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DIMENSION AND PRODUCTS OF TOPOLOGICAL GROUPS

Dedicated to Professor Akihiro Okuyama on his 60th birthday

By

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Summary. In this paper, we prove that, for each pair n, d of positive integers with $n\leqq d$, there is a subgroup G_{nd} of $\boldsymbol{R}^{n+1}$ satisfying $\dim G_{nd}=n$ and $\dim(G_{nd})^{\omega}=d$. We also prove that there is a separable metrizable precompact topological group H_{nd} satisfying the same property.

1. Introduction

In the present paper we consider the topological dimension of products in separable metrizable topological groups. One of the most interesting parts in dimension theory is to investigate the behaviour of dimension functions in producing product spaces. It is known that for finite dimensional separable metrizable spaces X and Y with $\dim X>0$ and $\dim Y>0$, the inequality $\dim(X\times Y)>\max\{\dim X, \dim Y\}$ holds if X or Y is compact. In 1967, Anderson and Keisler [1] have shown that the compactness can not be dropped in the theorem above. Indeed, they have proved that for each positive integer n there is a subspace X of $\boldsymbol{R}^{n+1}$ such that $\dim X=\dim X^{\omega}=n$. In 1985, Keesling [7] has shown that there is a subgroup of $\boldsymbol{R}^{n+1}$ having the same properties. On the other hand, Kulesza [8] improves the theorem of Anderson and Keisler in a different direction with Keesling's one.

Theorem ([8, Theorem 3]). *For each pair* n, d *of positive integers with* $n\leqq d$, *there is a subspace* X_{nd} *of* $\boldsymbol{R}^{n+1}$ *such that* $\dim X_{nd}=n$ *and* $\dim(X_{nd})^{\omega}=d$.

The purpose of the present paper is to show that there are a subgroup of $\boldsymbol{R}^{n+1}$ and a separable metrizable precompact topological group which have the same properties as Kulesza's space. All spaces considered here will be separable metrizable spaces. By a dimension we mean covering dimension dim. We

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refer the readers to [5] and [9] for dimension theory.

Let $\boldsymbol{R}$, $\boldsymbol{Q}$ and $\boldsymbol{Z}$ denote the reals, rationals and integers respectively. Let I denote the closed interval $[-1, 1]$. Let ω be the first infinite ordinal and $\mathfrak{c}$ denote the cardinality of the continuum that we think of as an initial ordinal. Follows from [1], by a *hyperplane* in $\boldsymbol{R}^n$ we mean a translation of a linear subspace of $\boldsymbol{R}^n$. For a hyperplane H in $\boldsymbol{R}^n$, let $\hat{H}$ be the set of all translations of H. Hyperplanes H and K of dimensions d_H and d_K are said to be *in general position* if for $H'\in\hat{H}$ and $K'\in\hat{K}$, $H'\cap K'\neq\phi$ implies $H'\cap K'$ is a hyperplane in $\boldsymbol{R}^n$ such that $\dim(H'\cap K')=\max\{0, d_H+d_K-n\}$.

2. A subgroup of the Euclidean space $\boldsymbol{R}^{n+1}$

The following theorem improves theorems due to Keesling and Kulesza mentioned in the introduction simultaneously.

Theorem 2.1. *For each pair n, d of positive integers with $n\leqq d$, there is a subgroup G_{nd} of $\boldsymbol{R}^{n+1}$ such that* $\dim G_{nd}=n$ *and* $\dim(G_{nd})^{\omega}=d$.

Proof. Follows from [8], for given n, $m\geqq 2$ and $n\leqq d\leqq nm-m$ we construct a subgroup G of $\boldsymbol{R}^n$ with $\dim G=n-1$ and $\dim G^q=d$ for all $q\geqq m$. The proof proceedes in six steps.

Step 1. We define a bijection

$$\varphi:\{(j, i): j=1, 2, \cdots, m-1, \text{ and } i=1, 2, \cdots, n\}\cup\{(m, n)\} \longrightarrow \{1, 2, \cdots, n(m-1)+1\}$$

as follows;

$$\varphi(j, i)=\begin{cases} j, & \text{if } i=n, \\ (m-1)(n-i)+j+1, & \text{if } i\leqq n-1. \end{cases}$$

For $j\leqq m$, we put

$$\lambda(j)=\begin{cases} \min\{i:\varphi(j, i)\leqq nm-d\}, & \text{if } 1\leqq j\leqq m-1, \\ n, & \text{if } j=m. \end{cases}$$

Furthermore, we put

$$J_j=\{\lambda(j), \lambda(j)+1, \cdots, n\},$$

$$M=\bigcup\{\{j\}\times J_j: j=1, 2, \cdots, m\}.$$

We consider

$$\boldsymbol{R}^{nm}=\prod_{j=1}^{m}\Big(\prod_{i=1}^{n}\boldsymbol{R}_{(j,i)}\Big) \quad\text{and}\quad I^{nm}=\prod_{j=1}^{m}\Big(\prod_{i=1}^{n}I_{(j,i)}\Big),$$

where $\boldsymbol{R}_{(j,i)}=\boldsymbol{R}$ and $I_{(j,i)}=I$ for each (j, i). Notice that we consider $\boldsymbol{R}^n=\prod_{i=1}^n \boldsymbol{R}_{(j,i)}$ for a suitable j to the occasion. Let $\{(A_{(j,i)}, B_{(j,i)}): j\leqq m \text{ and } i\leqq n\}$ be the standard essential family for I^{nm}, i.e.,

$$A_{(j,i)}=\pi_{(j,i)}^{-1}(-1) \quad \text{and} \quad B_{(j,i)}=\pi_{(j,i)}^{-1}(1),$$

where $\pi_{(j,i)}: I^{nm}\to I_{(j,i)}$ is the projection. We put

$$\mathcal{K}=\{\bigcap\{K_{(j,i)}:(j, i)\notin M\}: K_{(j,i)} \text{ is a separator of } A_{(j,i)} \text{ and } B_{(j,i)} \text{ in } I^{nm}\}.$$

Then we have two lemmas follow from Kulesza [8]. (Notice that the cardinality of the complement of M is $nm-(nm-d)=d$ in Lemma 2.2.)

Lemma 2.2. *Let Y be a subspace of $\boldsymbol{R}^{nm}$ with $Y\cap K\neq\phi$ for every $K\in\mathcal{K}$. Then* $\dim Y\geqq d$.

Lemma 2.3. $\pi_M(K)=\prod\{I_{(j,i)}:(j, i)\in M\}$ *for each $K\in\mathcal{K}$, where $\pi_M: I^{nm}\to \prod\{I_{(j,i)}:(j, i)\in M\}$ is the projection.*

Step 2. Let

$$\mathcal{C}=\{C: C \text{ is a continuum from } A_{(1,1)} \text{ to } B_{(1,1)} \text{ in } \boldsymbol{R}^n\}.$$

The following lemma is well known (cf. [1], [11]).

Lemma 2.4. *Let Y be a subspace of $\boldsymbol{R}^n$ with $Y\cap C\neq\phi$ for every $C\in\mathcal{C}$. Then* $\dim Y\geqq n-1$.

Step 3. For each $q>m$, we construct hyperplanes in $\boldsymbol{R}^{nq}$. Let

$$H_0=\{(x, x, \cdots, x)\in\boldsymbol{R}^{nq}: x=(x(1), x(2), \cdots, x(n))\in\boldsymbol{R}^n\},$$

and for each $j=1, 2, \cdots, m-1$,

$$H_j=\{(x, x, \cdots, x)\in H_0: x(k)=0 \text{ for } k\in J_j\}.$$

Then H_0 is a linear subspace of $\boldsymbol{R}^{nq}$ with $\dim H_0=n$ and H_j is a linear subspace of $\boldsymbol{R}^{nq}$ with $\dim H_j=\lambda(j)-1$ for $j=1, 2, \cdots, m-1$. For each $j=0, 1, 2, \cdots, m-1$ and each $p=(p_1, p_2, \cdots, p_m)\in\boldsymbol{Q}^{qm}$, where $p_i=(p_i(1), p_i(2), \cdots, p_i(q))\in\boldsymbol{Q}^q$, let

$$p_{j+1}H_j=\{(p_{j+1}(1)x, p_{j+1}(2)x, \cdots, p_{j+1}(q)x):(x, x, \cdots +x)\in H_j\}.$$

Let

$$H(p)=p_1H_0+p_2H_1+\cdots+p_mH_{m-1}.$$

Then $H(p)$ is a linear subspace of $\boldsymbol{R}^{nq}$ satisfying

$$\dim H(p) \leqq n + \sum_{j=1}^{m-1} (\lambda(j)-1) = n + n(m-1) - (nm-d-1) = d+1 .$$

Finally, we put

$$\mathcal{H} = \{H(p) : p \in \boldsymbol{Q}^{qm}\} .$$

Step 4. The following lemmas can be proved by arguments similar to that of [1] and [8].

Lemma 2.5. *For each* $q > m$ *there are countably many spheres* S_i, $i \in \omega$, *in* $\boldsymbol{R}^{nq}$ *satisfying the following conditions:*

(0) $S_i \cap \boldsymbol{Z}^{nq} = \phi$ *for each* $i \in \omega$.

(1) $\dim S_i = nq-d-1$ *for each* $i \in \omega$.

(2) *For each* $i \in \omega$, S_i *is contained in a hyperplane* H_i *in* $\boldsymbol{R}^{nq}$ *with* $\dim H_i = nq-d$.

(3) H_i *is in general position with respect to each* $H' \in \hat{H}$ *and each* $H \in \mathcal{H}$.

(4) $\dim (\boldsymbol{R}^{nq} - \bigcup \{S_i : i \in \omega\}) \leqq d$.

Lemma 2.6. *For each* $H \in \mathcal{H}$, $H' \in \hat{H}$ *and each* $i \in \omega$, $|H' \cap S_i| \leqq 2$.

Now, for each $q > m$, we put

$$T_{nq} = \bigcup \{S_i : i \in \omega\} .$$

Step 5. We enumerate $\mathcal{K}$ by $\mathcal{K} = \{K_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{C}$ by $\mathcal{C} = \{C_\alpha : \alpha < \mathfrak{c}\}$ such that both K_0 and C_0 contain the origin. We shall inductively construct subgroups Y_α and G_α of $\boldsymbol{R}^n$ satisfying the following conditions: For each $\alpha < \mathfrak{c}$,

(5) $Y_\alpha \cap C_\alpha \neq \phi$,

(6) $(G_\alpha)^m \cap K_\alpha \neq \phi$,

(7) $(G_\alpha)^q \cap T_{nq} = \phi$ for each $q > m$,

(8) $|G_\alpha| \leqq \omega \cdot |\alpha+1|$, and

(9) $\boldsymbol{Z}^n \subset G_\beta \subset Y_\alpha \subset G_\alpha$ if $\beta < \alpha$.

We put $Y_0 = G_0 = \boldsymbol{Z}^n$. Then it is clear that Y_0 and G_0 satisfy the conditions (5)-(9) for $\alpha = 0$. Suppose that $\alpha > 0$ and for each $\beta < \alpha$ subgroups Y_β and G_β of $\boldsymbol{R}^n$ satisfying the conditions above are constructed. First, we shall construct Y_α. We put

$$U_\alpha = \bigcup \{G_\beta : \beta < \alpha\} .$$

It follows that U_α is a subgroup of $\boldsymbol{R}^n$ satisfying

(10) $(U_\alpha)^q \cap T_{nq} = \phi$ for each $q > m$,

(11) $|U_\alpha| \leqq \omega \cdot |\alpha+1|$.

For each $q > m$, we put

$$B_q = \bigcup \{\pi_k(\boldsymbol{Q} \cdot (H(p) \cap (S_i + (U_\alpha)^q))) : p \in \boldsymbol{Q}^{qm},\ i \in \omega, \text{ and } k \leqq q\} ,$$

where $\pi_k : \boldsymbol{R}^{nq}=\prod_{k=1}^{q}(\boldsymbol{R}^n)_k \to (\boldsymbol{R}^n)_k$ is the projection. If follows from Lemma 2.6 that

$$|H(p)\cap(S_i+u)|\leqq 2 \qquad \text{for each } p\in\boldsymbol{Q}^{qm} \text{ and each } u\in(U_\alpha)^q .$$

Hence

$$|H(p)\cap(S_i+(U_\alpha)^q)|\leqq|(U_\alpha)^q|\leqq\omega\cdot|\alpha+1| .$$

Therefore $|B_q|\leqq\omega\cdot|\alpha+1|<\mathfrak{c}$. Thus there is $y_\alpha\in C_\alpha-\bigcup\{B_q : q>m\}$. Let

$$Y_\alpha=U_\alpha+\boldsymbol{Q}y_\alpha .$$

Then Y_α is a subgroup of $\boldsymbol{R}^n$ satisfying the condition (5). Furthermore, we have the following claim.

Claim 1. *$(Y_\alpha)^q\cap T_{nq}=\phi$ for each $q>m$.*

Proof. Fix $q>m$. Suppose that $(Y_\alpha)^q\cap T_{nq}\neq\phi$. Then there are

$$u=(u_1, u_2, \cdots, u_q)\in(U_\alpha)^q, \quad p_1=(p_1(1), p_1(2), \cdots, p_1(q))\in\boldsymbol{Q}_b \text{ and } i\in\omega$$

such that $u+p_1y_\alpha^*\in S_i$, where $y_\alpha^*=(y_\alpha, y_\alpha, \cdots, y_\alpha)\in H_0$. Hence

$$p_1y_\alpha^*\in S_i-u\subset S_i+(U_\alpha)^q .$$

We put $p=(p_1, 0, 0, \cdots, 0)\in\boldsymbol{Q}^{qm}$. Then we have $p_1y_\alpha^*\in p_1H_0\subset H(p)$. Hence

$$p_1y_\alpha^*\in H(p)\cap(S_i+(U_\alpha)^q) .$$

On the other hand, it follows from (10) that there is $k\leqq q$ such that $p_1(k)\neq 0$. Thus

$$y_\alpha=\pi_k(p_1y_\alpha^*/p_1(k))\in\pi_k(\boldsymbol{Q}\cdot(H(p)\cap(S_i+(U_\alpha)^q)))\subset B_q .$$

This completes the proof of the claim.

Now, we shall construct G_α. We inductively find, for $j=1, 2, \cdots, m$,

$$r_j=(r_j(\lambda(j)), r_j(\lambda(j)+1), \cdots, r_j(n))\in\prod_{i=\lambda(j)}^{n} I_{(j,i)}$$

such that, at each stage $k\leqq m$, letting

$$E_j=\{x\in\boldsymbol{R}^n : \pi_{J_j}(x)=r_j\} \qquad \text{for } j=1, 2, \cdots, k ,$$

where

$$\pi_{J_j} : \boldsymbol{R}^n=\prod_{i=1}^{n}\boldsymbol{R}_{(j,i)} \longrightarrow \prod_{i=\lambda(j)}^{n}\boldsymbol{R}_{(j,i)}$$

is the projection, and choosing $x_j\in E_j$ arbitrarily, for all $q>m$,

$$(*) \qquad (Y_\alpha+\boldsymbol{Q}x_1+\boldsymbol{Q}x_2+\cdots+\boldsymbol{Q}x_k)^q\cap T_{nq}=\phi .$$

Assume that $t\leqq m$ and that we have $r_1, r_2, \cdots, r_{t-1}$.

Claim 2. *There is a subfamily* $\mathcal{H}^{\natural}$ *of* $\bigcup\{\hat{H}: H\in\mathcal{H}\}$ *such that*

(12) $|\mathcal{H}^{\natural}|\leqq\omega\cdot|\alpha+1|$, *and*

(13) $\bigcup\{(Y_\alpha+\boldsymbol{Q}x+\boldsymbol{Q}x_1+\boldsymbol{Q}x_2+\cdots+\boldsymbol{Q}x_{t-1})^q : x\in\boldsymbol{R}^n,\ x_j\in E_j,\ 1\leqq j\leqq t-1\}$
$\subset\bigcup\{H' : H'\in\mathcal{H}^{\natural}\}$.

Proof. We put

$$\mathcal{H}^{\natural}=\left\{H(p)+y+\delta : p\in\boldsymbol{Q}^{qm},\ y\in(Y_\alpha)^q,\text{ and }\delta\in\Big(\sum_{k=1}^{t-1}\sum_{i=\lambda(k)}^{n}\boldsymbol{Q}r_k(i)\Big)^{nq}\right\}.$$

It is clear that $|\mathcal{H}^{\natural}|\leqq\omega\cdot|\alpha+1|$. To show that $\mathcal{H}^{\natural}$ satisfies the condition (13), let $x\in\boldsymbol{R}^n$, $x_j\in E_j$ for each $j\leqq t-1$, and

$$\sigma=(\sigma_1,\sigma_2,\cdots,\sigma_q)\in(Y_\alpha+\boldsymbol{Q}x+\boldsymbol{Q}x_1+\boldsymbol{Q}x_2+\cdots+\boldsymbol{Q}x_{t-1})^q,$$

where

$$\sigma_k=y_k+s_kx+s_k^1x_1+s_k^2x_2+\cdots+s_k^{t-1}x_{t-1}$$

for some $y_k\in Y_\alpha$ and $s_k, s_k^1, s_k^2, \cdots, s_k^{t-1}\in\boldsymbol{Q}$. Then we put

$$y=(y_1,y_2,\cdots,y_q)\in(Y_\alpha)^q,$$

$$p_1=(s_1,s_2,\cdots,s_q)\in\boldsymbol{Q}^q,$$

$$p_{j+1}=\begin{cases}(s_1^j,s_2^j,\cdots,s_q^j)\in\boldsymbol{Q}^q, & \text{if } 1\leqq j\leqq t-1,\\ (0,0,\cdots,0)\in\boldsymbol{Q}^q, & \text{if } t\leqq j\leqq m-1,\end{cases}$$

$$p=(p_1,p_2,\cdots,p_m)\in\boldsymbol{Q}^{qm},$$

$$x^*=(x,x,\cdots,x)\in H_0,\ \text{and}$$

$$x_j^*=\begin{cases}(0,0,\cdots,0)\in H_j, & \text{if } t\leqq j\leqq m-1,\\ (x_j',x_j',\cdots,x_j')\in H_j, & \text{if } 1\leqq j\leqq t-1,\end{cases}$$

where

$$x_j'(i)=\begin{cases}x_j(i), & \text{for } i\leqq\lambda(j)-1,\\ 0, & \text{for } \lambda(j)\leqq i\leqq n.\end{cases}$$

Furthermore we define

$$\delta=(\delta_1,\delta_2,\cdots,\delta_q)\in\Big(\sum_{k=1}^{t-1}\sum_{i=\lambda(k)}^{n}\boldsymbol{Q}r_k(i)\Big)^{nq}$$

as follows: For each $j\leqq q$ and $i\leqq n$,

$$\delta_j(i)=\begin{cases}\sum_k\{s_j^kr_k(i) : i\in J_k\}, & \text{if } i\in J_k \text{ for some } k,\\ 0, & \text{otherwise.}\end{cases}$$

Since $x_j\in E_j$ for each $j\leqq t-1$, it follows that

$$\begin{aligned}\sigma &= p_1x^*+p_2x_1^*+\cdots+p_mx_{m-1}^*+y+\delta\\ &\in p_1H_0+p_2H_1+\cdots+p_mH_{m-1}+y+\delta\\ &=H(p)+y+\delta\in\mathcal{H}^\natural .\end{aligned}$$

This completes the proof of the claim.

For each $q>m$, we put

$$F_{qt}=\{x\in\boldsymbol{R}^n : (Y_\alpha+\boldsymbol{Q}x+\boldsymbol{Q}x_1+\boldsymbol{Q}x_2+\cdots+\boldsymbol{Q}x_{t-1})^q\cap T_{nq}\neq\phi \text{ for some } x_j\in E_j,\ j=1, 2, \cdots, t-1\},$$

and

$$B_{qt}=\pi_{J_t}(F_{qt}),$$

where

$$\pi_{J_t}:\boldsymbol{R}^n=\prod_{i=1}^n\boldsymbol{R}_{(t,i)}\longrightarrow\prod_{i=\lambda(t)}^n\boldsymbol{R}_{(t,i)}$$

is the projection. Then we have

Claim 3. $|B_{qt}|\leqq\omega\cdot|\alpha+1|$ *for each* $q>m$.

Proof. Let

$$F_{qt}^\natural=\bigcup\{(Y_\alpha+\boldsymbol{Q}x+\boldsymbol{Q}x_1+\boldsymbol{Q}x_2+\cdots+\boldsymbol{Q}x_{t-1})^q\cap T_{nq}: x\in\boldsymbol{R}^n,\ x_j\in E_j,\ 1\leqq j\leqq t-1\}$$

and $\psi: F_{qt}\to F_{qt}^\natural$ be a mapping satisfying

$$\psi(x)\in(Y_\alpha+\boldsymbol{Q}x+\boldsymbol{Q}x_1+\boldsymbol{Q}x_2+\cdots+\boldsymbol{Q}x_{t-1})^q\cap T_{nq}$$

for some $x_j\in E_j$, $j=1, 2, \cdots, t-1$. For each $a=(a(\lambda(t)), a(\lambda(t)+1), \cdots, a(n))\in B_{qt}$, there is $x_a\in F_{qt}$ such that $\pi_{J_t}(x_a)=a$. We define a mapping $\Psi: B_{qt}\to F_{qt}^\natural$ as

$$\Psi(a)=\psi(x_a)\qquad\text{for each } a\in B_{qt}.$$

To show that

$$|\Psi^{-1}\Psi(a)|\leqq\omega\cdot|\alpha+1|\qquad\text{for each } a\in B_{qt},$$

let a, $b\in B_{qt}$ with $\Psi(a)=\Psi(b)$. Let $\Psi(a)=(\sigma_1, \sigma_2, \cdots, \sigma_q)$ and $\Psi(b)=(\tau_1, \tau_2, \cdots, \tau_q)$, where for each $k=1, 2, \cdots, q$,

$$\sigma_k=y_k+s_kx_a+s_k^1x_1+s_k^2x_2+\cdots+s_k^{t-1}x_{t-1},$$

$$\tau_k=y_k'+v_kx_b+v_k^1z_1+v_k^2z_2+\cdots+v_k^{t-1}z_{t-1},$$

for some y_k, $y_k'\in Y_\alpha$, x_j, $z_j\in E_j$ and s_k, v_k, s_k^j, $v_k^j\in\boldsymbol{Q}$. It follows from the inductive assumption (*) and Claim 1 that there is $k\leqq q$ such that $v_k\neq0$. Since x_j, $z_j\in E_j$ for each $j=1, 2, \cdots, t-1$, it follows that

$$\pi_{J_t}(\sigma_k)=(\pi_{J_t}(y_k)+s_k a+\sum_{j=1}^{t-1} s_k^j(r_j(\lambda(t)), r_j(\lambda(t)+1), \cdots, r_j(n)), \quad \text{and}$$

$$\pi_{J_t}(\tau_k)=(\pi_{J_t}(y_k')+v_k b+\sum_{j=1}^{t-1} v_k^j(r_j(\lambda(t)), r_j(\lambda(t)+1), \cdots, r_j(n)).$$

Thus we have the following:

$$b=\frac{1}{v_k}\Big(\pi_{J_t}(y_k)-\pi_{J_t}(y_k')+s_k a+\sum_{j=1}^{t-1}(s_k^j-v_k^j)(r_j(\lambda(t)), r_j(\lambda(t)+1), \cdots, r_j(n))\Big).$$

Therefore, it follows that

(14) $$|\Psi^{-1}\Psi(a)|\leqq\omega\cdot|a+1| \qquad \text{for each } a\in B_{qt}.$$

On the other hand, it follows from (13) that

$$F_{qt}^{\natural}\subset\bigcup\{H\cap T_{nq}: H\in\mathcal{H}^{\natural}\}$$
$$=\bigcup\{H\cap S_i: H\in\mathcal{H}^{\natural} \text{ and } i\in\omega\}.$$

Thus it follows that $|F_{qt}^{\natural}|\leqq\omega\cdot|\alpha+1|$ by Lemma 2.6 and (12). Hence we have $|B_{qt}|\leqq\omega\cdot|\alpha+1|$.

It follows from Claim 3 that there is

$$r_t\in\prod_{i=\lambda(t)}^{n} I_{(t,i)}-\bigcup\{B_{qt}: q>m\}.$$

It is clear that the condition (*) is satisfied for $r_1, r_2, \cdots, r_t$. Therefore we complete the inductive procedure for finding $r_1, r_2, \cdots, r_m$.

It follows from Lemma 2.3 that there is $x=(x_1, x_2, \cdots, x_m)\in K_\alpha$ such that $\pi_{J_t}(x_j)=r_j$ for each $j=1, 2, \cdots, m$. Let

$$G_\alpha=Y_\alpha+\boldsymbol{Q}x_1+\boldsymbol{Q}x_2+\cdots+\boldsymbol{Q}x_m.$$

Then G_α is a subgroup of $\boldsymbol{R}^n$ with $|G_\alpha|\leqq\omega\cdot|\alpha+1|$. Since $x_j\in E_j$ for each j, it follows that $(G_\alpha)^q\cap T_{nq}=\phi$. Furthermore it is clear that $(G_\alpha)^m\cap K_\alpha\ni x$. The subgroups Y_α and G_α of $\boldsymbol{R}^n$ satisfy the conditions (5)-(9).

Step 6. Finally we put

$$G=\bigcup\{G_\alpha: \alpha<\mathfrak{c}\}.$$

Then G is a subgroup of $\boldsymbol{R}^n$ and it follows from (7) that $G^q\cap T_{nq}=\phi$ for all $q>m$. Hence $\dim G^q\leqq d$ by (4) in Lemma 2.5. It follows from (6) and Lemma 2.2 that $\dim G^m\geqq d$. Hence $\dim G^q=d$ for all $q\geqq m$ and hence $\dim G^\omega=d$ by [1, Lemma 4] or [10].

It follows from (5) and Lemma 2.4 that $\dim G\geqq n-1$. On the other hand, since G has no non-empty open set of $\boldsymbol{R}^n$, $\dim G\leqq n-1$. Hence the theorem is proved.

Remark. If we do not require that the group G is a subgroup of $\boldsymbol{R}^{n+1}$, then there is another way for finding a separable metrizable topological group G satisfying $\dim G=\dim G^{\omega}=n$. Indeed, let X be a separable metrizable space satisfying $\dim X=\dim X^{\omega}=n$ ([1]). Let $F_a(X)$ be the free topological group of X equipped with the Graev's metric topology (see [6] or [2]). It follows from [2] and [3] that $F_a(X)$ is a separable metric topological group satisfying

(a) $$\dim F_a(X)=\sup\{\dim X^m : m\in\omega\}=n .$$

But we can not apply this argument to find a separable metric topological group G_{nd} satisfying $\dim G_{nd}=n$ and $\dim (G_{nd})^{\omega}=d$ for $n<d$. Because, it follows from the condition (a) that $\dim F_a(X_{nd})=\dim (F_a(X_{nd}))^{\omega}=d$ for a space X_{nd} with $\dim X_{nd}=n$ and $\dim (X_{nd})^{\omega}=d$. Thus the topological group G_{nd} described in Theorem 2.1 is the first example of an n-dimensional separable metrizable topological group satisfying $\dim (G_{nd})^{\omega}=d$ even if one does not require the additional condition 'to be a subgroup of $\boldsymbol{R}^{n+1}$'.

3. A separable metrizable, precompact group

We shall turn to the precompact topological groups. A topological group is said to be *precompact* if it is isomorphic to a subgroup of a compact group. Shakhmatov has proved in [12] that for each positive integer n there is an n-dimensional separable metrizable, precompact topological group G satisfying $\dim G^{\omega}=n$ by use of an argument of free topological groups. W shall improve the Shakhmatov's theorem.

A continuous mapping f of a space X onto a space Y is said to be a *local homeomorphism* if each point x of X has an open neighborhood U such that $f(U)$ is an open set of Y and the restriction of f to U is a homeomorphism of U onto $f(U)$. Let $T^n=S^1\times S^1\times \cdots \times S^1$ be the n-dimensional torus. Let $h:\boldsymbol{R}\to S^1$ be a mapping defined by $h(t)=(\sin 2\pi t, \cos 2\pi t)$ for $t\in\boldsymbol{R}$, and $h^n=h\times h\times \cdots \times h:\boldsymbol{R}^n\to T^n$ be the product mapping. It is clear that h^n is a local homeomorphism. Notice that for a subgroup G of $\boldsymbol{R}^n$ the restriction of h^n to G need not be a local homeomorphism. However, we have the following simple lemma.

Lemma 3.1. *Let G be a subgroup of $\boldsymbol{R}^n$ containing $\boldsymbol{Z}^n$. Then the restriction of h^n to G is a local homeomorphism.*

Theorem 3.2. *For each pair n, d of positive integers with $n\leqq d$, there is a separable metrizable, precompact topological group H_{nd} such that $\dim H_{nd}=n$ and $\dim (H_{nd})^{\omega}=d$.*

Proof. Let G_{nd} be a subgroup of $\boldsymbol{R}^{n+1}$ constructed in Theorem 2.1. We put $H_{nd}=h^{n+1}(G_{nd})$. Since h^{n+1} is a homeomorphism, H_{nd} is a subgroup of T^{n+1}. Hence H_{nd} is precompact. If follows from the condition (9) in the proof of Theorem 2.1 that G_{nd} contains $\boldsymbol{Z}^{n+1}$. By Lemma 3.1, it follows that the restriction of h^{n+1} to G_{nd} is a local homeomorphism. Hence $\operatorname{locdim} G_{nd}=\operatorname{locdim} H_{nd}$, where $\operatorname{locdim} X$ is a *local dimension* of a space X introduced by Dowker [4]. Since G_{nd} and H_{nd} are metrizable, their covering dimension and local dimension coincide. Thus $\dim H_{nd}=\dim G_{nd}=n$. Similarly we can show that $\dim (H_{nd})^m=\dim (G_{nd})^m$ for every $m\in\omega$. Thus it follows from [10] that

$$\dim (H_{nd})^\omega=\max\{\dim (H_{nd})^m : m\in\omega\}=\max\{\dim (G_{nd})^m : m\in\omega\}$$
$$=\dim (G_{nd})^\omega=d .$$

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