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ON THE ISOVARIANCY OBSTRUCTION OF SOME INVOLUTIONS ON HOMOTOPY SPHERES

Dedicated to Professor Seiya Sasao on his sixtieth birthday

By

YASUHIKO KITADA

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Abstract. The present work treats certain orientation reversing involutions on $(4k+1)$-dimensional homotopy spheres with $2k$-dimenstional fixed point sets. The author calculates the explicit obstruction for these involutions to be isovariantly homotopy equivalent to the linear involution on the standard sphere.

1. Introduction and statement of results

The purpose of this paper is to calculate the isovariancy obstruction for maps between certain homotopy spheres with smooth involutions. Throughout this paper n is a positive even integer. By a Kervaire manifold, we mean a compact framed $(2n+2)$-manifold with non-zero Arf-Kervaire invariant. A $(2n+1)$-dimensional homotopy sphere that bounds a Kervaire $(2n+2)$-manifold (with boundary) is called a Kervaire sphere, which is not diffeomorphic to the standard sphere S^{2n+1} unless $n+2$ is a power of two. In this research we shall treat the involutions described below. In [4] there is another description of our involutions using the section of Brieskorn varieties around the origin. We shall denote the standard inner product of vectors $x, y \in \boldsymbol{R}^{n+1}$ by $\langle x, y\rangle$. Let D^{n+1} be the closed unit disk centered at the origin of $\boldsymbol{R}^{n+1}$ and its boundary is the unit sphere S^n. Given a unit vector $x \in S^n$, we define a linear transformation $\theta_x \in O(n+1)$ by

$$\theta_x(y)=2\langle x, y\rangle x-y\,, \qquad y \in \boldsymbol{R}^{n+1}\,.$$

Consider the diffeomorphism ψ of $S^n \times S^n$ defined by

$$\psi(x, y)=(\theta_x\theta_y(x),\ \theta_x\theta_y(y))\,.$$

From the relation $\theta_{Px}=P\theta_x P^{-1}$ for $P \in O(n+1)$, we can verify that $\psi^q(x, y)=((\theta_x\theta_y)^q x,\ (\theta_x\theta_y)^q y)$ holds. Suppose that we are given a map φ defined on a

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subset of C of a space B into another space A, then we shall denote by $A\cup_{\varphi}B$ the space obtained by attaching B to A via the map φ. If the map φ is an inclusion map of C into A, we shall simply write $A\cup_{C}B$ instead of $A\cup_{\varphi}B$. Consider the involution on $S^n\times S^n$ which maps the point (x, y) to $(x, -y)$. Since the map ϕ is equivariant with respect to this involution, we may introduce an involution on

$$\Sigma_q^{2n+1}=S^n\times D^{n+1}\cup_{\phi^q}D^{n+1}\times S^n,$$

given by $(x, y)\mapsto(x, -y)$, where (x, y) is an element of $S^n\times D^{n+1}$ or $D^{n+1}\times S^n$. It is clear that the involution on Σ_0^{2n+1} is equivariantly diffeomorphic to the linear involution on the standard sphere S^{2n+1}. Whereas the underlying space Σ_1^{2n+1} is a Kervaire sphere when we forget the involution. Among the involutions for all q, it is known that there are at most two equivariantly different diffeomorphism classes represented by Σ_0^{2n+1} and Σ_1^{2n+1} ([4]). That is, Σ_q^{2n+1} for $q\equiv 0$, 3 mod 4 (resp. $q\equiv 1$, 2 mod 4) is equivariantly diffeomorphic to Σ_0^{2n+1} (resp. Σ_1^{2n+1}). Let us recall that an equivariant map is transverse linearly isovariant (t-isovariant for short) if the map keeps the isotropy group at each point and in addition the map can be regarded as a linear bundle map around the fixed point sets ([3]). It is known that when $n+2$ is not a power of 2, that Σ_0 and Σ_1 are not t-isovariantly homotopy equivalent ([4]). In this paper, we shall calculate the explicit obstruction for the existence of a t-isovariant map of degree one from Σ_1 to Σ_0 by constructing a t-isovariant map up to a $2n$-skeleton of Σ_1 under a certain equivariant cell complex structure. About the final obstruction for extending this t-isovariant map, we have the following theorem.

Theorem. *Under a certain construction of a t-isovariant map from* Σ_1 *to* Σ_0 *in lower dimensions, the final obstruction to building a degree one t-isovariant map is given by the Whitehead product* $[\iota_{n+1}, \iota_{n+1}]\in\pi_{2n+1}(S^{n+1})$.

The vanishing of the Whitehead product obstruction is a sufficient condition for the existence of a t-isovariant map. Moreover, as we shall see later, we construct such a map starting from a homotopy equivalence around the fixed point set. So if the final obstruction vanishes, we have a t-isovariant homotopy equivalence. In particular when $n=2$ or 6, S^{n+1} is an H-space and $[\iota_{n+1}, \iota_{n+1}]$ vanishes. Thus we have the following corollary.

Corollary. *If* $n=2$ *or* $n=6$, Σ_q^{2n+1} *for all* q *are mutually t-isovariantly homotopy equivalent.*

2. Preliminary construction

Let χ be an equivariant diffeomorphism of $S^n \times S^n$ with respect to the involution $(x, y) \mapsto (x, -y)$ and we assume that χ induces the identity map on homology groups. We consider two homotopy spheres with involutions:

$$\Sigma_0 = S^n \times D^{n+1} \cup_{id} D^{n+1} \times S^n$$

$$\Sigma_\chi = S^n \times D^{n+1} \cup_\chi D^{n+1} \times S^n .$$

Given a map $\rho : S^n \to SO(n+1)$, we define a diffeomorphism b_ρ of $S^n \times D^{n+1}$ by $b_\rho(x, y) = (x, \rho(x)y)$.

Lemma 1. *If there exists a t-isovariant homotopy equivalence* $\Sigma_\chi \to \Sigma_0$, *then there exists a t-isovariant homotopy equivalence* $f : \Sigma_\chi \to \Sigma_0$ *such that* $f | S^n \times D^{n+1}$ *coincides with* $b_\rho : S^n \times D^{n+1} \to S^n \times D^{n+1} \subset \Sigma_0$ *for some* $\rho : S^n \to SO(n+1)$ *and* $f(D^{n+1} \times S^n) \subset D^{n+1} \times S^n$.

Proof. Since the t-isovariant homotopy equivalence induces a homotopy equivalence at the fixed point set, the map has degree one up to sign. If the degree at the fixed point set is minus one, we replace the t-isovariant map by its composition with an equivariant orientation preserving diffeomorphism of Σ_0 which reverses the orientations of the fixed point set and its normal bundle. Then we have a t-isovariant homotopy equivalence which induces a map of degree one at the fixed point sphere. Since the degree one map of a sphere is homotopic to the identity, we may deform this map by homotopy to obtain a t-isovariant map which is the "identity" at the fixed point sphere $S^n \times \{0\}$. The t-isovariant map is a linear bundle map around the fixed point set and we may write $f | S^n \times D^{n+1} = b_\rho$ for some $\rho : S^n \to SO(n+1)$. Since the map f is isovariant, $f(D^{n+1} \times S^n)$ does not touch $S^n \times \{0\}$ and we may deform $f | D^{n+1} \times S^n$ rel. boundary to a map into $D^{n+1} \times S^n \subset \Sigma_0$ since $D^{n+1} \times S^n$ is an equivariant deformation retract of $\Sigma_0 - S^n \times \{0\}$. □

We shall try to construct a t-isovariant map $f : \Sigma_\chi \to \Sigma_0$ starting from a map b_ρ for some $\rho : S^n \to SO(n+1)$. Its restriction $\phi_\rho = b_\rho | S^n \times S^n$ to the boundary induces a diffeomorphism $\bar{\phi}_\rho$ of the quotient space $S^n \times P^n$, where P^n is the n-dimensional real projective space.

Lemma 2. *The map* b_ρ *extends to a t-isovariant map* $f : \Sigma_\chi \to \Sigma_0$ *if and only if* $p_2 \bar{\phi}_\rho \bar{\chi}$ *is homotopic to* p_2. *Here* $p_2 : S^n \times P^n \to P^n$ *is the second projection and* $\bar{\chi}$ *is the diffeomorphism of* $S^n \times P^n$ *induced by* χ.

Proof. The map f can be isovariant if and only if $f(D^{n+1} \times S^n)$ does not meet the fixed point set $S^n\times\{0\}\subset\sum_0$. This is possible if and only if the map $\bar\phi_\rho: S^n\times P^n\to S^n\times P^n$ can be extended to a map

$$D^{n+1}\times P^n \longrightarrow (\textstyle\sum_0 - S^n\times\{0\})/involution \simeq D^{n+1}\times P^n \simeq P^n .$$

Since χ induces the identity map on homology, we may assume that $\chi(*, y)=(*, y)$ where $*$ is the base point of S^n. Since we may assume that $\rho(*)=1\in SO(n+1)$, $\phi_\rho\chi$ is the identity map of $\{*\}\times S^n$. On the quotient space, this shows that $p_2\bar\phi_\rho\bar\chi\,|\,\{*\}\times P^n$ is the second projection, and the extended map $D^{n+1}\times P^n\to P^n$ gives the homotopy between $p_2\bar\phi_\rho\bar\chi$ and p_2. □

In what follows the Hopf-construction for a map $g: X\times Y\to W$ will be denoted by

$$\Gamma(g): X*Y \longrightarrow SW ,$$

where $X*Y$ is the reduced join of X and Y and SW is the reduced suspension of W. Points of $X*Y$ and SW are written as $[x, t, y]$ and $[t, w]$ ($t\in[0, 1]$) respectively. The homotopy class of $\Gamma(g)$ is denoted by $[\Gamma(g)]$. The following lemma will be used in the proof of Lemma 4. Its proof follows directly from the definition of the sum of homotopy classes and will be omitted.

Lemma 3. *Let f, g be maps $S^m=D^m_+\cup_{S^{m-1}} D^m_-\to S^q$ where the base point of S^m is on the equator S^{m-1}. Suppose that $f\,|\,D^m_+$ coincides with $g\,|\,D^m_-$ under the canonical identification of D^m_+ with D^m_-. Then the sum $[f]+[g]$ of the homotopy classes $[f]$ and $[g]$ can be represented by a map h defined by $h\,|\,D^m_-=f\,|\,D^m_-$ and $h\,|\,D^m_+=g\,|\,D^m_+$.*

Lemma 4. *Let $J: \pi_n(O(n+1))\to\pi_{2n+1}(S^{n+1})$ be the J-homomorphism. Then we have*

$$[\Gamma(p_2\phi_\rho\chi)]=[\Gamma(p_2\chi)]+J(\rho)$$

in $\pi_{2n+1}(S^{n+1})$.

Proof. Using the hemisphere decomposition $S^n\times S^n=(D^n_+\cup_{S^{n-1}} D^n_-)\times S^n$, we have an identification

$$S^{2n+1}=D^{2n+1}_+\cup_{S^{2n}} D^{2n+1}_-=D^n_+*S^n\cup_{S^{n-1}*S^n} D^n_-*S^n .$$

We may assume that $\rho(D^n_-)=1\in O(n+1)$ and $p_2\chi(D^n_+, y)=y$ for all $y\in S^n$. Then $J(\rho)$ maps $[x, t, y]\in D^n_-*S^n$ to $[t, y]\in SS^n=S^{n+1}$ and $p_2\chi$ maps $[x, t, y]\in D^n_+*S^n$ to $[t, y]\in S^{n+1}$. From the definition of Hopf-construction, $\Gamma(p_2\phi_\rho\chi)$ is represented by a map which coincides with $J(\rho)$ on D^{2n+1}_+ and with $\Gamma(p_2\chi)$ on D^{2n+1}_-. In view of Lemma 3, this shows our assertion. □

Proposition 5. *$\sum^{2n+1}_0$ and $\sum^{2n+1}_1$ are t-isovariantly homotopy equivalent if* *and only if the map*

$$h: S^n \times P^n \longrightarrow P^n, \qquad h(x, [y])=[(\theta_x \theta_y)^2 y]$$

is homotopic to the second projection $p_2: S^n \times P^n \to P^n$.

Proof. Suppose that there exists a t-isovariant homotopy equivalence f for $\chi=\psi$

$$f: \Sigma_\chi=\Sigma_1=S^n \times D^{n+1} \cup_\psi D^{n+1} \times S^n \longrightarrow \Sigma_0=S^n \times D^{n+1} \cup_{id} D^{n+1} \times S^n$$

where $f | S^n \times D^{n+1}=b_\rho$ for some $\rho \in \pi_n(O(n+1))$ and f maps $D^{n+1} \times S^n$ into $D^{n+1} \times S^n$. When we forget the involution, the map $p_2 \phi_\rho \psi: S^n \times S^n \to S^n$ has an extension $D^{n+1} \times S^n \to S^n$. Therefore its Hopf-construction $\Gamma(p_2 \phi_\rho \psi)$ must be null homotopic. By Lemma 4, we must have $[\Gamma(p_2 \psi)]+[J(\rho)]=0$ in $\pi_{2n+1}(S^{n+1})$. It is well-known that $[\Gamma(p_2 \psi)]=[J(\theta)]=[\iota_{n+1}, \iota_{n+1}]$ is a nonzero element of order two unless $n=0$, 2 or 6 and it vanishes under suspension. The homotopy group $\pi_n(O(n+1))$ vanishes for $n=2$ or 6, and in other cases for even n, the group is isomorphic to $\boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2$ if $n \equiv 0 \bmod 8$ and to $\boldsymbol{Z}/2$ otherwise. When $n \neq 2$, 6, θ gives one generator, and when $n \equiv 0 \bmod 8$, there is another $\boldsymbol{Z}/2$ generator that survives to a nonzero stable class of $\pi_n(O)$. But since the stable J-map $\pi_n(O) \to \pi_n^S$ is injective when $n \equiv 0 \bmod 8$, we cannot take this element as ρ. Thus the only choice for ρ is θ. It is not hard to verify that $p_2 \phi_\theta \psi: S^n \times S^n \to S^n$ is given by $(x, y) \mapsto (\theta_x \theta_y)^2 y$. If we consider the map on the quotient space, we get the conclusion by Lemma 2. □

3. The last isovariancy obstruction

Now we have come to a quite concrete problem about the extensibility of the map

$$h: S^n \times P^n \longrightarrow P^n, \qquad h(x, [y])=[(\theta_x \theta_y)^2 y]$$

over $D^{n+1} \times P^n$. Our next step is to calculate this obstruction. To do this we shall introduce the following notations:

$$X=D^{n+1} \times P^n, \quad \tilde{X}=D^{n+1} \times S^n, \quad X^\perp=\{(x, [y]) \in X \mid \langle x, y \rangle=0\},$$

$$\tilde{X}_+=\{(x, y) \in \tilde{X} \mid \langle x, y \rangle \geqq 0\}, \qquad \tilde{X}_-=\{(x, y) \in \tilde{X} \mid \langle x, y \rangle \leqq 0\},$$

$$Y=S^n \times P^n, \qquad \tilde{Y}=S^n \times S^n,$$

$$\tilde{Y}_+=\{(x, y) \in \tilde{Y} \mid \langle x, y \rangle \geqq 0\}, \qquad \tilde{Y}_-=\{(x, y) \in \tilde{Y} \mid \langle x, y \rangle \leqq 0\},$$

$$Y^\perp=Y \cap X^\perp, \qquad \tilde{Y}^\perp=\tilde{Y}_+ \cap \tilde{Y}_-.$$

The next proposition says that the map h is 'almost' extensible and the suspension of the extension obstruction is given by a Whitehead product.

Proposition 6. *The map $h: Y \to P^n$ has a t-isovariant extension to a 2n-skeleton of X and the last obstruction ζ that lies in $\pi_{2n}(P^n)$ may be regarded as an element of $\pi_{2n}(S^n)$. We can find an extension of h in such a way that the suspension of the last obstruction ζ is equal to $[\iota_{n+1}, \iota_{n+1}]$.*

Proof. On $Y^{\perp}$ we have $h(x, [y])=[y]$. Hence we have an extension h' over $X^{\perp}$ by defining $h'(x, [y])=[y]$. The given involution exchanges $\tilde{Y}_+$ with $\tilde{Y}_-$ on $\tilde{Y}$ and $\tilde{X}_+$ with $\tilde{X}_-$ on $\tilde{X}$ that are separated by $\tilde{X}^{\perp}$. The map h' maps the point $(x, y) \in \partial \tilde{X}_+ = \tilde{Y}_+ \cup_{\tilde{Y}^{\perp}} \tilde{X}^{\perp}$ to $[(\theta_x \theta_y)^2 y]$ if $(x, y) \in \tilde{Y}_+$ and to $[y]$ if $(x, y) \in \tilde{X}^{\perp}$. When x and y are linearly independent vectors, we may write

$$y=(\cos u)x+(\sin u)e$$

using the unit vector e which is normal to x and is contained in the plane spanned by x and y. Then we can write

$$(\theta_x \theta_y)^2 y = \cos(-3u)x + \sin(-3u)e\,.$$

This formula facilitates our understanding of the map $(\theta_x \theta_y)^2$. If we could extend the map h' to the interior of $\tilde{X}_+$, then the map h has an extension to the whole X. We should remark that this extension problem is equivalent to the following extension problem without actions: Is the map $h_1: S^n \times S^n \to S^n$ defined by

$$h_1(x, y)=\begin{cases} (\theta_x \theta_y)^2 y & (\text{if } \langle x, y\rangle \geqq 0) \\ y & (\text{if } \langle x, y\rangle \leqq 0) \end{cases}$$

extensible to whole $D^{n+1} \times S^n$?

If $\langle x, y\rangle \leqq 1/2$, then $\langle x, h_1(y)\rangle \leqq 1/2$ holds. Therefore h_1 is homotopic to h_2 defined by

$$h_2(x, y)=\begin{cases} (\theta_x \theta_y)^2 y & (\text{if } \langle x, y\rangle \geqq 1/2) \\ -x & (\text{if } \langle x, y\rangle \leqq 1/2)\,. \end{cases}$$

since the set $\{(x, y) \mid \langle x, y\rangle \leqq 1/2\}$ has $\{(x, -x) \mid x \in S^n\}$ as a deformation retract. The well-definedness of h_2 is clear from the definition of θ or from the explicit formula for $(\theta_x \theta_y)^2 y$ mentioned above. Using the notation $y = \cos u\, x + \sin u\, e$, we find that the homotopy

$$H(x, y, t)=\cos(2t-3)u x+\sin(2t-3)u e, \qquad (0 \leqq t \leqq 1)$$

deforms h_2 to h_3 which is defined by $h_3(x, y) = -\theta_x y$. Since $h_3 | S^n \times \{pt\}$ is null homotopic, we take one of its null homotopies as an extension over $D^{n+1} \times \{pt\}$. Thus we have obtained an extension over a $2n$-skeleton of $D^{n+1} \times S^n$.

This defines the last obstruction $\zeta \in \pi_{2n}(S^n)$. If we take another choice of null homotopies as an extension over $D^{n+1}\times\{pt\}$ by an element of $\pi_{n+1}(S^n)\cong \boldsymbol{Z}/2$ generated by a stable class η, the obstruction ζ is changed by $[\eta, \iota_n]$ from the definition of Whitehead products. From the EHP exact sequence

$$\pi_{2n+2}(S^{2n+1}) \xrightarrow{P} \pi_{2n}(S^{2n}) \xrightarrow{E} \pi_{2n+1}(S^{n+1}) \xrightarrow{H} \pi_{2n+1}(S^{2n+1})\cong \boldsymbol{Z},$$

we find that the obstruction of extending h_1 over $D^{n+1}\times S^n$ is equal to the suspension $E\zeta\in\pi_{2n+1}(S^{n+1})$ and this is equal to the final obstruction of extending h_3, which is equal to $J(-\theta)=[\iota_{n+1}, \iota_{n+1}]$ by the property of the J-homomorphism. □

Proof of Theorem. The obstruction $[\iota_{n+1}, \iota_{n+1}]$ in Proposition 5 was calculated fixing an extension on $X^{\perp}$. This obstruction might vanish if other extensions were to be chosen on $X^{\perp}$. However, if $n+2$ is not a power of two, this obstruction cannot vanish from the result of [5], Corollary 5. This completes the proof when $n+2$ is not a power of two. Even when $n+2$ is a power of two, the same argument applies, but in this case, if the Whitehead product vanishes, the obstruction is zero and we can construct a t-isovariant homotopy equivalence from Σ_1 to Σ_0. □

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Department of Applied Mathematics
Faculty of Engineering
Yokohama National University
Tokiwadai, Hodogaya-ku
Yokohama, 240 Japan