aligned} D^{\alpha}(t)-(-U^{\alpha}(t))=&-\left(\frac{1}{\alpha}\right)\frac{1}{\kappa+\gamma}\int_0^t D^{\alpha}(s)g'\left(\zeta^{\alpha}\left(\frac{s}{\alpha}\right)\right)ds \\ &-\left(\frac{1}{\alpha}\right)\frac{\gamma}{\kappa(\kappa+\gamma)}\int_0^t E\left[D^{\alpha}(s)g'\left(\zeta^{\alpha}\left(\frac{s}{\alpha}\right)\right)\right]ds+\sqrt{\alpha}\,r^{\alpha}\left(\frac{t}{\alpha}\right). \end{aligned} \tag{3.22}$$

Now, $W^{\alpha}(t)$ is the solution of (3.1), and so $U^{\alpha}(t)$ satisfies

$$dU^{\alpha}(t)=-(\kappa+\gamma)U^{\alpha}(t)dt+\frac{\delta}{\kappa+\gamma}d\tilde{w}(t) \qquad \text{with} \qquad U^{\alpha}(0)=0\,,$$

where $\tilde{w}(t)$ is the one-dimensional Brownian motion process defined by $\tilde{w}(t)=\sqrt{\alpha}\,w(t/\alpha)$. Namely, $U^{\alpha}(t)$ is the solution of (2.2) which has the pathwise unique solution $U(t)$, and hence $U^{\alpha}(t)=U(t)$. Take the square of (3.22). Then, using the Schwarz inequality and appealing to the boundedness condition on $g'(x)$, we can find a positive constant C being independent of α, such that

$$\begin{aligned} &E[|D^{\alpha}(t)-(-U(t))|^2] \\ &=E[|D^{\alpha}(t)-(-U^{\alpha}(t))|^2] \\ &\leqq C\left\{\left(\frac{1}{\alpha^2}\right)t\int_0^t E[|D^{\alpha}(s)|^2]ds+\alpha E\left[\left|r^{\alpha}\left(\frac{t}{\alpha}\right)\right|^2\right]\right\}. \end{aligned} \tag{3.23}$$

Let $T<\infty$ be arbitrary and fixed. Then, according to the estimates (7.11) and

(7.7), there exist constants $M(T)>0$ and $K(T)>0$ depending on T and being independent of α, such that

$$\sup_{0\leq u\leq T} E[|\Delta^{\alpha}(u)|^{2}]\leqq M(T)\Big(\frac{1}{\alpha^{2}}+\frac{1}{\alpha}\Big)$$

and

$$\sup_{0\leq u\leq T} E[|r^{\alpha}(u)|^{2}]\leqq K(T)\Big(\frac{1}{\alpha^{2}}\Big).$$

Choose α so large that $\alpha\gg 1$. Then, since

$$0\leqq\frac{s}{\alpha}\leqq\frac{t}{\alpha}\leqq t\leqq T \qquad \text{for } 0\leqq s\leqq t\leqq T,$$

the following estimates hold:

$$\begin{aligned}E[|D^{\alpha}(s)|^{2}]&=\alpha E\Big[\Big|\Delta^{\alpha}\Big(\frac{s}{\alpha}\Big)\Big|^{2}\Big]\\ &\leqq\alpha\Big(\sup_{0\leq u\leq T}E[|\Delta^{\alpha}(u)|^{2}]\Big)\leqq M(T)\Big(\frac{1}{\alpha}+1\Big) \quad \text{for } 0\leqq s\leqq t\leqq T.\end{aligned} \tag{3.24}$$

$$\alpha E\Big[\Big|r^{\alpha}\Big(\frac{t}{\alpha}\Big)\Big|^{2}\Big]\leqq\alpha\Big(\sup_{0\leq u\leq T}E[|r^{\alpha}(u)|^{2}]\Big)\leqq K(T)\Big(\frac{1}{\alpha}\Big) \quad \text{for } 0\leqq t\leqq T. \tag{3.25}$$

Substituting (3.24) and (3.25) into (3.23), we obtain

$$E[|D^{\alpha}(t)-(-U(t))|^{2}]\leqq C\Big\{\Big(\frac{1}{\alpha^{2}}\Big)t\int_{0}^{t}M(T)\Big(\frac{1}{\alpha}+1\Big)ds+K(T)\Big(\frac{1}{\alpha}\Big)\Big\} \quad \text{for } 0\leqq t\leqq T,$$

showing (2.1). Hence the proof is complete.

4. Limit behavior of rescaled velocity process

Here we prove Theorem 2.3, Theorem 2.4 and Theorem 2.5.

Proof of Theorem 2.3. As follows from the formula (6.7), $y^{\alpha}(t)$ satisfies

$$\frac{1}{\sqrt{\alpha}}y^{\alpha}(t)=Z^{\alpha}(t)+(\kappa+\gamma)W^{\alpha}(t),$$

where

$$Z^{\alpha}(t)=\frac{1}{\sqrt{\alpha}}\eta\exp[-\alpha(\kappa+\gamma)t]-(\kappa+\gamma)\sqrt{\alpha}I_{1}^{\alpha}(t)+(\kappa+\gamma)\sqrt{\alpha}I_{2}^{\alpha}(t),$$

$$\eta=y^{\alpha}(0), \qquad W^{\alpha}(t)=\sqrt{\alpha}I_{3}^{\alpha}(t),$$

and $I_{1}^{\alpha}(t)$, $I_{2}^{\alpha}(t)$ and $I_{3}^{\alpha}(t)$ are given in Notation 1.1. Namely

$$(\kappa+\gamma)W^{\alpha}(t)-\frac{1}{\sqrt{\alpha}}y^{\alpha}(t)=-Z^{\alpha}(t). \tag{4.1}$$

First, Lemma 3.1 implies

$$W^{\alpha}(t) \Longrightarrow \overline{W} \qquad \text{as } \alpha\to\infty \quad \text{for each } t>0\,,$$

where $\overline{W}$ is the Gaussian random variable with mean 0 and variance

$$\sigma^2=\left(\frac{\delta}{\kappa+\gamma}\right)^2\frac{1}{2(\kappa+\gamma)}\,.$$

So, Example 25.8 in [1] (p. 288) implies

$$(4.2)\qquad (\kappa+\gamma)W^{\alpha}(t) \Longrightarrow (\kappa+\gamma)\overline{W} \qquad \text{as } \alpha\to\infty \quad \text{for each } t>0\,.$$

Next we will evaluate the moment of $Z^{\alpha}(t)$. Since $(a+b+c)^2\leqq 3(a^2+b^2+c^2)$ for real numbers a, b and c, $Z^{\alpha}(t)$ satisfies

$$E[|Z^{\alpha}(t)|^2]\leqq 3\Big[\Big(\frac{1}{\alpha}\Big)E[|\eta|^2]\exp[-2\alpha(\kappa+\gamma)t]$$
$$+(\kappa+\gamma)^2\alpha E[|I_1^{\alpha}(t)|^2]+(\kappa+\gamma)^2\alpha E[|I_2^{\alpha}(t)|^2]\Big]\,.$$

Under the assumptions, all conditions of Lemma 7.1 are satisfied, and hence Lemma 7.1 implies

$$E[|Z^{\alpha}(t)|^2]\leqq K\Big[\Big(\frac{1}{\alpha}\Big)\exp[-2\alpha(\kappa+\gamma)t]$$
$$+\alpha\Big\{E\Big[\Big(1+\sup_{0\leqq u\leqq t}|x^{\alpha}(u)|\Big)^2\Big]\Big\}\Big\{\lambda(t\,;\,\alpha(\kappa+\gamma))\Big\}^2\Big]$$

for all $t\geqq 0$ with a constant $K>0$ being independent of t and α. Here

$$\lambda(t\,;\,\alpha(\kappa+\gamma))=\frac{1}{\alpha(\kappa+\gamma)}\Big\{1-\exp[-\alpha(\kappa+\gamma)t]\Big\}\leqq\frac{1}{\alpha(\kappa+\gamma)}$$

for all $t\geqq 0$. Note that the estimate (i) of Theorem 1.1 holds for $x^{\alpha}(t)$. Then we can find a constant $M(t)>0$ depending on t such that

$$E[|Z^{\alpha}(t)|^2]\leqq M(t)\Big[\Big(\frac{1}{\alpha}\Big)\exp[-2\alpha(\kappa+\gamma)t]+\Big(\frac{1}{\alpha}\Big)\Big]$$

for all $t\geqq 0$. Thus, $Z^{\alpha}(t)\to 0$ in mean square as $\alpha\to\infty$, and hence

$$(4.3)\qquad Z^{\alpha}(t)\Longrightarrow 0 \qquad \text{as } \alpha\to\infty \quad \text{for all } t\geqq 0\,.$$

Therefore, combining Theorem 25.4 in [1] (p. 285) with the relations (4.1), (4.2) and (4.3), we can conclude that

$$\frac{1}{\sqrt{\alpha}}\,y^{\alpha}(t)\Longrightarrow(\kappa+\gamma)\overline{W} \qquad \text{as } \alpha\to\infty \quad \text{for each } t>0\,.$$

Hence the proof is complete.

Proof of Theorem 2.4. Set

$$Y^{\alpha}(t)=y^{\alpha}\Big(\frac{t}{\alpha^2}\Big)-\eta\exp\Big(-\frac{(\kappa+\gamma)t}{\alpha}\Big), \qquad \text{where} \quad \eta=y^{\alpha}(0).$$

Then, as follows from (6.7), $Y^{\alpha}(t)$ satisfies

$$Y^{\alpha}(t)=-(\kappa+\gamma)\alpha\Big\{I_1^{\alpha}\Big(\frac{t}{\alpha^2}\Big)-I_2^{\alpha}\Big(\frac{t}{\alpha^2}\Big)-I_3^{\alpha}\Big(\frac{t}{\alpha^2}\Big)\Big\}. \tag{4.4}$$

Introduce a one-dimensional Brownian motion process $\hat{W}(t)$ by

$$\hat{W}(t)=\alpha w(t/\alpha^2),$$

where $w(t)$ is the same Brownian motion process as in (1.1). Then, by (1.7) since

$$dI_3^{\alpha}(t)=-\alpha(\kappa+\gamma)I_3^{\alpha}(t)dt+\frac{\delta}{\kappa+\gamma}dw(t) \qquad \text{with} \quad I_3^{\alpha}(0)=0,$$

the following equation holds:

$$\alpha I_3^{\alpha}\Big(\frac{t}{\alpha^2}\Big)=-(\kappa+\gamma)\int_0^t I_3^{\alpha}\Big(\frac{u}{\alpha^2}\Big)du+\frac{\delta}{\kappa+\gamma}\hat{W}(t).$$

Substituting this form into the third term of the right-hand side of (4.4), we get

$$Y^{\alpha}(t)-\delta\hat{W}(t)=-(\kappa+\gamma)\alpha I_1^{\alpha}\Big(\frac{t}{\alpha^2}\Big)+(\kappa+\gamma)\alpha I_2^{\alpha}\Big(\frac{t}{\alpha^2}\Big)-(\kappa+\gamma)^2\int_0^t I_3^{\alpha}\Big(\frac{u}{\alpha^2}\Big)du.$$

Thus the Schwarz inequality yields

$$\begin{aligned} E[|Y^{\alpha}(t)-\delta\hat{W}(t)|^2]\leqq 3\Big\{&(\kappa+\gamma)^2\alpha^2 E\Big[\Big|I_1^{\alpha}\Big(\frac{t}{\alpha^2}\Big)\Big|^2\Big]+(\kappa+\gamma)^2\alpha^2 E\Big[\Big|I_2^{\alpha}\Big(\frac{t}{\alpha^2}\Big)\Big|^2\Big] \\ &+(\kappa+\gamma)^4 t\int_0^t E\Big[\Big|I_3^{\alpha}\Big(\frac{u}{\alpha^2}\Big)\Big|^2 du\Big]\Big\} \end{aligned} \tag{4.5}$$

for all $t\geqq 0$. By the assumption, since $|g'(x)|\leqq B$ for all $x\in R^1$, the function $g(x)$ satisfies Assumption 1.1. Further, the random vector $\phi=(\xi, \eta)$ satisfies Assumption 1.2, and so $E[|\eta|]<\infty$. Therefore, by Lemma 7.1 we can find a constant $K>0$ such that

$$\Big|I_i^{\alpha}\Big(\frac{t}{\alpha^2}\Big)\Big|\leqq K\Big(1+\sup_{0\leq s\leq t/\alpha^2}|x^{\alpha}(s)|\Big)\lambda\Big(\frac{t}{\alpha^2};\, \alpha(\kappa+\gamma)\Big)$$

for all $t\geqq 0$, where $i=1$ and 2. Here

$$\lambda\Big(\frac{t}{\alpha^2};\, \alpha(\kappa+\gamma)\Big)=\frac{1}{\alpha(\kappa+\gamma)}\Big\{1-\exp\Big(-\frac{(\kappa+\gamma)t}{\alpha}\Big)\Big\}.$$

Choose α so large that $\alpha\gg 1$. Let $T<\infty$ be arbitrary and fixed. Then, since $0\leqq s/\alpha^2\leqq t/\alpha^2\leqq t\leqq T$ for $0\leqq s\leqq t\leqq T$, the estimate (i) of Theorem 1.1 implies

$$E\Big[\sup_{0\leqq s\leqq t/\alpha^2}|x^\alpha(s)|^{2m}\Big]\leqq E\Big[\sup_{0\leqq s\leqq T}|x^\alpha(s)|^{2m}\Big]\leqq C(T)$$

with a constant $C(T)>0$ depending on T, where m is the same exponent as in Assumption 1.2. Accordingly, we can find a constant $\bar{C}(T)>0$ depending on T such that

$$\begin{aligned} &E\Big[\Big|I_1^\alpha\Big(\frac{t}{\alpha^2}\Big)\Big|^2\Big]+E\Big[\Big|I_2^\alpha\Big(\frac{t}{\alpha^2}\Big)\Big|^2\Big] \\ &\leqq\bar{C}(T)\Big\{\lambda\Big(\frac{t}{\alpha^2};\,\alpha(\kappa+\gamma)\Big)\Big\}^2 \qquad (4.6)\\ &=\bar{C}(T)\Big(\frac{1}{\alpha(\kappa+\gamma)}\Big)^2\Big\{1-\exp\Big(-\frac{(\kappa+\gamma)t}{\alpha}\Big)\Big\}^2 \qquad \text{for } 0\leqq t\leqq T\,. \end{aligned}$$

Moreover, by Lemma 7.2 we see that for all $u\geqq 0$

$$\begin{aligned} E\Big[\Big|I_3^\alpha\Big(\frac{u}{\alpha^2}\Big)\Big|^2\Big]&=\Big(\frac{\delta}{\kappa+\gamma}\Big)^2\lambda\Big(\frac{u}{\alpha^2};\,2\alpha(\kappa+\gamma)\Big) \qquad (4.7)\\ &=\Big(\frac{\delta}{\kappa+\gamma}\Big)^2\Big(\frac{1}{2\alpha(\kappa+\gamma)}\Big)\Big\{1-\exp\Big(-\frac{2(\kappa+\gamma)u}{\alpha}\Big)\Big\}\,. \end{aligned}$$

Substituting (4.6) and (4.7) into the right-hand side of (4.5), we obtain (2.3). Hence the proof is complete.

For the proof of Theorem 2.5 we prepare the following lemma.

Lemma 4.1. *Suppose that $g(x)$ satisfies the global Lipschitz condition in $x\in R^1$ and that $|g(x)|\leqq C$ for all $x\in R^1$ with a constant $C>0$. Let $\phi=(\xi,\eta)$ be the same random vector as in Assumption* 1.2. *For the solution $y^\alpha(t)$ of* (1.1), *set $n^\alpha(t)=E[y^\alpha(t)]$. Then*

$$|n^\alpha(t)|\leqq E[|\eta|]\exp[-\alpha\kappa t]+\frac{C}{\kappa}\Big(1-\exp[-\alpha\kappa t]\Big) \qquad (4.8)$$

for all $t\geqq 0$, where $\eta=y^\alpha(0)$. Moreover

$$\sup_{\alpha\geqq 1}\Big\{\sup_{t\geqq 0}|n^\alpha(t)|\Big\}\equiv\bar{N}<\infty\,.$$

Proof. Since $|g(x)|\leqq C$ for all $x\in R^1$, the expression (6.3) implies

$$\begin{aligned} |n^\alpha(t)|&\leqq E[|\eta|]\exp[-\alpha\kappa t]+\alpha\exp[-\alpha\kappa t]\int_0^t\exp[\alpha\kappa u]\,|E[g(x^\alpha(u))]|\,du \\ &\leqq E[|\eta|]\exp[-\alpha\kappa t]+\alpha C\exp[-\alpha\kappa t]\int_0^t\exp[\alpha\kappa u]du\,, \end{aligned}$$

which yields (4.8). Hence the proof is complete.

Proof of Theorem 2.5. The expression (6.7) implies

$$\frac{1}{\sqrt{\alpha}}y^{\alpha}(t)=\eta\frac{1}{\sqrt{\alpha}}\exp[-\alpha(\kappa+\gamma)t]-(\kappa+\gamma)\sqrt{\alpha}I_1^{\alpha}(t)$$
$$+(\kappa+\gamma)\sqrt{\alpha}I_2^{\alpha}(t)+(\kappa+\gamma)\sqrt{\alpha}I_3^{\alpha}(t),$$

and so

$$E\left[\left|\frac{1}{\sqrt{\alpha}}y^{\alpha}(t)\right|^2\right]$$
$$\tag{4.9} \leqq 4\left[\left(\frac{1}{\alpha}\right)E[|\eta|^2]\exp[-2\alpha(\kappa+\gamma)t]\right.$$
$$\left.+(\kappa+\gamma)^2\alpha\left\{E[|I_1^{\alpha}(t)|^2]+E[|I_2^{\alpha}(t)|^2]+E[|I_3^{\alpha}(t)|^2]\right\}\right].$$

Hereafter, for $p>0$ and $t\geqq 0$, put

$$\lambda(t;\alpha p)=\frac{1}{\alpha p}(1-\exp[-\alpha pt]), \qquad \text{so that} \qquad \lambda(t;\alpha p)\leqq\frac{1}{\alpha p}.$$

In the definition of $I_1^{\alpha}(t)$ and $I_2^{\alpha}(t)$ cited in Notation 1.1, use the condition that $|g(x)|\leqq C$ for all $x\in R^1$. Then we have

$$|I_1(t)|\leqq\frac{C}{\kappa+\gamma}\lambda(t;\alpha(\kappa+\gamma))\leqq\frac{C}{\kappa+\gamma}\frac{1}{\alpha(\kappa+\gamma)} \qquad \text{for all } t\geqq 0.$$

Further, under the assumptions, Lemma 4.1 applies to $I_2^{\alpha}(t)$, so that

$$|I_2^{\alpha}(t)|\leqq\frac{\gamma}{\kappa+\gamma}\exp[-\alpha(\kappa+\gamma)t]\int_0^t\exp[\alpha(\kappa+\gamma)u]\,|n^{\alpha}(u)|\,du$$
$$\leqq\frac{\gamma}{\kappa+\gamma}\bar{N}\lambda(t;\alpha(\kappa+\gamma))\leqq\frac{\gamma}{\kappa+\gamma}\bar{N}\frac{1}{\alpha(\kappa+\gamma)} \qquad \text{for all } t\geqq 0.$$

On the other hand, Lemma 7.2 implies

$$E[|I_3^{\alpha}(t)|^2]=\left(\frac{\delta}{\kappa+\gamma}\right)^2\lambda(t;2\alpha(\kappa+\gamma))$$
$$\leqq\left(\frac{\delta}{\kappa+\gamma}\right)^2\frac{1}{2\alpha(\kappa+\gamma)} \qquad \text{for all } t\geqq 0.$$

Therefore, substituting these estimates into the right-hand side of (4.9), we get

$$E\left[\left|\frac{1}{\sqrt{\alpha}}y^{\alpha}(t)\right|^2\right]\leqq M\left[\left(\frac{1}{\alpha}\right)\exp[-2\alpha(\kappa+\gamma)t]+\alpha\left\{\frac{1}{\alpha^2}+\frac{1}{\alpha}\right\}\right]$$

for all $t\geqq 0$ with a constant $M>0$ being independent of t and α. Therefore we get (2.4). Hence the proof is complete.

5. Exponential estimate for remainder term

In order to prove Theorem 2.6 we prepare two lemmas.

Lemma 5.1. *Let $I_3^\alpha(t)$ be the process given by Notation 1.1, and by μ_t^α denote the probability distribution in R^1 of $I_3^\alpha(t)$. Then, for each $t>0$, the family $\{\mu_t^\alpha\}_{\alpha\geq 1}$ satisfies the large deviation principle with the action functional $\lambda(\alpha)L(x)$ in the sense of Definition 2.1, where*

$$\lambda(\alpha)=\alpha, \quad L(x)=\frac{1}{2\sigma^2}x^2 \quad \text{and} \quad \sigma^2=\left(\frac{\delta}{\kappa+\gamma}\right)^2\frac{1}{2(\kappa+\gamma)}.$$

Proof. As follows from Remark 6.1 and Lemma 7.2, $I_3^\alpha(t)$ is a Gaussian stochastic process with mean 0 and variance

$$v^\alpha(t)=E[|I_3^\alpha(t)|^2]=\frac{1}{\alpha}\sigma^2\Big(1-\exp[-2\alpha(\kappa+\gamma)t]\Big),$$

where

$$\sigma^2=\left(\frac{\delta}{\kappa+\gamma}\right)^2\frac{1}{2(\kappa+\gamma)}.$$

By $M^\alpha(z)$ denote the moment generating function of $I_3^\alpha(t)$, and put

$$H^\alpha(z)=\log M^\alpha(z) \quad \text{and} \quad \lambda(\alpha)=\alpha.$$

Then, since $M^\alpha(z)=E[\exp\{zI_3^\alpha(t)\}]=\exp\{v^\alpha(t)z^2/2\}$, we see

$$\frac{1}{\lambda(\alpha)}H^\alpha(\lambda(\alpha)z)=\frac{1}{2}\sigma^2(1-\exp[-2\alpha(\kappa+\gamma)t])z^2$$

$$\longrightarrow\frac{1}{2}\sigma^2z^2 \quad \text{as } \alpha\to\infty \quad \text{for each } t>0.$$

Define the function $H(z)$ by

$$H(z)=(\sigma^2/2)z^2.$$

To this function $H(z)$, the Legendre transform is defined by

$$L(x)=\sup_z\{zx-H(z)\}.$$

Since $z=z(x)=x/\sigma^2$ is the solution of the equation $H'(z)=x$, $L(x)$ is determined from the formula

$$L(x)=z(x)x-H(z(x)).$$

Namely

$$L(x)=\frac{1}{2\sigma^2}x^2.$$

The function $L(x)$ is continuous and strictly convex. Thus, Theorems 1.2 and

1.1 in [2] (p. 140 and p. 138) apply to the family $\{\mu_t^\alpha\}_{\alpha\geqq 1}$ for each $t>0$, which implies that Conditions (I) and (II) of Definition 2.1 are satisfied. Hence the proof is complete.

Lemma 5.2. *Suppose that $g(x)$ is a bounded and differentiable function satisfying*

$$|g(x)|+|g'(x)|\leqq D \qquad \text{for all } x\in R^1$$

with a constant $D>0$. Let $\phi=(\xi, \eta)$ be any two-dimensional random vector independent of the two-dimensional Brownian motion process, such that

$$|\phi|\leqq a \qquad \text{with probability } 1$$

with a constant $a>0$. Let $R^\alpha(t)$ be the remainder term defined by Notation 1.1, and let $R_0^\alpha(t)$ be the solution of the following Langevin equation:

$$dR_0^\alpha(t)=-\alpha(\kappa+\gamma)R_0^\alpha(t)dt-\frac{\delta}{\kappa+\gamma}dw(t), \qquad R_0^\alpha(0)=0, \tag{5.1}$$

where $w(t)$ is the same Brownian motion process as in (1.1). Then

$$|R^\alpha(t)-R_0^\alpha(t)|\leqq \bar{K}\left(\frac{1}{\alpha}\right) \qquad \text{for all } t\geqq 0 \text{ with probability } 1, \tag{5.2}$$

where $\bar{K}$ is a positive constant being independent of t and α.

Proof. For $R^\alpha(t)$ and $R_0^\alpha(t)$ in the hypothesis, set

$$V^\alpha(t)=R^\alpha(t)-R_0^\alpha(t).$$

Then, since $R^\alpha(t)$ satisfies the equation (6.8), $V^\alpha(t)$ is the solution of the following equation:

$$V^\alpha(t)=-\alpha(\kappa+\gamma)\int_0^t V^\alpha(u)du+\int_0^t A^\alpha(u)du,$$

where

$$A^\alpha(t)=\frac{\gamma}{\kappa}\Big\{E[\eta]-n^\alpha(t)\Big\}+\Big\{\eta+\frac{1}{\kappa+\gamma}g(x^\alpha(t))\Big\}+\frac{\gamma}{\kappa(\kappa+\gamma)}E[g(x^\alpha(t))], \tag{5.3}$$

$$\eta=y^\alpha(0) \qquad \text{and} \qquad n^\alpha(t)=E[y^\alpha(t)].$$

Namely

$$\frac{d}{dt}V^\alpha(t)=-\alpha(\kappa+\gamma)V^\alpha(t)+A^\alpha(t) \qquad \text{with} \quad V^\alpha(0)=0.$$

The solution is given by the form

$$V^\alpha(t)=\exp[-\alpha(\kappa+\gamma)t]\int_0^t \exp[\alpha(\kappa+\gamma)u]A^\alpha(u)du. \tag{5.4}$$

The boundedness condition on $g(x)$ and $\phi=(\xi, \eta)$ applies to (5.3), and so

$$|A^{\alpha}(t)|\leqq\overline{M}(1+|n^{\alpha}(t)|)\qquad\text{for all } t\geqq 0$$

with a constant $\overline{M}>0$ being independent of t and α. Further, under the assumptions, all conditions of Lemma 4.1 are satisfied, and hence Lemma 4.1 implies

$$|n^{\alpha}(u)|\leqq\overline{N}\qquad\text{uniformly in } u\geqq 0 \text{ and } \alpha\geqq 1$$

with a constant $\overline{N}>0$. Accordingly

$$|A^{\alpha}(u)|\leqq\overline{M}(1+\leqq\overline{N})\qquad\text{uniformly in } u\geqq 0 \text{ and } \alpha\geqq 1\,. \tag{5.5}$$

Therefore, by (5.4) we see

$$\begin{aligned}|V^{\alpha}(t)|&\leqq\overline{M}(1+\overline{N})\exp[-\alpha(\kappa+\gamma)t]\int_0^t\exp[\alpha(\kappa+\gamma)u]du\\&=\overline{M}(1+\overline{N})\frac{1}{\alpha(\kappa+\gamma)}(1-\exp[-\alpha(\kappa+\gamma)t]),\end{aligned}$$

showing (5.2). Hence the proof is complete.

Remark 5.1. The processes $I_3^{\alpha}(t)$ and $R_0^{\alpha}(t)$ are the pathwise unique solutions of the Langevin equations (1.7) and (5.1), respectively, and hence

$$R_0^{\alpha}(t)=-I_3^{\alpha}(t)\,.$$

Therefore, $R_0^{\alpha}(t)$ and $I_3^{\alpha}(t)$ have the same normal distribution.

Proof of Theorem 2.6. By ν_t^{α} denote the probability distribution in R^1 of $R^{\alpha}(t)$. Then we must show that for each $t>0$ the family $\{\nu_t^{\alpha}\}_{\alpha\geq 1}$ satisfies Conditions (I) and (II) of Definition 2.1. In the following, let $\varepsilon>0$, $\theta>0$, $x\in R^1$ and $s>0$ be arbitrary. Since $g(x)$, $g'(x)$ and $\phi=(\xi,\eta)$ are bounded, Lemma 5.2 implies

$$\begin{aligned}|R^{\alpha}(t)-x|&\leqq|R^{\alpha}(t)-R_0^{\alpha}(t)|+|R_0^{\alpha}(t)-x|\\&\leqq\overline{K}\Big(\frac{1}{\alpha}\Big)+|R_0^{\alpha}(t)-x|\qquad\text{for all } t\geqq 0\end{aligned}\tag{5.6}$$

with a constant $\overline{K}>0$. Choose α so large that $\varepsilon-\overline{K}(1/\alpha)>0$. Then

$$\{|R^{\alpha}(t)-x|<\varepsilon\}\supseteqq\Big\{|R_0^{\alpha}(t)-x|<\varepsilon-\overline{K}\Big(\frac{1}{\alpha}\Big)\Big\}\,.$$

Let μ_t^{α} be the probability distribution of $I_3^{\alpha}(t)$. Then, by Remark 5.1, the probability distribution of $R_0^{\alpha}(t)$ is equivalent to μ_t^{α}, for which Lemma 5.1 holds. Thus

$$\begin{aligned}\nu_t^\alpha\{y\,;\,\rho(y,\,x)<\varepsilon\}&=P(|R^\alpha(t)-x|<\varepsilon)\\&\geqq P\Big(|R_0^\alpha(t)-x|<\varepsilon-\bar{K}\Big(\frac{1}{\alpha}\Big)\Big)\\&=\mu_t^\alpha\Big\{y\,;\,\rho(y,\,x)<\varepsilon-\bar{K}\Big(\frac{1}{\alpha}\Big)\Big\}\\&\geqq\exp\{-\alpha[L(x)+\theta]\}\qquad\text{for each } t>0\,,\end{aligned}$$

where

$$L(x)=\frac{1}{2\sigma^2}x^2\quad\text{and}\quad\sigma^2=\Big(\frac{\delta}{\kappa+\gamma}\Big)^2\frac{1}{2(\kappa+\gamma)}\,.$$

Here and hereafter, for $y\in R^1$ and $x\in R^1$, $\rho(y,\,x)$ is defined by

$$\rho(y,\,x)=|y-x|\,.$$

Namely, for each $t>0$, $\{\nu_t^\alpha\}_{\alpha\geq 1}$ satisfies Condition (I) of Definition 2.1.

Next, put $\Phi(s)=\{x\,;\,L(x)\leqq s\}$. Then it follows from (5.6) that

$$\begin{aligned}\rho(R^\alpha(t),\,\Phi(s))&=\inf\{|R^\alpha(t)-x|\,;\,x\in\Phi(s)\}\\&\leqq\bar{K}\Big(\frac{1}{\alpha}\Big)+\inf\{|R_0^\alpha(t)-x|\,;\,x\in\Phi(s)\}\\&=\bar{K}\Big(\frac{1}{\alpha}\Big)+\rho(R_0^\alpha(t),\,\Phi(s))\end{aligned}$$

and so

$$\{\rho(R^\alpha(t),\,\Phi(s))\geqq\varepsilon\}\subseteqq\Big\{\rho(R_0^\alpha(t),\,\Phi(s))\geqq\varepsilon-\bar{K}\Big(\frac{1}{\alpha}\Big)\Big\}\,.$$

By Remark 5.1, Lemma 5.1 applies to the family of probability distributions of $R_0^\alpha(t)$. Thus

$$\begin{aligned}\nu_t^\alpha\{y\,;\,\rho(y,\,\Phi(s))\geqq\varepsilon\}&=P(\rho(R^\alpha(t),\,\Phi(s))\geqq\varepsilon)\\&\leqq P\Big(\rho(R_0^\alpha(t),\,\Phi(s))\geqq\varepsilon-\bar{K}\Big(\frac{1}{\alpha}\Big)\Big)\\&=\mu_t^\alpha\Big\{y\,;\,\rho(y,\,\Phi(s))\geqq\varepsilon-\bar{K}\Big(\frac{1}{\alpha}\Big)\Big\}\\&\leqq\exp\{-\alpha(s-\theta)\}\qquad\text{for each } t>0\,.\end{aligned}$$

Namely, for each $t>0$, $\{\nu_t^\alpha\}_{\alpha\geq 1}$ satisfies Condition (II) of Definition 2.1. Evidently, the function $L(x)=x^2/(2\sigma^2)$ satisfies Condition (0) of Definition 2.1. Hence the proof is complete.

Proof of Theorem 2.7. For $\alpha\gg 1$, put $\varepsilon=1/\alpha$, so that ε is a small parameter. Consider the following one-dimensional stochastic differential equations:

(5.7) $$dR_x^\varepsilon(t)=[-(\kappa+\gamma)R_x^\varepsilon(t)+\varepsilon A^\varepsilon(t)]dt+\sqrt{\varepsilon}\left(-\frac{\delta}{\kappa+\gamma}\right)d\tilde{w}(t), \quad R_x^\varepsilon(0)=x\in R^1.$$

(5.8) $$dI_x^\varepsilon(t)=-(\kappa+\gamma)I_x^\varepsilon(t)dt+\sqrt{\varepsilon}\left(-\frac{\delta}{\kappa+\gamma}\right)d\tilde{w}(t), \quad I_x^\varepsilon(0)=x .$$

Here $\tilde{w}(t)$ is the one-dimensional Brownian motion process given by $\tilde{w}(t)=\sqrt{\alpha}\,w(t/\alpha)$ with the same Brownian motion process $w(t)$ as in (1.1). For $x\in R^1$ and $T<\infty$, define $S_{x,T}(\varphi)$ by the following equation:

(5.9) $$S_{x,T}(\varphi)=\begin{cases}\displaystyle\frac{1}{2}\int_0^T\left(\frac{\delta}{\kappa+\gamma}\right)^{-2}\left\{\frac{d}{dt}\varphi(t)+(\kappa+\gamma)\varphi(t)\right\}^2dt & \\ \qquad \text{if } \varphi(0)=x \text{ and } \varphi(t) \text{ is absolutely} \\ \qquad \text{continuous on } 0\leqq t\leqq T\,, \\ \infty \qquad \text{otherwise.}\end{cases}$$

Let $\{Q_x^\varepsilon\}_{0<\varepsilon<1}$ be the family of probability measures on $C([0, \infty); R^1)$ of the space of continuous functions $\omega: [0, \infty)\to R^1$, induced by $I_x^\varepsilon(\cdot)=\{I_x^\varepsilon(t)\}_{0<\varepsilon<1}$, where $I_x^\varepsilon(t)$ is the solution of (5.8). Then $\{Q_x^\varepsilon\}_{0<\varepsilon<1}$ satisfies the large deviation principle with the action functional $(1/\varepsilon)S_{x,T}(\varphi)$, where $S_{x,T}(\varphi)$ is defined by (5.9), in the sense of [2] (p. 80 and p. 146). Now, by $\{P_x^\varepsilon\}_{0<\varepsilon<1}$ denote the family of probability measures on $C([0, \infty); R^1)$, induced by $R_x^\varepsilon(\cdot)=\{R_x^\varepsilon(t)\}_{0<\varepsilon<1}$ with the solution $R_x^\varepsilon(t)$ of (5.7).

Under the assumptions, Lemma 4.1 implies that

$$|n^\alpha(t)|=|E[y^\alpha(t)]|\leqq\bar{N}<\infty \qquad \text{uniformly in } \alpha\geqq 1 \text{ and } t\geqq 0\,.$$

For $\alpha\gg 1$, let $A^\alpha(t)$ be the process given by (5.3). Define $A^\varepsilon(t)$ by $A^\varepsilon(t)=A^\alpha(t/\alpha)$, emphasizing the dependence on the small parameter $0<\varepsilon=1/\alpha\ll 1$. Then, since $0\leqq t/\alpha\leqq t$ for all $t\geqq 0$, it follows from (5.5) that

$$|A^\varepsilon(t)|=|A^\alpha(t/\alpha)|\leqq\bar{A} \qquad \text{for all } t\geqq 0 \text{ uniformly in } \varepsilon$$

with the same constant $\bar{A}=\bar{M}(1+\bar{N})>0$ as in (5.5). Accordingly

$$\sup_{t\geq 0}|\varepsilon A^\varepsilon(t)|\longrightarrow 0 \qquad \text{as } \varepsilon\to 0\,.$$

Therefore, by only minor change of the proof of Theorem 3.1 in [2] (p. 154), we can obtain that $\{P_x^\varepsilon\}_{0<\varepsilon<1}$ satisfies the large deviation principle with the same action functional $(1/\varepsilon)S_{x,T}(\varphi)$ as in (5.9). If we set $x=0$ in (5.7), then we have that $R_0^\varepsilon(t)=R^\varepsilon(t)$, where $R^\varepsilon(t)$ is the solution of (2.7) with the initial state $R^\varepsilon(0)=0$. Namely, $(1/\varepsilon)S_{0,T}(\varphi)$ is the action functional corresponding to the measures induced by $R^\varepsilon(t)$. Hence the proof is complete.

6. Appendix A (Decomposition of processes)

Under Assumptions 1.1 and 1.2, let $(x^\alpha(t), y^\alpha(t))$ be the solution of (1.1) with the initial state $(x^\alpha(0), y^\alpha(0))=(\xi, \eta)=\phi$, and set $\eta^\alpha(t)=E[y^\alpha(t)]$. For $0\leqq i\leqq 4$, let $I_i^\alpha(t)$ be the same processes as in Notation 1.1.

First, $y^\alpha(t)$ and $n^\alpha(t)$ satisfy the following linear equations:

$$dy^\alpha(t)=[-\alpha(\kappa+\gamma)y^\alpha(t)-\alpha g(x^\alpha(t))+\alpha\gamma n^\alpha(t)]dt+\alpha\delta dw(t), \quad y^\alpha(0)=\eta, \tag{6.1}$$

$$\frac{d}{dt}n^\alpha(t)=-\alpha\kappa n^\alpha(t)-\alpha E[g(x^\alpha(t))], \qquad n^\alpha(0)=E[\eta]. \tag{6.2}$$

Solving (6.1) and (6.2), we have the following expressions:

$$n^\alpha(t)=\exp[-\alpha\kappa t]E[\eta]-\alpha\exp[-\alpha\kappa t]\int_0^t \exp[\alpha\kappa u]E[g(x^\alpha(u))]du, \tag{6.3}$$

$$\begin{aligned} y^\alpha(t)=\exp[-\alpha(\kappa+\gamma)t]\times\Big[&\int_0^t \exp[\alpha(\kappa+\gamma)u]\{-\alpha g(x^\alpha(u))+\alpha\gamma n^\alpha(u)\}du \\ &+\alpha\delta\int_0^t \exp[\alpha(\kappa+\gamma)u]dw(u)+\eta\Big]. \end{aligned} \tag{6.4}$$

Secondly, substitute (6.4) into

$$x^\alpha(t)=\xi+\int_0^t y^\alpha(u)du,$$

and change the order of integration in the double integral. Then we have the following expression of the position process:

$$x^\alpha(t)=\xi-\frac{1}{\kappa+\gamma}\int_0^t g(x^\alpha(u))du-\frac{\gamma}{\kappa(\kappa+\gamma)}\int_0^t E[g(x^\alpha(u))]du+\frac{\delta}{\kappa+\gamma}w(t)+R^\alpha(t), \tag{6.5}$$

where $R^\alpha(t)$ is the remainder term given by Notation 1.1. Next, let $x(t)$ be the solution of (1.2), and put $\Delta^\alpha(t)=x^\alpha(t)-x(t)$. If $g(x)$ is differentiable, then by the mean value theorem we can rewrite (1.3) as follows:

$$\Delta^\alpha(t)=-\frac{1}{\kappa+\gamma}\int_0^t \Delta^\alpha(u)g'(\zeta^\alpha(u))du-\frac{\gamma}{\kappa(\kappa+\gamma)}\int_0^t E[\Delta^\alpha(u)g'(\zeta^\alpha(u))]du+R^\alpha(t), \tag{6.6}$$

where

$$\zeta^\alpha(u)=x(u)+\theta^\alpha(u)\Delta^\alpha(u) \quad \text{and} \quad 0<\theta^\alpha(u)<1.$$

On the other hand, (6.4) can be rewritten as follows:

$$y^\alpha(t)=\eta\exp[-\alpha(\kappa+\gamma)t]-\alpha(\kappa+\gamma)I_1^\alpha(t)+\alpha(\kappa+\gamma)I_2^\alpha(t)+\alpha(\kappa+\gamma)I_3^\alpha(t). \tag{6.7}$$

Remark 6.1. For $0\leqq i\leqq 4$, each $I_i^\alpha(t)$ satisfies the following linear equations:

$$\frac{d}{dt}I_0^\alpha(t)=-\alpha(\kappa+\gamma)I_0^\alpha(t)+\eta\,, \qquad I_0^\alpha(0)=0 \quad (\eta=y^\alpha(0))\,.$$

$$\frac{d}{dt}I_1^\alpha(t)=-\alpha(\kappa+\gamma)I_1^\alpha(t)+\frac{1}{\kappa+\gamma}g(x^\alpha(t))\,, \qquad I_1^\alpha(0)=0\,.$$

$$\frac{d}{dt}I_2^\alpha(t)=-\alpha(\kappa+\gamma)I_2^\alpha(t)+\frac{\gamma}{\kappa+\gamma}E[y^\alpha(t)]\,, \qquad I_2^\alpha(0)=0\,.$$

$$dI_3^\alpha(t)=-\alpha(\kappa+\gamma)I_3^\alpha(t)dt+\frac{\delta}{\kappa+\gamma}dw(t), \qquad I_3^\alpha(0)=0\,.$$

$$\frac{d}{dt}I_4^\alpha(t)=-\alpha\kappa I_4^\alpha(t)+\left\{\frac{\gamma}{\kappa+\gamma}E[\eta]+\frac{\gamma}{\kappa(\kappa+\gamma)}E[g(x^\alpha(t))]\right\}, \qquad I_4^\alpha(0)=0.$$

In fact, consider the linear equation of the form

$$dX(t)=[-\alpha pX(t)+A(t)]dt+bdw(t)$$

with positive constants α, p and b, integrable function $A(t)$ and Brownian motion process $w(t)$. Then the solution $X(t)$ is given by

$$X(t)=\exp[-\alpha pt]\left[\int_0^t \exp[\alpha pu]\{A(u)du+bdw(u)\}+X(0)\right].$$

This formula together with Notation 1.1 gives the above remark.

Remark 6.2. The remainder term $R^\alpha(t)$ satisfies the following stochastic differential equation:

$$dR^\alpha(t)=-\alpha(\kappa+\gamma)R^\alpha(t)dt+A^\alpha(t)dt-\frac{\delta}{\kappa+\gamma}dw(t)\,, \qquad R^\alpha(0)=0\,, \tag{6.8}$$

where

$$A^\alpha(t)=\frac{\gamma}{\kappa}\left\{E[\eta]-n^\alpha(t)\right\}+\left\{\eta+\frac{1}{\kappa+\gamma}g(x^\alpha(t))\right\}+\frac{\gamma}{\kappa(\kappa+\gamma)}E[g(x^\alpha(t))]$$

and $w(t)$ is the same Brownian motion process as in (1.1).

In fact, Remark 6.1 implies

$$\begin{aligned}dR^\alpha(t)=&-\alpha(\kappa+\gamma)\{I_0^\alpha(t)+I_1^\alpha(t)-I_2^\alpha(t)-I_3^\alpha(t)\}\,dt-\alpha\kappa I_4^\alpha(t)dt\\&+\left[\left\{\eta+\frac{1}{\kappa+\gamma}g(x^\alpha(t))\right\}+\frac{\gamma}{\kappa+\gamma}\left\{E[\eta]-n^\alpha(t)\right\}\right.\\&\left.+\frac{\gamma}{\kappa(\kappa+\gamma)}E[g(x^\alpha(t))]\right]dt-\frac{\delta}{\kappa+\gamma}dw(t).\end{aligned} \tag{6.9}$$

Note that $I_0^\alpha(t)+I_1^\alpha(t)-I_2^\alpha(t)-I_3^\alpha(t)=R^\alpha(t)-I_4^\alpha(t)$. Observe the definition of $I_4^\alpha(t)$

and the form (6.3) for $n^{\alpha}(t)$, so that

$$I_4^{\alpha}(t)=\frac{\gamma}{\kappa+\gamma}\frac{1}{\alpha\kappa}\left\{E[\eta]-n^{\alpha}(t)\right\}. \tag{6.10}$$

Then, by substituting (6.10) into (6.9), we can obtain (6.8).

7. Appendix B (Estimate for processes)

Under Assumptions 1.1 and 1.2, let $(x^{\alpha}(t), y^{\alpha}(t))$ be the solution of (1.1) with the initial state $(x^{\alpha}(0), y^{\alpha}(0))=(\xi, \eta)=\phi$, and set $n^{\alpha}(t)=E[y^{\alpha}(t)]$. For $0\leqq i\leqq 4$, let $I_i^{\alpha}(t)$ be the same processes as in Notation 1.1.

Notation 7.1. For $t\geqq 0$ and $p>0$, define the function $\lambda(t\,;p)$ by

$$\lambda(t\,;p)=\frac{1}{p}(1-\exp[-pt]).$$

Remark 7.1. The following inequalitis hold for $\lambda(t\,;p)$:
(i) $\lambda(t\,;p)\leqq t$ for all $t\geqq 0$ uniformly in $p>0$.
(ii) $\lambda(t\,;p)\leqq 1/p$ for all $t\geqq 0$.

Lemma 7.1. *Suppose that the function $g(x)$ and the initial vector $\phi=(\xi, \eta)$ satisfy Assumptions 1.1 and 1.2, respectively. Then*

$$|g(x)|\leqq c(1+|x|) \qquad \textit{for all } x\in R^1,$$

where $c=\max\{l, |g(0)|\}$ with the Lipschitz constant l for $g(x)$. Further, for $i\neq 3$, each $I_i^{\alpha}(t)$ satisfies the following estimates:

$$|I_0^{\alpha}(t)|\leqq|\eta|\lambda(t\,;\alpha(\kappa+\gamma)) \quad \textit{for all } t\geqq 0, \textit{ where } \eta=y^{\alpha}(0).$$

$$|I_1^{\alpha}(t)|\leqq\frac{c}{\kappa+\gamma}\Big(1+\sup_{0\leq u\leq t}|x^{\alpha}(u)|\Big)\lambda(t\,;\alpha(\kappa+\gamma)) \quad \textit{for all } t\geqq 0.$$

$$|n^{\alpha}(t)|\leqq\exp[-\alpha\kappa t]E[|\eta|]+cE\Big[1+\sup_{0\leq u\leq t}|x^{\alpha}(u)|\Big]\Big\{\alpha\lambda(t\,;\alpha\kappa)\Big\} \quad \textit{for all } t\geqq 0.$$

$$|I_2^{\alpha}(t)|\leqq\frac{\gamma}{\kappa+\gamma}\Big\{E[|\eta|]+\frac{c}{\kappa}E\Big[1+\sup_{0\leq u\leq t}|x^{\alpha}(u)|\Big]\Big\}\lambda(t\,;\alpha(\kappa+\gamma)) \quad \textit{for all } t\geqq 0.$$

$$|I_4^{\alpha}(t)|\leqq\frac{\gamma}{\kappa+\gamma}\Big\{E[|\eta|]+\frac{c}{\kappa}E\Big[1+\sup_{0\leq u\leq t}|x^{\alpha}(u)|\Big]\Big\}\lambda(t\,;\alpha\kappa) \quad \textit{for all } t\geqq 0.$$

Proof. If α and p are positive constants and $h(t)$ is a function satisfying $\sup_{0\leq u\leq t}|h(u)|<\infty$ for every $t<\infty$, then

$$(7.1)\qquad \left|\exp[-\alpha pt]\int_0^t \exp[\alpha pu]h(u)du\right| \leqq \left(\sup_{0\leqq u\leqq t}|h(u)|\right)\lambda(t;\alpha p).$$

The above inequality (7.1) applies to Notation 1.1 as follows:
In $I_0^\alpha(t)$, use (7.1) with $h(u)=\eta$, where $\eta=y^\alpha(0)$.
In $I_1^\alpha(t)$, use (7.1) with $h(u)=g(x^\alpha(u))$.
In the formula (6.3) for $n^\alpha(t)$, use (7.1) with $h(u)=E[g(x^\alpha(u))]$.
In $I_2^\alpha(t)$, use (7.1) with $h(u)=n^\alpha(u)$.
In $I_4^\alpha(t)$, use (7.1) with $h(u)=E[\eta]+(1/\kappa)E[g(x^\alpha(u))]$.
Then the proof is complete.

Lemma 7.2. *The process $I_3^\alpha(t)$ has the following moments:*

$$(7.2)\qquad E[|I_3^\alpha(t)|^2]=\left(\frac{\delta}{\kappa+\gamma}\right)^2\lambda(t;2\alpha(\kappa+\gamma)) \quad \textit{for all } t\geqq 0.$$

$$(7.3)\qquad E[|I_3^\alpha(t)|^4]=6\left(\frac{\delta}{\kappa+\gamma}\right)^4\frac{1}{2\alpha(\kappa+\gamma)}\left\{\lambda(t;2\alpha(\kappa+\gamma))-\lambda(t;4\alpha(\kappa+\gamma))\right\} \quad \textit{for all } t\geqq 0.$$

For any integer $m\geqq 2$ and every $T<\infty$,

$$(7.4)\qquad \begin{aligned} &E\left[\sup_{0\leqq t\leqq T}|I_3^\alpha(t)|^{2m}\right] \\ &\leqq 2\sqrt{2}\left[\left(\frac{\delta}{\kappa+\gamma}\right)^2 m(2m-1)T(J_{2m-2}^\alpha(T))^{1/2}+\frac{\delta}{\kappa+\gamma}2m(TJ_{2m-1}^\alpha(T))^{1/2}\right]\end{aligned}$$

where

$$J_n^\alpha(t)=\left(\frac{\delta}{\kappa+\gamma}\right)^{2n}(n(2n-1))^n t^{n-1}\lambda(t;2n\alpha(\kappa+\gamma)).$$

Proof. Put

$$f_2(t)=E[|I_3^\alpha(t)|^2] \quad \text{and} \quad f_4(t)=E[|I_3^\alpha(t)|^4].$$

Then

$$\begin{aligned} E[|I_3^\alpha(t)|^2]&=E\left[\left(\frac{\delta}{\kappa+\gamma}\exp[-\alpha(\kappa+\gamma)t]\int_0^t\exp[\alpha(\kappa+\gamma)u]dw(u)\right)^2\right] \\ &=\left(\frac{\delta}{\kappa+\gamma}\right)^2\exp[-2\alpha(\kappa+\gamma)t]\int_0^t\exp[2\alpha(\kappa+\gamma)u]du,\end{aligned}$$

which shows (7.2). Since $I_3^\alpha(t)$ satisfies the Langevin equation of Remark 6.1, Ito's formula yields

$$\begin{aligned} d[|I_3^\alpha(t)|^4]=&\left[-4\alpha(\kappa+\gamma)|I_3^\alpha(t)|^4+6\left(\frac{\delta}{\kappa+\gamma}\right)^2|I_3^\alpha(t)|^2\right]dt \\ &+4\left(\frac{\delta}{\kappa+\gamma}\right)I_3^\alpha(t)^3dw(t).\end{aligned}$$

Taking expectations, we have the linear ordinary differential equation of first order:

$$\frac{d}{dt}f_4(t)=-4\alpha(\kappa+\gamma)f_4(t)+6\Big(\frac{\delta}{\kappa+\gamma}\Big)^2 f_2(t), \qquad f_4(0)=0\,.$$

The solution is given by

$$f_4(t)=\exp\left[-4\alpha(\kappa+\gamma)t\right]\int_0^t \exp\left[4\alpha(\kappa+\gamma)u\right]6\Big(\frac{\delta}{\kappa+\gamma}\Big)^2 f_2(u)du\,,$$

where $f_2(u)$ has the explicit form (7.2), which shows (7.3). The estimate (7.4) follows from [5] (p. 309). Hence the proof is complete.

Lemma 7.3. *Suppose that the function* $g(x)$ *and the initial vector* $\phi=(\xi,\eta)$ *satisfy Assumptions* 1.1 *and* 1.2, *respectively. Then, for every* $T>0$ *there exists a constant* $K(T)>0$, *for which the following estimates hold*:

$$E[|R^\alpha(t)|^2]\leqq K(T)\Big(\frac{1}{\alpha^2}+\frac{1}{\alpha}\Big) \qquad \textit{for } 0\leqq t\leqq T. \tag{7.5}$$

$$E[|R^\alpha(t)|^4]\leqq K(T)\Big(\frac{1}{\alpha^4}+\frac{1}{\alpha^2}\Big) \qquad \textit{for } 0\leqq t\leqq T. \tag{7.6}$$

$$E[|r^\alpha(t)|^2]\leqq K(T)\Big(\frac{1}{\alpha^2}\Big) \qquad \textit{for } 0\leqq t\leqq T. \tag{7.7}$$

Proof. By (ii) of Remark 7.1, since for $\alpha>0$ and $p>0$

$$\lambda(t\,;\alpha p)\leqq\frac{1}{\alpha p} \qquad \text{for all } t\geqq 0\,,$$

Lemma 7.1 implies that for $i\neq 3$, each $I_i^\alpha(t)$ satisfies

$$|I_i^\alpha(t)|\leqq \Psi_i^\alpha(t)\Big(\frac{1}{\alpha}\Big) \qquad \text{for all } t\geqq 0 \tag{7.8}$$

with random processes $\Psi_i^\alpha(t)$. Under Assumptions 1.1 and 1.2, the moment estimate (i) of Theorem 1.1 holds for $x^\alpha(t)$ and the moment condition such that $E[|\phi|^{2n}]<\infty$ with $n=1$ and 2 holds for $\phi=(\xi,\eta)$. So, we can find a constant $H(T)>0$ such that for $i\neq 3$,

$$E[|\Psi_i^\alpha(t)|^{2n}]\leqq H(T) \qquad \text{for } 0\leqq t\leqq T \tag{7.9}$$

with $n=1$ and 2. On the other hand, by Lemma 7.2, there exists a constant $C>0$ being independent of t and α such that

$$E[|I_3^\alpha(t)|^{2n}]\leqq C\Big(\frac{1}{\alpha^n}\Big) \qquad \text{for all } t\geqq 0 \tag{7.10}$$

with $n=1$ and 2. Further, the Schwarz inequality implies that

$$E[|R^{\alpha}(t)|^2]\leqq 5\sum_{i=0}^{4}E[|I_i^{\alpha}(t)|^2]\,,\qquad E[|R^{\alpha}(t)|^4]\leqq 5^3\sum_{i=0}^{4}E[|I_i^{\alpha}(t)|^4]$$

and

$$E[|r^{\alpha}(t)|^2]\leqq 4\sum_{i\neq 3}E[|I_i^{\alpha}(t)|^2]\,.$$

Thus, by (7.8)～(7.10) we get (7.5)～(7.7), showing the proof.

Lemma 7.4. *Suppose that the function $g(x)$ and the initial vector $\phi=(\xi,\eta)$ satisfy Assumptions* 1.1 *and* 1.2, *respectively. For the solutions $x^{\alpha}(t)$ of* (1.1) *and $x(t)$ of* (1.2), *set*

$$\Delta^{\alpha}(t)=x^{\alpha}(t)-x(t)\,.$$

Then, for every $T>0$ there exists a constant $D(T)>0$ depending on T and being independent of α, for which the following estimates hold:

$$E[|\Delta^{\alpha}(t)|^2]\leqq D(T)\Big(\frac{1}{\alpha^2}+\frac{1}{\alpha}\Big)\exp[D(T)t]\qquad for\ 0\leqq t\leqq T\,. \tag{7.11}$$

$$E[|\Delta^{\alpha}(t)|^4]\leqq D(T)\Big(\frac{1}{\alpha^4}+\frac{1}{\alpha^2}\Big)\exp[D(T)t]\qquad for\ 0\leqq t\leqq T\,. \tag{7.12}$$

Proof. Let $T<\infty$ be arbitrary and fixed, and consider the interval $0\leqq t\leqq T$. Then the Schwarz inequality applies to (1.3) with the following estimates:

$$\begin{aligned}&|\Delta^{\alpha}(t)|^2\\ &\leqq 3\Big[\Big(\frac{l}{\kappa+\gamma}\Big)^2T\int_0^t|\Delta^{\alpha}(u)|^2du+\Big(\frac{\gamma l}{\kappa(\kappa+\gamma)}\Big)^2T\int_0^tE[|\Delta^{\alpha}(u)|^2]du+|R^{\alpha}(t)|^2\Big],\end{aligned} \tag{7.13}$$

$$\begin{aligned}&|\Delta^{\alpha}(t)|^4\\ &\leqq 3^23\Big[\Big(\frac{l}{\kappa+\gamma}\Big)^4T^3\int_0^t|\Delta^{\alpha}(u)|^4du+\Big(\frac{\gamma l}{\kappa(\kappa+\gamma)}\Big)^4T^3\int_0^tE[|\Delta^{\alpha}(u)|^4]du+|R^{\alpha}(t)|^4\Big]\end{aligned} \tag{7.14}$$

for $0\leqq t\leqq T$. Here, the constant l in (7.13) and (7.14) is the Lipschitz constant for $g(x)$. Take expectations on (7.13) and (7.14), and use the estimates (7.5) and (7.6) of Lemma 7.3 for $R^{\alpha}(t)$. Then we have

$$E[|\Delta^{\alpha}(t)|^2]\leqq 3\Big[\Big\{\Big(\frac{l}{\kappa+\gamma}\Big)^2+\Big(\frac{\gamma l}{\kappa(\kappa+\gamma)}\Big)^2\Big\}T\int_0^tE[|\Delta^{\alpha}(u)|^2]du+K(T)\Big(\frac{1}{\alpha^2}+\frac{1}{\alpha}\Big)\Big]$$

and

$$E[|\Delta^{\alpha}(t)|^4]\leqq 3^23\Big[\Big\{\Big(\frac{l}{\kappa+\gamma}\Big)^4+\Big(\frac{\gamma l}{\kappa(\kappa+\gamma)}\Big)^4\Big\}T^3\int_0^tE[|\Delta^{\alpha}(u)|^4]du+K(T)\Big(\frac{1}{\alpha^4}+\frac{1}{\alpha^2}\Big)\Big]$$

for $0\leqq t\leqq T$. Therefore, by the Gronwall-Bellman inequality we get (7.11) and (7.12), showing the proof.

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