Yokohama Mathematical
Journal Vol. 42, 1994

RANDOM WALKS WITH STOCHASTICALLY BOUNDED INCREMENTS: RENEWAL THEORY VIA FOURIER ANALYSIS

By

Gerold Alsmeyer

(Received March 23, 1993)

Summary. Random walks $S_N=(S_n)_{n\geq 0}$ with stochastically bounded increments $X_0, X_1, \cdots$ have been introduced in [2], [3] as natural generalizations of those with i.i.d. increments. In this article we present Blackwell-type renewal theorems proved by means of Fourier analysis. In the special case of independent $X_0, X_1, \cdots$ these results lead to generalizations of earlier ones in the literature, notably in [3] where proofs were based on coupling technique which is a purely probabilistic device. As a further application we prove Blackwell's renewal theorem for certain random walks with stationary 1-dependent increments that appear in Markov renewal theory as subsequences of Markov random walks.

1. Introduction

Random walks with stochastically lower and/or upper bounded increments, see Definition 1.1 below, are a natural generalization of those with i.i.d. increments and have been introduced in [2], [3]. Certain drift bounds describing the mean growth of these random walks over finite remote time intervals as well as related characterization results are given in [2], whereas [3] is devoted to the proof of Blackwell-type renewal theorems under appropriate additional assumptions. Of principal importance there is the use of the coupling method, a probabilistic device which has regained great importance since the seventies. In this article we will derive Blackwell-type renewal theorems via the more classical approach based upon Fourier analysis.

We keep the basic notation of [2] and [3] which is briefly summarized below. Let $X_N=(X_n)_{n\geq 0}$ be a sequence of real-valued, integrable random variables on a probability space $(\Omega, \mathfrak{F}, P)$ with associated random walk S_N, defined through $S_n=X_0+\cdots+X_n$ for all $n\in N$. Let $\mathfrak{F}_N$ be an arbitrary filtration to

1991 Mathematics Subject Classification: Primary 60K05; Secondary 60G40, 60G42, 60G50, 60J15.

Key words and phrases: Random walk, stochastic boundedness, stochastic stability, maximal minorant, minimal majorant, Blackwell's renewal theorem, sequences of type AC and CC, Markov renewal theory, Fourier analysis.

which X_N is adapted and $\mathcal{G}_N$ the canonical filtration of X_N, i.e. $\mathcal{G}_n=\sigma(X_0, \cdots, X_n)$. For each measure F, we use the same letter for its "distribution function" and thus write $F(t)$ for $F((-\infty, t])$. If F is a probability measure, let $\bar{F}(t)=1-F(t)$ and $\mu(F)$ its mean value provided it exists. For n, $k\in\boldsymbol{N}$ and $0\leqq j\leqq n$, we further define

$$S_{n,k}=S_{n+k}-S_n, \qquad m_{n+1}=E(X_{n+1}\mid\mathcal{F}_n),$$

$$L_0=X_0, \qquad L_{n+1}=m_1+\cdots+m_{n+1}, \qquad L_{n,k}=L_{n+k}-L_n,$$

$$L_{n,k}^j=E(S_{n,k}\mid\mathcal{F}_j)=E(L_{n,k}\mid\mathcal{F}_j),$$

$$Q_n(dx)=P(S_n\in dx), \qquad Q_{n,k}(\omega, dx)=P(S_{n,k}\in dx\mid\mathcal{F}_n)(\omega),$$

where $Q_{n,k}$ is chosen to be a regular conditional distribution. Finally, $\mathcal{B}$ always denotes the Borel-σ-field over $\boldsymbol{R}$ and $\|\cdot\|_\infty$ the supremum norm on the vector space $L_\infty(\Omega, \mathcal{F}, P)$.

Definition 1.1. A sequence X_N, adapted to a filtration $\mathcal{F}_n$, is called
—*stochastically bounded* (*s.b.*) *w.r.t.* $\mathcal{F}_N$, if for distributions F, G with finite means

(A.1) $\qquad G(t)\leqq Q_{n,1}(\cdot, t)\leqq F(t) \qquad$ a.s. for all $t\in\boldsymbol{R}$ and $n\in\boldsymbol{N}$.

—*stochastically stable w.r.t.* $\mathcal{F}_N$, if it is s.b. w.r.t. $\mathcal{F}_N$ and if additionally

(A.2) $$\lim_{k\to\infty}\sup_{n\geq 0}\|k^{-1}L_{n,k}^n-\theta\|_\infty=0$$

holds for some $\theta\in\boldsymbol{R}$ which is then called the *mean* of X_N.
—*ultimately stochastically bounded w.r.t.* $\mathcal{F}_N$, if $X_{\tau+N}$ is s.b. w.r.t. $\mathcal{F}_{\tau+N}$ for some $\mathcal{F}_N$-time τ, such that $E\tau<\infty$ and $E|S_\tau|<\infty$. τ is then called an *entrance time* of X_N.
—*ultimately stochastically stable w.r.t.* $\mathcal{F}_N$, if it is ultimately s.b. with a sequence τ_N of entrance times such that

(A.3) $$\lim_{j\to\infty}\limsup_{k\to\infty}\sup_{n\geq 0}\|k^{-1}L_{\tau_j+n,k}^{\tau_j+n}-\theta\|_\infty=0$$

for some $\theta\in\boldsymbol{R}$ which again is called the *mean* of X_N.

The distributions F and G in (A.1) are called a *minorant* and a *majorant* of X_N and of $Q_{N,1}$, resp.

Note that the definition of an entrance time τ differs in [2] and [3]. We have chosen the more restrictive one of [3] because its additional requirements $E\tau<\infty$ and $E|S_\tau|<\infty$ are indispensable for renewal theory. We denote by $\mathcal{E}(X_N, \mathcal{F}_N)$ the class of entrance times of X_N w.r.t. $\mathcal{F}_N$, and we simply write $\mathcal{E}$ where this is not ambiguous.

It is not difficult to see that stochastic boundedness w.r.t. $\mathcal{F}_N$ is slightly stronger than uniform integrability of

$$\{P(|X_{n+1}|\in\cdot\,|\,\mathcal{F}_n)(\omega)\,;\ n\geqq 1,\ \omega\in\Omega-N\}$$

for some P-null set N and slightly weaker than uniform $L_{1+\delta}$-boundedness of this family for some $\delta>0$, i.e.

$$\sup_{n\geq 0}\|E(|X_{n+1}|^{1+\delta}|\mathcal{F}_n)\|_\infty<\infty\,.$$

It holds particularly true if, for each $n\geqq 0$, the conditional distribution of X_{n+1} given $S_0, \cdots, S_n$ is chosen from a finite set, a situation which occurs, for instance, in certain stochastic control problems (e.g. treatment allocation). We refer the reader to an article by Lalley and Lorden [10] for a typical application of this type. Further examples of random walks with s.b. increments may e.g. be found in [1], Section 4, and further in Section 4 of this paper. While stochastic boundedness guarantees a uniform tail decrease of the conditional increment distributions, it does not make at all for a uniform mean growth of the associated random walk over finite remote time intervals, formally measured through $k^{-1}L^n_{n,k}$ for large n and $k\to\infty$. Blackwell's renewal theorem, however, just demands for such a uniform growth behavior, and it is for that reason that condition (A.2) (stochastic stability) is introduced. It already occurs in earlier works by Smith [13], Williamson [14] and Maejima [11].

The "optimal" choices for F and G in (A.1)−(A.3) are obviously

$$F(t)\doteq\sup_{n\geq 0}\|Q_{n,1}(\cdot,\,t)\|_\infty \quad\text{and}\quad G(t)\doteq 1-\sup_{n\geq 0}\|\bar{Q}_{n,1}(\cdot,\,t)\|_\infty \tag{1.1}$$

and called *maximal minorant* and *minimal majorant*, resp., of X_N and of $Q_{N,1}$ (w.r.t. $\mathcal{F}_N$, if this is to be emphasized). [$f(t)\doteq g(t)$ means that $f(t)$ is the right continuous modification of $g(t)$]. For an arbitrary entrance time τ let F^τ_k and G^τ_k be the maximal minorant and the minimal majorant of $Q_{\tau+N,k}$ w.r.t. $\mathcal{F}_{\tau+N}$, respectively. With the help of these distributions, we can define

$$\vartheta_*(S_{\tau+N},\,\mathcal{F}_{\tau+N})=\sup_{k\geq 1}k^{-1}\mu(F^\tau_k) \quad\text{and}\quad \vartheta^*(S_{\tau+N},\,\mathcal{F}_{\tau+N})=\inf_{k\geq 1}k^{-1}\mu(G^\tau_k)$$

which are called the *lower* and *upper asymptotic drift* of $S_{\tau+N}$ w.r.t. $\mathcal{F}_{\tau+N}$. It is shown in [2], see Lemma 4.1 there, that supremum and infimum in (1.2) yield as limits, i.e.

$$\vartheta_*(S_{\tau+N},\,\mathcal{F}_{\tau+N})=\lim_{k\to\infty}k^{-1}\mu(F^\tau_k) \quad\text{and}\quad \vartheta^*(S_{\tau+N},\,\mathcal{F}_{\tau+N})=\lim_{k\to\infty}k^{-1}\mu(G^\tau_k) \tag{1.2}$$

Let us write ϑ_*, ϑ^* for $\vartheta_*(S_N,\,\mathcal{F}_N)$, $\vartheta^*(S_N,\,\mathcal{F}_N)$. Next, for ultimately s.b. X_N,

$$\begin{aligned}\eta_*&=\eta_*(S_N,\,\mathcal{F}_N)=\sup_{\tau\in\mathcal{E}}\vartheta_*(S_{\tau+N},\,\mathcal{F}_{\tau+N}),\\ \eta^*&=\eta^*(S_N,\,\mathcal{F}_N)=\inf_{\tau\in\mathcal{E}}\vartheta_*(S_{\tau+N},\,\mathcal{F}_{\tau+N}),\end{aligned} \tag{1.3}$$

are, resp., the *maximal lower* and the *minimal upper asymptotic drift* of S_N (w.r.t. $\mathcal{F}_N$). Evidently,

$$\vartheta_* \leqq \eta_* \leqq \eta^* \leqq \vartheta^*,$$

and they are all equal to some θ iff X_N is s.s. w.r.t. $\mathcal{F}_N$ with mean θ, see Theorem 5.1 in [2]. The latter means that a random walk with s.s. increments with positive mean θ has almost constant average drift θ over finite remote time sets $\{n, \cdots, n+k\}$ if k is large. Indeed, it also satisfies a uniform weak law of large numbers as Theorem 5.1 in [2] further states. We already mentioned earlier, that these facts give rise to the conjecture that such a random walk forms a natural candidate for satisfying Blackwell's renewal theorem.

For any given random walk S_N we denote by $U=\sum_{n\geq 0} P(S_n \in \cdot)$ its associated renewal measure which is locally finite (finite on bounded subsets of $\boldsymbol{R}$) whenever X_N is ultimately s.b. with $\eta_*>0$, see Lemma 6.4 in [3]. Finally, the span $d(S_N)$ of S_N is defined through

$$d(S_N)=\sup\{d>0 ; P(S_n \in d\boldsymbol{Z})=1 \text{ for all } n \geqq 0\}.$$

The paper is organized as follows. In Section 2 we state and prove a basic Blackwell-type renewal theorem for random walks with ultimately s.b. increments and positive drift. Its intrinsic assumption, a technical integrability condition on the Fourier transform of the appearing renewal measure, see (2.2), is further discussed in Section 3 leading to the definition of two suitable subclasses of random walks (in fact, their increments) for which it can be verified. The resulting renewal theorem (Theorem 3.2) is proved in Section 5. Section 4 contains a number of applications, notably to random walks with independent increments satisfying a local limit theorem and to Markov-modulated random walks which arise in Markov renewal theory.

2. The basic Blackwell-type renewal theorem

The Fourier-analytic nature of Theorem 2.1 below first requires further notation. For $n \geqq 0$, $k \geqq 1$, $t \in \boldsymbol{R}$ and $\omega \in \Omega$, let

$$\psi_n(t)=E(e^{itS_n}),$$

$$\phi_{n,k}(\omega, t)=E(e^{itS_{n,k}} | \mathcal{G}_n)(\omega)=\int_\Omega e^{itx} Q_{n,k}(\omega, dx),$$

$$\phi_{n+1}(t)=E(e^{itX_{n+1}})=E\phi_{n,1}(\cdot, t)$$

be the Fourier transforms (F.t.) of S_n, of $S_{n,k}$ given $\mathcal{G}_n$, and of X_{n+1}, resp. Note that $\phi_{n,k}(\omega, t)$ can be factorized as

$$\phi_{n,k}(\omega, t)=\tilde{\phi}_{n,k}(\boldsymbol{X}_n(\omega), t), \qquad \boldsymbol{X}_n=(X_0, \cdots, X_n).$$

The so-called discounted renewal measures $U(s, \cdot)$, $0<s<1$ associated with S_N

are defined through

$$U(s,\ \cdot)=\sum_{n\geq 0} s^n P(S_n\in\cdot)=\sum_{n\geq 0} s^n Q_n .$$

$U(s,\ \cdot)$ has finite total mass $1/(1-s)$ and F.t.

$$\hat{U}(s,\ t)=\sum_{n\geq 0} s^n \phi_n(t) .$$

Abel's theorem implies

$$\lim_{s\uparrow 1} \hat{U}(s,\ t)=\hat{U}(t)\overset{\text{def}}{=}\sum_{n\geq 0} \phi_n(t) \tag{2.1}$$

for all $t\in \boldsymbol{D}(\hat{U})\overset{\text{def}}{=}\{y\in \boldsymbol{R}:\ |\sum_{n\geq 0}\phi_n(y)|<\infty\}$. Since $\phi_n(0)=1$ for all $n\geqq 0$, we have $\boldsymbol{D}(\hat{U})\subset\boldsymbol{R}-\{0\}$. If S_N has span $d>0$ then all ϕ_n are $(2\pi/d)$-periodic so that even $\boldsymbol{D}(\hat{U})\subset\boldsymbol{R}-(2\pi/d)\boldsymbol{Z}$ holds.

With l_0 and l_d, $d>0$ denoting Lebesgue measure and d times counting measure on $d\boldsymbol{Z}$, resp., our basic renewal theorem now takes the following form.

Theorem 2.1. *Let S_N be a random walk with span d and ultimately s. b. increments $X_0, X_1, \cdots$ with $\eta_*>0$. Suppose that*

$$\int_{(-a,a)} t^2 \sum_{n\geq 0} |Re(\phi_n)(t)|\,dt<\infty \qquad \textit{for all} \quad 0<a<\frac{2\pi}{d}. \tag{2.2}$$

Then for all bounded intervals I

$$\eta^{*-1} l_d(I)\leqq \liminf_{t\to\infty} U(t+I)\leqq \limsup_{t\to\infty} U(t+I)\leqq \eta_*^{-1} l_d(I). \tag{2.3}$$

If X_N is even ultimately s. s. with positive mean θ, then

$$\lim_{t\to\infty} U(t+I)=\theta^{-1} l_d(I). \tag{2.4}$$

In all cases t runs through $d\boldsymbol{Z}$ only if $d>0$.

Remarks. (a) Condition (2.2) as it stands is obviously difficult to verify in applications and therefore further discussed in the following section. Note that the integrand in (2.2) always has a singularity at $t=0$, but in contrast to the i.i.d. case it may have further ones in $(-2\pi/d,\ 2\pi/d)$, necessarily of integrable order.

(b) It follows from Proposition 5.1 (a) in [3] that under the assumptions of Theorem 2.1 above

$$\sup_{t\in\boldsymbol{R}} U(t+B)<\infty \quad \text{and} \quad \lim_{t\to-\infty} U(t+B)=0 \tag{2.5}$$

for all bounded $B\in\mathcal{B}$. As a consequence it is enough to prove the assertions of Theorem 2.1 with U replaced by its symmetrization

$$V\overset{\text{def}}{=}U+U(-\cdot).$$

Proof of Theorem 2.1. We restrict ourselves to the nonarithmetic case ($d=0$) but note that for $d>0$ the following arguments apply after minor modifications due to the $(2\pi/d)$-periodicity of all involved F.t.

So let X_N be nonarithmetic and ultimately s.b. with $\eta_*>0$. By Proposition 5.1 in [3], for each $\mu\in(0, \eta_*)$ and $\nu\in(\eta^*, \infty)$ there is a nonarithmetic distribution F_0 such that for all bounded $B\in\mathcal{B}$

$$\nu^{-1}l_0(B)\leqq\liminf_{t\to\infty} F_0*U(t+B)\leqq\limsup_{t\to\infty} F_0*U(t+B)\leqq\mu^{-1}l_0(B). \tag{2.6}$$

As in [3], this is the key result which considerably simplifies the subsequent arguments because it allows us to prove the theorem's assertions by examining $U-F_0*U$ instead of U itself. As one can easily verify, (2.6) remains true when replacing F_0, U by their symmetrizations

$$F_0^s \overset{\text{def}}{=} F_0*F_0(-\cdot)$$

and V, resp. It is now enough to show for (2.3) that

$$W(t+\cdot)\overset{\text{def}}{=}V(t+\cdot)-F_0^s*V(t+\cdot)$$

tends vaguely to 0, i.e.

$$\lim_{t\to\infty}\int_{\mathbf{R}} h(x)W(t+dx)=0 \tag{2.7}$$

for all continuous functions $h:\mathbf{R}\to\mathbf{C}\in\mathcal{C}_0$, the vector space of continuous functions with compact support. To this end we let, for $0<s<1$ and $a\in\mathbf{R}$,

$$V(s, a, \cdot)=U(s, a+\cdot)+U(s, -a-\cdot) \quad\text{and}\quad W(s, \cdot)=V(s, a, \cdot)-F_0^s*V(s, a, \cdot)$$

be the discounted versions of $V(a+\cdot)$ and $W(a+\cdot)$, resp. Their F.t. are easily computed as, resp.,

$$\hat{V}(s, a, t)\overset{\text{def}}{=}2e^{-iat}Re(\hat{U}(s, t)) \quad\text{and}\quad \hat{W}(s, a, t)\overset{\text{def}}{=}2e^{-iat}(1-\hat{F}_0^s(t))Re(\hat{U}(s, t)),$$

where $\hat{F}_0^s$ denotes the F.t. of F_0^s. Note also that $W(s, a, \cdot)$ is a finite signed measure with total mass 0, and that (2.5) implies for the total variation $|W|$ of W

$$C_B\overset{\text{def}}{=}\sup_{a\in\mathbf{R}}|W|(a+B)<\infty \tag{2.8}$$

for all bounded $B\in\mathcal{B}$. Now let $\mathcal{C}_0^\infty$ be the space of all infinitely differentiable functions with compact support and $\mathcal{D}=\{f: \hat{f}\in\mathcal{C}_0^\infty\}$ which is a subspace of all infinitely differentiable and l_0-integrable functions. We first prove (2.7) for $h\in\mathcal{D}$ and under the assumption that F_0^s has finite second moment $\mu_2(F_0^s)$, in which case its (nonnegative) F.t. is twice continuously differentiable with Taylor expansion

$$\hat{F}_0^s(t)=1+r(t)t^2,$$

where $r(t)$ is continuous and equals $\mu_2(F_0^s)/2$ at 0. We will show later how to remove the second moment assumption. Parseval's relation (see e.g. [9], p. 619) gives us the key relation

$$\int_R h(x)W(s, a, dx)=\int_{K(\hat{h})} \hat{h}(t)\hat{W}(s, a, t)dt \tag{2.9}$$

for $a, y\in \boldsymbol{R}$, $0<s<1$, where K denotes the compact support of $\hat{h}$. The left-hand side of (2.9) converges to

$$\int_R h(x)W(a+dx), \qquad \text{as} \quad s\uparrow 1,$$

because h is l_0-integrable and continuous (thus directly Riemann integrable) and by (2.8). The right-hand side of (2.9) equals

$$\int_{K(\hat{h})} 2\hat{h}(t)e^{-iat}r(t)t^2 Re(\hat{U}(s, t))dt,$$

is bounded in absolute value by

$$\|\hat{h}\|_\infty r(t)t^2\sum_{n\geq 0}|Re(\psi_n)(t)| \qquad \text{for all} \quad 0<s<1$$

and hence converges to

$$\int_{K(\hat{h})} 2\hat{h}(t)e^{-iat}r(t)t^2 Re(\hat{U}(t))dt, \qquad \text{as} \quad s\uparrow 1$$

by (2.1), (2.2) and dominated convergence. We have thus obtained

$$\int_R h(x)W(a+dx)=\int_{K(\hat{h})} \hat{h}(t)e^{-iat}r(t)t^2 Re(\hat{U}(t))dt. \tag{2.10}$$

A further appeal to (2.2) together with the Riemann-Lebesgue lemma shows that the right-hand side of (2.10) converges to 0, as $a\to\infty$, yielding the desired result (2.7).

For arbitrary $h\in C_0$ with compact support $K(h)$ and $\varepsilon>0$, we can choose a function $h_\varepsilon\in\mathcal{D}$ such that $\|h-h_\varepsilon\|_\infty<\varepsilon/C_{K(h)}$ (see e.g. [15], p. 114f) and, by (2.8),

$$\sup_{a\in R}\int_{R-K(h)}|h_\varepsilon(x)|\,|W|(a+dx) < \varepsilon.$$

Consequently, (2.7) follows from

$$\begin{aligned}\left|\int_R h(x)W(a+dx)\right| &\leqq \left|\int_R h_\varepsilon(x)W(a+dx)\right| + C_{K(h)}\|h-h_\varepsilon\|_\infty \\ &\quad + \int_{R-K(h)}|h_\varepsilon(x)|\,|W|(a+dx) \\ &\leqq o(1)+2\varepsilon, \qquad \text{as} \quad a\to\infty.\end{aligned}$$

In case where F_0^s has infinite second mean the following argument will show that it can be replaced by a suitable truncated version H without loosing much in (2.6) if B is replaced by some arbitrary bounded interval there and F_0

by H. Let $\varepsilon, b>0$ be arbitrary and $M=\sup_{t\in R}U[t, t+b]$, which is finite by (2.5). Choose $z>0$ so large that $F_0^s(-z)<\varepsilon/(2M)$ and define

$$H=F_0^s((-z, z)\cap\cdot)+F_0^s(-z)(\delta_{-z}+\delta_z),$$

δ_z being the Dirac measure at z. H is then again symmetric and it clearly has finite second mean. Moreover,

$$\begin{aligned}&\limsup_{t\to\infty} H*U([t, t+b])\\ (2.11)\qquad &\leqq 2MF_0^s(-z)+\limsup_{t\to\infty}\int_{(-z, z)}U([t-x, t+b-x])F_0^s(dx)\\ &\leqq \limsup_{t\to\infty} F_0^s*U([t, t+b])+\varepsilon\leqq\frac{b}{\mu}+\varepsilon, \qquad [d=0]\end{aligned}$$

and similarly

$$\liminf_{t\to\infty} H*U([t, t+b])\geqq\frac{b}{\nu}-\varepsilon. \tag{2.12}$$

If we now define $W=V-H*V$ we infer from the previous part of the proof that $W(t+\cdot)$ still converges vaguely to 0, as $t\to\infty$, and together with (2.11), (2.12) this implies

$$\frac{b}{\nu}-\varepsilon\leqq\liminf_{t\to\infty} H*U([t, t+b])\leqq\limsup_{t\to\infty} U([t, t+b])\leqq\frac{b}{\mu}+\varepsilon$$

proving (2.3) because $\varepsilon, b>0$ and $\mu\in(0, \eta_*)$, $\nu\in(\eta^*, \infty)$ were arbitrarily chosen. (2.4) is now a trivial consequence of (2.3) because $\eta_*=\eta^*=\theta$ under the holding assumption there.

3. Discussion of condition (2.2)

This section is devoted to a discussion of the intricate analytic condition (2.2) of Theorem 2.1. The problem with it is obviously that the occuring infinite series $\sum_{n\geq 0}|Re(\phi_n)(t)|$ cannot easily be estimated about its singularity 0. As a consequence we must be after more transparent, probabilistic alternatives. Definition 3.1 below introduces two appropriate subclasses of increment sequences X_N which contain in particular most non-trivial independent sequences. Special cases are considered in Section 4. The essential property of these increment sequences is that infinitely many of its variables contain a distributional component which is independent of the "rest of the world". Such an assumption is by now standard, for instance in the definition of Harris-recurrent Markov chains.

We begin with some further notation which is needed to present the results of this section. $B(1, \alpha)$, $\alpha\in(0, 1)$, denotes the Bernoulli distribution on $\{0, 1\}$ with α being the probability of $\{1\}$. For each random variable Y let Y^s be a symmetrization, i.e. $Y^s=Y-Y'$ with Y' being an independent copy of Y. We call Y completely d-arithmetic if $Y+z$ is d-arithmetic *for all* $z\in\boldsymbol{R}$. As one can easily verify, this holds true iff Y and Y^s are both d-arithmetic.

A sequence Y_N of random variables is called

—*tight* if $\sup_{n\in N}P(|Y_n|>t)\to 0$, as $t\to\infty$;

—*non-reducible* if all weakly convergent subsequences have non-degenerate limits (in particular, all Y_n are non-degenerate);

—*completely d-arithmetic*, $d\geqq 0$, if all weakly convergent subsequences have completely d-arithmetic limits (in particular, all Y_n are completely d-arithmetic).

As one can easily verify, each of the three previously defined properties holds for Y_N iff it does so for Y_N^s. Moreover, a completely d-arithmetic sequence is necessarily non-reducible.

Let us finally stipulate that all hereafter occuring, not explicitly specified random variables with index 0 are supposed to be 0.

Definition 3.1. A sequence X_N is called to be of

—*type AC* (*Additive Component*), if for an increasing (possibly random) sequence $0\leqq\xi_0<\xi_1<\cdots$

$$X_n=\begin{cases} Z_kY_k+(1-Z_k)\tilde{X}_n & \text{if } \xi_k=n \\ \hat{X}_n & \text{otherwise} \end{cases} \qquad \text{a.s. for all } n\geqq 0, \tag{A.6}$$

where Y_N is a non-reducible sequence of independent random variables, Z_1, $Z_2, \cdots$ are i.i.d. with common distribution $B(1, \alpha)$ for some $\alpha\in(0, 1]$ and Y_N, Z_N and $(\hat{X}_N, \xi_N)$ are mutually independent.

—*type CC* (*Convolution Component*), if for an increasing (possibly random) sequence $0\leqq\xi_0<\xi_1<\cdots$

$$X_n=\begin{cases} Y_k+\hat{X}_n & \text{if } \xi_k=n \\ \hat{X}_n & \text{otherwise} \end{cases} \qquad \text{a.s. for all } n\geqq 0, \tag{A.7}$$

where again Y_N is a non-reducible sequence of independent random variables which is further independent of $(\hat{X}_N, \xi_N)$.

In both cases ξ_N is called a *decomposition sequence* for X_N.

Remarks. (a) As we are always dealing with distributional properties of X_N in the following, results where X_N is assumed to be of type AC or type CC remain of course unchanged if only a copy of X_N (constructed on a suitable probability space) is of this type.

(b) In [3] sequences of type IAC [ICC] (identical additive [convolution] component) were introduced which are further specialized versions of the ones defined above. Namely, in [3] $(Y_1, Z_1), (Y_2, Z_2), \cdots$ $[Y_1, Y_2, \cdots]$ must even be i.i.d. and $\xi_0, \xi_1, \cdots$ stopping times. These are natural requirements for a coupling approach towards Blackwell-type renewal theorems but can be relaxed if Fourier analysis is used. On the other hand we will here need assumptions on the occurrence rate of ξ_N, see (3.1) and (3.2) below, which can be dispensed with in the former approach. A further discussion can be found at the end of Section 4.

(c) Clearly, each non-reducible sequence X_N of independent random variables is of type AC as well as of type CC. More generally, if $X_n = X_n' + X_n''$ for each $n \geqq 0$, where X_N' and X_N'' are independent and X_N' is a non-reducible sequence of independent random variables, then X_N is of type CC.

As for validity of condition (2.2), we will separately give sufficient conditions for

(C.1) $\Psi_{abs}(t) \stackrel{\text{def}}{=} \sum_{n\geq 0} |\phi_n(t)|$ to be continuous on $\boldsymbol{R}_0 = \boldsymbol{R} - \{0\}$ $(d=0)$, resp. $\boldsymbol{R}_d \stackrel{\text{def}}{=} \boldsymbol{R} - (2\pi/d)\boldsymbol{Z}$ $(d>0)$, where d denotes the span of X_N;

(C.2) $t^2\Psi_{abs}(t)$ to be integrable in some neighborhood of 0.

Validity of both, (C.1) and (C.2), then clearly implies that of (2.2). Our result is stated in the following proposition the proof of which we defer to Section 5.

Theorem 3.2. *Let X_N be of type AC or CC with span d.*

(a) *If there exists a decomposition sequence ξ_N with associated sequence Y_N such that*

$$\sum_{n\geq 1} s^n E(\xi_n - \xi_{n-1}) < \infty \quad \textit{for all} \quad s \in (0, 1), \tag{3.1}$$

and Y_N is either completely d-arithmetic and tight, or weakly convergent to a completely d-arithmetic limit, then Ψ and Ψ_{abs} are both continuous on $\boldsymbol{R}_d$, i.e. (C.1) *holds true.*

(b) *If there exists a decomposition sequence ξ_N with associated sequence Y_N such that*

$$\sum_{n\geq 1} n^{-3/2} E(\xi_n - \xi_{n-1}) < \infty \tag{3.2}$$

and Y_N is tight, then $t^2\Psi_{abs}$ is integrable at 0, *i.e.* (C.2) *holds true.*

We can now easily combine Theorem 2.1 with Theorem 3.2 to get the following renewal theorem for random walks with AC- or CC-type increments.

Corollary 3.3. *Let X_N be d-arithmetic and of type AC or CC with decomposition sequence ξ_N satisfying* (3.2) *and associated tight sequence Y_N. Suppose further that there is a subsequence ξ_N' of ξ_N satisfying* (3.1) *and such that its*

associated subsequence Y'_N *of* Y_N *is additionally either completely d-arithmetic or weakly convergent to a completely d-arithmetic limit. Then, if* X_N *is further*
—ultimately s.b. with $\eta_*>0$, (2.3) *of Theorem* 2.1 *holds true.*
—ultimately s.s. with positive mean θ, (2.4) *of Theorem* 2.1 *holds true, i.e.*

$$\lim_{t\to\infty} U(t+I)=\theta^{-1}l_d(I).$$

In either case t *runs through* $d\boldsymbol{Z}$ *only if* $d>0$.

4. Examples and discussion

In this section we want to look at a number of special cases to which Theorem 2.1 or Corollary 3.3 are applicable. Four examples are picked from the class of random walks with AC- or CC-type increments, a further one deals with a certain subclass of Markov-modulated random walks which arise in Markov renewal theory.

Random walks with increments of type AC or CC

It is evident that AC- and CC-type sequences may be found in abundance in the class of sequences of independent random variables. As a consequence, three of the following four examples have been chosen from this class. Note also that in all these examples Blackwell's renewal theorem cannot be concluded from the results in [3], at least not to the same extent. A further discussion is given at the end of the section.

Example 4.1. [*Random walks satisfying a local limit theorem*]

It has been shown by Maejima [11], and under more restrictive conditions already by Cox and Smith [8], how Blackwell's renewal theorem can be deduced from a uniform local limit theorem. The result in [11] looks as follows: Let S_N be a random walk with s.s. increments with positive mean θ and variances satisfying

$$\lim_{n\to\infty}\frac{\mathrm{Var}\,S_n}{n}=\sigma^2\in(0,\ \infty). \tag{4.1}$$

Fix $h>0$ and suppose further that with $a_n=ES_n$, $b_n^2=\mathrm{Var}\,S_n$ and with f denoting the standard normal density

$$\lim_{n\to\infty}\sup_{x\in\boldsymbol{R}} x^m\left|\frac{b_n}{h}P\Big(-\frac{h}{2}+a_n+xb_n<S_n\leqq\frac{h}{2}+a_n+xb_n\Big)-f(x)\right|=0 \tag{4.2}$$

for $m\in\{0,\ 2\}$. Then

(4.3) $$\lim_{x\to\infty} U\left(\left[x-\frac{h}{2},\ x+\frac{h}{2}\right]\right)=\frac{h}{\theta}.$$

At first sight it is hard to compare this result with ours due to the intrinsic assumption (4.2). However, Maejima further shows that it holds true if X_0, $X_1, \cdots$ satisfy (4.1) and the classical Lindeberg condition, have finite third moments, and if their F.t. $\phi_0, \phi_1, \cdots$ satisfy for some $\varepsilon>0$

(4.4) $$\phi_{\max}(t) \stackrel{\text{def}}{=} \sup_{n\geq 0}|\phi_n(t)|<1 \qquad \text{for all } |t|\geqq\varepsilon \text{ and } n\geqq 1.$$

As for this set of conditions, we can now easily argue that it is much more restrictive than actually needed for (4.3) to be valid. Indeed, (4.4) alone is already equivalent with X_N to be completely nonarithmetic. Moreover, since X_N is s.s. it is of type AC and CC with trivial decomposition sequence $\xi_n\equiv n$, $n\geqq 0$ and associated sequence $Y_N=X_N$ which is thus clearly tight. Consequently, Corollary 3.3 applies, i.e. (4.3) holds true without needing any of (4.1), the Lindeberg condition and finite third moments.

Example 4.2. [*Asymptotically i.i.d. increments*]
Suppose that $X_0, X_1, \cdots$ are independent, d-arithmetic, ultimately s.s. with positive mean θ and weakly convergent to a completely d-arithmetic limit. Such a sequence may be roughly characterized as "asymptotically i.i.d.". It is an immediate consequence of the previous corollary that under the given assumptions Blackwell's renewal theorem (2.4) holds true for the associated random walk. If $X_0, X_1, \cdots$ are e.g. response variables in a sequential medical trial with treatment allocation, where each treatment corresponds to a certain response distribution, then this sequence will be asymptotically i.i.d. under each allocation sequence which in the long-run chooses always the same treatment (preferably the superior one).

Example 4.3. [*Increments with pairwise singular distributions*]
Two distributions Q_1, Q_2 on $\boldsymbol{R}$ are called *singular* if there is a Borel set B such that $Q_1(B)=0$ and $Q_2(B^c)=0$. In other words, Q_1 and Q_2 must "live" on disjoint subsets of $\boldsymbol{R}$. We call two random variables singular if there distributions are so. In classical renewal theory, random walks with nonarithmetic increment distributions which are singular with respect to Lebesgue measure (e.g. the Cantor distribution) turn out to be bad as to convergence rates in Blackwell's and other renewal theorems even if all moments are finite, see [7]. A typical example is the Laplace distribution on $\{\alpha, 1\}$ with $\alpha>0$ an irrational number. Now consider a sequence X_N of independent, pairwise singular random variables which is further completely nonarithmetic and ultimately s.s. with positive mean θ. With view to the previous remark we might conjecture that Blackwell's renewal theorem (2.4) fails under these assumptions, but Corollary

3.3 immediately sets us right. This is also interesting because it cannot be concluded from the results of [3] where $X_0, X_1, \cdots$ or a subsequence of it must share a distributional component which is clearly excluded by pairwise singularity. For illustration we finally note a simple, more concrete example: Let α_N be a sequence of pairwise different irrationals $\in(0, 1)$ with no rational limit point. Let $X_0, X_1, \cdots$ be independent and each X_n-1 Laplacian on $\{\pm\alpha_n, \pm(\alpha_n+1)\}$ so that $EX_n=1$ for all $n\in\boldsymbol{N}$. It is then easily verified that X_N is completely nonarithmetic, s.s. with mean 1 and with pairwise singular distributions. Thus Corollary 3.3 applies.

Example 4.4. [*Linear growth processes with i.i.d. perturbances*]

Our fourth example shall give an application of Corollary 3.3 to random walks S_N with dependent increment sequences X_N. Let $\theta>0$ and M_N be an ultimately s.b. (thus ultimately s.s.) d-arithmetic martingale. Let ξ_N be a sequence of random, but not necessarily stopping times for M_N which satisfies (3.2). At these "shock" or "perturbance" epochs our random walk is perturbed by i.i.d. zero-mean and completely d-arithmetic random variables $Y_1, Y_2, \cdots$ which are further independent of (M_N, ξ_N). S_n is now defined through

$$S_n=n\theta+M_n+\sum_{j\geq 1} Y_j\mathbf{1}(\xi_j\leqq n)$$

for each $n\geqq 0$. Then again it is easily verified that Corollary 3.3 applies yielding (2.4).

Markov-modulated random walks

In Markov renewal theory we are given a bivariate Markov chain (M_N, X_N) with state space $(S\times\boldsymbol{R}, \mathcal{S}\otimes\mathcal{B})$ and transition kernel $\boldsymbol{P}: S\times(\mathcal{S}\otimes\mathcal{B})\to[0, 1]$, i.e. (M_{n+1}, X_{n+1}) depends on the past only through M_n. Suppose that S is Polish with Borel σ-field $\mathcal{S}$ and that M_N forms a Harris chain with transition kernel $\boldsymbol{P}^*(x, dy)\overset{\text{def}}{=}\boldsymbol{P}(x, dy\times\boldsymbol{R})$ and regeneration set $\mathfrak{R}$. This implies that for some $r\geqq 1$ and $\alpha>0$ and some probability measure ν on S with $\nu(R)=1$ the r-step transition kernel $\boldsymbol{P}_r^*$ satisfies the minorization condition

$$\boldsymbol{P}_r^*(x, \cdot)\geqq\alpha\nu \quad \text{for all} \quad x\in\mathfrak{R}. \tag{4.5}$$

For any distribution λ on $S\times\boldsymbol{R}$ let P_λ be such that $P_\lambda((M_0, X_0)\in\cdot)=\lambda$. If λ denotes a distribution on S only then $P_\lambda\overset{\text{def}}{=}P_{\lambda\otimes\delta_0}$. Finally, $P_{x,y}\overset{\text{def}}{=}P_{\delta_{x,y}}$ and $P_x\overset{\text{def}}{=}P_{\delta_{x,0}}$ for $(x, y)\in S\times\boldsymbol{R}$.

By using the regeneration technique of Athreya and Ney [5] one can define (on a possibly enlarged probability space) a version of (M_N, X_N) together with a sequence T_N of randomized stopping times for M_N, $T_0=0$, such that for each initial distribution λ and all $n \geqq 1$

$$P_\lambda(M_{T_n+N} \in \cdot) = P_\nu(M_N \in \cdot), \tag{4.6}$$

and $(M_j, X_j)_{0 \le j \le T_{n-1}}$ and (M_{T_n+N}, X_{T_n+N}) are independent. Thus T_N forms a sequence of regeneration epochs for M_N, and its unique (up to a multiplicative constant) stationary measure is given by

$$\xi(A) \overset{\text{def}}{=} E_\nu\Big(\sum_{j=0}^{T_1-1} 1(M_j \in A)\Big), \qquad A \in \mathcal{S}. \tag{4.7}$$

Induced by T_N and under P_ν, we are now given an i.i.d. sequence $\hat{M}_N$, $\hat{M}_n = M_{T_n}$, together with a random walk $\hat{S}_n = S_{T_n}$, $n \geqq 0$ whose increments $\hat{X}_n = S_{T_n} - S_{T_{n-1}}$, $n \geqq 1$ form a stationary, 1-dependent sequence with mean

$$\mu = \int_S E(X_1 \mid M_0 = x)\xi(dx), \tag{4.8}$$

as one can easily verify. Under arbitrary P_λ, the same holds true for $(M_{N+1}, \hat{S}_{N+1} - \hat{S}_1)$. Consequently, if $\hat{X}_N$ is s.b. under any P_λ, then it is also s.s. with mean μ due to the stationarity and 1-dependence. In the following, we want to show that $\hat{S}_N$ then satisfies Blackwell's renewal theorem provided $\mu > 0$ and an additional nonlatticeness assumption on X_N holds true, see (4.9) below. The result can be used for the derivation of a general Markov renewal theorem, see [4].

So suppose $\mu \in (0, \infty)$ and furthermore for all $t \neq 0$

$$\inf_{n \geq 1} |E(e^{itS_n} \mid M_0, M_n)| < 1 \qquad P_\xi\text{-a.s.} \tag{4.9}$$

It can be shown that this condition implies $\boldsymbol{P}(x, \cdot)$ be nonlattice as defined in [12], see also Lemma 3.3 and the subsequent Remark in [4]. Denote by U_λ the renewal measure of $\hat{S}_N$ under P_λ, i.e.

$$U_\lambda(B) = \sum_{n \geq 0} P_\lambda(\hat{S}_n \in B) = \sum_{n \geq 0} P_\lambda(S_{T_n} \in B).$$

We will prove now

Theorem 4.5. *Let λ be an arbitrary distribution on $S \times \boldsymbol{R}$. If $\hat{X}_N$ is s.s. under P_λ with mean $\mu \in (0, \infty)$ and if (4.9) holds true, then for all bounded intervals I*

$$\lim_{t \to \infty} U_\lambda(t + I) = \mu^{-1} l_0(I). \tag{4.10}$$

Proof. Let ν be as given in (4.5). We define

$$\varphi(x, y, t) = E(e^{itX_1} \mid M_0 = x, M_1 = y), \qquad \phi_n(x, y, t) = E(e^{itS_n} \mid M_0 = x, M_n = y),$$

and similarly $\hat{\varphi}$, $\hat{\phi}_n$ for $(\hat{M}_N, \hat{X}_N)$. It is now verified that condition (2.2) holds for $\hat{S}_N$, more precisely

$$\int_{(-a,a)} t^2 \sum_{n\geq 0} |E_\nu e^{it\hat{S}_n}|\, dt < \infty \qquad \text{for all} \quad a>0.$$

If $\gamma_n(M_0, M_n)=\inf\{t>0: |\psi_n(M_0, M_n, t)|=1\}$, where $\inf\phi \overset{\text{def}}{=} \infty$, then (4.9) implies $P_\xi(\gamma_p(M_0, M_p)>0)>0$ for some $p\geqq 1$. Hence there are some $b>0$ and $A\in S^2$ such that $P_\xi((M_0, M_p)\in A)>0$ and

$$\varphi_{\max}(t) \overset{\text{def}}{=} \sup_{(x,y)\in A} |\psi_p(x, y, t)|<1 \qquad \text{for all} \quad 0<|t|<b.$$

The latter particularly implies that $\{P(S_p\in\cdot\,|M_0=x, M_p=y): (x, y)\in A\}$ is nonreducable, and since S_p has a.s. finite conditional mean given M_0, M_p, we can choose A even in such a way that the former distribution family is also tight. It then follows by Lemma 5.4 in the next section that

$$(4.11) \qquad 1-\varphi_{\max}(t)\geqq at^2 \qquad \text{for some } a>0 \text{ and all } t\in(-b, b).$$

Next observe that A is a recurrence set for (M_N, M_{p+N}) whence

$$\beta \overset{\text{def}}{=} P_\nu((M_m, M_{m+p})\in A)>0 \qquad \text{for some} \quad m\geqq 0.$$

We infer

$$\begin{aligned} |E_\nu e^{it\hat{S}_{m+p}}| &\leqq E_\nu |E(e^{it\hat{S}_{m+p}}|\hat{M}_0, \hat{M}_{m+p})| \leqq E_\nu |E(e^{it(S_{m+p}-S_m)}|M_m, M_{m+p})| \\ &= E_\nu |\psi_p(M_m, M_{m+p}, t)| \leqq \beta\varphi_{\max}(t)+(1-\beta), \end{aligned}$$

where $T_{m+p}\geqq m+p$ and the conditional independence of S_m, $S_{m+p}-S_m$ and $S_{T_{m+p}}-S_{m+p}$ given M_0, M_m, M_{m+p}, $M_{T_{m+p}}$ has been utilized. This yields for $k\geqq 1$ and $0<|t|<b$

$$\begin{aligned} |E_\lambda e^{it\hat{S}_k}| &\leqq E_\lambda\Big(\prod_{j=2}^{l(k)+1} \Big|E(e^{it(\hat{S}_{j(m+p)}-\hat{S}_{(j-1)(m+p)})}|\hat{M}_{j(m+p)}, \hat{M}_{(j-1)(m+p)})\Big|\Big) \\ &= E_\nu\Big(\prod_{j=1}^{l(k)} \Big|E(e^{it(\hat{S}_{j(m+p)}-\hat{S}_{(j-1)(m+p)})}|\hat{M}_{j(m+p)}, \hat{M}_{(j-1)(m+p)})\Big|\Big) \quad \text{by (4.6)} \\ &\leqq \prod_{j=1}^{l(k)} (\beta\varphi_{\max}(t)+(1-\beta))^{l(k)}, \end{aligned}$$

where $l(k) \overset{\text{def}}{=} \sup\{j\geqq 0: j(m+p)\leqq k\}-1$, and then after a simple calculation

$$\Phi(t) \overset{\text{def}}{=} \sum_{n\geq 0} |E_\nu e^{it\hat{S}_n}| \leqq \frac{C}{1-\varphi_{\max}(t)}, \qquad 0<|t|<b,$$

for a suitable constant $C>0$. Together with (4.11) we obtain integrability of $t^2\Phi(t)$ on $(-b, b)$. Moreover, since even

$$(4.12) \qquad \sum_{n\geq k} E_\lambda e^{it\hat{S}_n} \leqq \frac{C\varphi_{\max}(t)^k}{1-\varphi_{\max}(t)} \qquad \text{for all } k\geqq 0 \text{ on } (-b, b),$$

we also infer continuity of Φ on $(-b, b)-\{0\}$ by uniform convergence of the associated finite partial sums on each compact subset. Finally, we must argue that $\hat{\phi}$ is also continuous outside $(-b, b)$. But for each $|t|>b$ we can proceed as before by choosing some $p\geqq 1$ according to (4.9) (which may depend on t) such that (4.12) holds true in some neighborhood of t for all $k\geqq 0$, of course with in general different $\varphi_{\max}$ and C. It follows local continuity of $\hat{\phi}(t)$ for each $t\neq 0$ and thus validity of (2.2). Assertion (4.10) is now a consequence of our Theorem 2.1.

Without strong distributional assumptions like stationarity or Markovian transitions, Blackwell-type renewal theorems for random walks with non-i.i.d. increments have been discussed earlier in a number of papers, most notably (besides [11]) in [13] and [14]. A discussion of this literature can be found in our companion paper [3] and we therefore restrict ourselves now to some brief remarks on how the results in that latter article compare with the present ones. Due to the totally different approaches it turns out that a number of applications there cannot be included here and vice versa. In fact, in [3], as already pointed out, the main condition on the increments $X_0, X_1, \cdots$ is that a subsequence $X_{\xi_{N+1}}$ shares a common component for an arbitrarily thin sequence of *entrance times* ξ_{N+1}. Loosely speaking, $X_{\xi_{N+1}}$ must contain a sequence of i.i.d. random variables Y_{N+1}. Here we have replaced i.i.d. sequences by more general sufficiently regular ones of independent random variables and the ξ_n need not be stopping times. On the other hand, ξ_N cannot be arbitrarily thin in that conditions (3.1) and (3.2) are imposed. It appears to be an interesting but probably very difficult problem to combine both said approaches to come up with a result which applies to *all* applications presented here and in [3].

5. Proof of Theorem 3.2

It is always assumed in the following that X_N is of type AC or CC with decomposition sequence ξ_N and associated sequence Y_N, as given by Definition 3.1. Let φ_n be the F.t. of Y_n for each $n\geqq 1$ and

(5.1) $$\varphi_{\max}(t)=\sup_{n\geq 1}|\varphi_n(t)|.$$

We further keep the notation of Sections 1–3.

The following lemma forms the basis for the proof of Theorem 3.2. Recall that ϕ_n denotes the F.t. of S_n.

Lemma 5.1. *Let* $\rho(n)=\sup\{k\geqq 0: \xi_k\leqq n\}$ *for* $n\in \boldsymbol{N}$.

(a) *If* X_N *is of type AC (with* $Z_1, Z_2, \cdots \sim B(1, \alpha)$*), then*

(5.2) $$|\phi_n(t)|\leqq E((\alpha\varphi_{\max}(t)+(1-\alpha))^{\rho(n)}) \qquad \text{for all} \quad n\geqq 0,\ t\in \boldsymbol{R}.$$

(b) *If* X_N *is of type CC, then*

(5.3) $$|\phi_n(t)|\leqq E(\varphi_{\max}(t)^{\rho(n)}) \qquad \text{for all} \quad n\geqq 0,\ t\in \boldsymbol{R}.$$

Proof. The proof of (5.3) is very easy and thus given first. Let $\hat{S}_n = \hat{X}_1 + \cdots + \hat{X}_n$ and $W_n = Y_1 + \cdots + Y_n$ for $n \geqq 1$. If X_N is of type CC then (A.7) implies $S_n = \hat{S}_n + W_{\rho(n)}$. It follows from the independence of W_N and $(\hat{S}_N, \rho(N))$, see Definition 3.1, that for all $n \in \boldsymbol{N}$ and $t \in \boldsymbol{R}$

$$\psi_n(t) = E(e^{itW_{\rho(n)}}) \cdot E(e^{it\hat{S}_n}) = E\Big(\prod_{j=1}^{\rho(n)} \varphi_j(t)\Big) \cdot E(e^{it\hat{S}_n}),$$

which in turn obviously yieds (5.3).

Now suppose X_N to be of type AC. Let $I = \{1, \cdots, \rho(n)\}$, $W_\phi = 0$, $Z_\phi = 1$, and for $\phi \neq J \subset \boldsymbol{N}$

$$W_J = \sum_{j \in J} Y_j \quad \text{and} \quad Z_J = (Z_j)_{j \in J}.$$

We write $Z_J = z$ to mean $Z_j = z$ for all $j \in J$. Finally, let $\tilde{S}_n = S_n - \sum_{j=1}^{\rho(n)} Z_j Y_j$ and observe that $S_n = \tilde{S}_n + W_J$ on $\{Z_J = 1, Z_{I-J} = 0\}$. By mutual independence of Y_N, Z_N and $(\hat{X}_N, \rho(N))$, it follows for all $n \in \boldsymbol{N}$ and $t \in \boldsymbol{R}$

$$\begin{aligned}
\psi_n(t) &= \sum_{k \geq 0} \sum_{j=0}^{k} \sum_{J \subset I, |J| = j} \int_{\{\rho(n)=k, Z_J=1, Z_{I-J}=0\}} e^{it(W_J + \tilde{S}_n)} dP \\
&= \sum_{k \geq 0} \sum_{j=0}^{k} \sum_{J \subset I, |J| = j} E(e^{itW_J}) \cdot \int_{\{\rho(n)=k, Z_J=1, Z_{I-J}=0\}} e^{it\tilde{S}_n} dP \\
&= \sum_{k \geq 0} \sum_{j=0}^{k} \sum_{J \subset I, |J| = j} \Big(\prod_{m \in J} \varphi_m(t)\Big) \cdot \int_{\{\rho(n)=k, Z_J=1, Z_{I-J}=0\}} e^{it\tilde{S}_n} dP,
\end{aligned}$$

and then further

$$\begin{aligned}
|\psi_n(t)| &\leqq \sum_{k \geq 0} \sum_{j=0}^{k} \sum_{J \subset I, |J| = j} \varphi_{\max}(t)^j P(\rho(n)=k, Z_J=1, Z_{I-J}=0) \\
&= \sum_{k \geq 0} \sum_{j=0}^{k} \binom{k}{j}_{J \subset I, |J|=j} \sum \varphi_{\max}(t)^j \alpha^j (1-\alpha)^{k-j} P(\rho(n)=k) \\
&= \sum_{k \geq 0} (\alpha \varphi_{\max}(t) + (1-\alpha))^k P(\rho(n)=k) \\
&= E((\alpha \varphi_{\max}(t) + (1-\alpha))^{\rho(n)})
\end{aligned}$$

which is the desired result.

In order for the previous lemma to be useful for our purposes we clearly have to provide conditions which ensure $\varphi_{\max}(t) < 1$ for all $t \in \boldsymbol{R}_d$. The next lemma does so and is a simple consequence of Levy's continuity theorem and the fact that distributional limits of completely d-arithmetic sequences are by definition again completely d-arithmetic. It is therefore stated without proof.

Lemma 5.2. *If Y_{N+1} is completely d-arithmetic and tight, then*

(5.4) $$\sup_{t \in K} \varphi_{\max}(t) < 1 \qquad \textit{for each compact } K \subset \boldsymbol{R}_d.$$

Proof of Theorem 3.2 (a). Let X_N be of type CC with ξ_N satisfying (3.1) and Y_{N+1} being completely d-arithmetic and tight. Combining Lemmata 5.1 and 5.2, we infer for all $t\in\boldsymbol{R}_d$

$$\begin{aligned}|\Psi(t)| &\leqq \Psi_{abs}(t)=\sum_{n\geqq 0}|\varphi_n(t)|\leqq\sum_{n\geqq 0}E(\varphi_{\max}(t)^{\rho(n)}) \\ &=\sum_{k\geqq 0}\varphi_{\max}(t)^k\sum_{n\geqq 0}P(\rho(n)=k)=\sum_{k\geqq 0}\varphi_{\max}(t)^k\sum_{n\geqq m}P(\xi_k\leqq n<\xi_{k+1}) \\ &=\sum_{k\geqq 0}E(\xi_{k+1}-\xi_k)\varphi_{\max}(t)^k<\infty,\end{aligned} \tag{5.5}$$

i.e. $\Psi(t)$ and $\Psi_{abs}(t)$ are both finite on $\boldsymbol{R}_d$. A similar estimation shows that on compact subsets both functions are uniform limits of their corresponding finite partial sums which are clearly continuous. Thus Ψ and Ψ_{abs} must be so, too.

The same arguments apply for AC-type sequences X_N. Just replace $\varphi_{\max}(t)$ by $\alpha\varphi_{\max}(t)+(1-\alpha)$ there and note that (5.4) also holds for the latter function since $\alpha>0$.

If Y_{N+1} is weakly convergent with completely d-arithmetic limit, let φ be its F.t. By compact convergence of φ_n to φ we infer for each compact $K\subset\boldsymbol{R}_d$ the existence of $N\in\boldsymbol{N}$ and some $C_K<1$ such that

$$\varphi_{N,\max}(t)\overset{\text{def}}{=}\sup_{n\geqq N}|\varphi_n(t)|\leqq C_K \qquad \text{for all } t\in K.$$

Thus, by using Lemma 5.1 with ξ_N, $\varphi_{\max}$, $\rho(n)$ replaced by ξ_{N+N}, $\varphi_{N,\max}$, $\rho_N(n)\overset{\text{def}}{=}\sup\{k\geqq 0:\xi_{N+k}\leqq n\}$, the desired conclusions follow almost the same way as above for the case when Y_{N+1} is completely d-arithmetic and tight. We omit further details.

For the proof of Theorem 3.2 (b), we must first examine $\varphi_{\max}$ in a small neighborhood of 0. The result is stated in Lemma 5.4 below which in turn is furnished by an auxiliary one stated next.

Lemma 5.3. *Let X be a random variable with F.t. φ and*

$$\Gamma(X,x,t)\overset{\text{def}}{=}\sum_{n\geqq 0}P\left(x+\frac{2n\pi}{t}<X\leqq x+\frac{(2n+1)\pi}{t}\right) \quad \textit{for } t\neq 0 \textit{ and } x>0. \tag{5.6}$$

Then for all $t\neq 0$

$$\frac{1-Re(\varphi(t))}{t}=\int_0^{\pi/t}\sin(tx)\Gamma(|X|,x,t)dx \quad \textit{and} \tag{5.7}$$

$$\begin{aligned}\frac{Im(\varphi(t))}{t}&=\int_0^{\pi/2t}\cos(tx)(P(X^+>x)-P(X^->x))dx \\ &-\int_{-\pi/2t}^{\pi/2t}\cos(tx)\left(\Gamma\left(X^+,x+\frac{\pi}{t},t\right)-\Gamma\left(X^-,x+\frac{\pi}{t},t\right)\right)dx\end{aligned} \tag{5.8}$$

Proof. Since

$$1-Re(\varphi(t))=1-E(\cos(tX))=1-E(\cos(t|X|))$$

and

$$Im(\varphi(t))=E(\sin(tX))=E(\sin(tX^{+}))-E(\sin(tX^{-}))\,,$$

it suffices to prove the assertions for nonnegative X which is therefore assumed in the following. Suppose further first that X is bounded by some $a\in\boldsymbol{R}$. Then

$$1-Re(\varphi(t))=1-E(\cos(tX))=E\left(\int_0^X t\sin(tx)dx\right)$$

$$=\int_0^\infty t\sin(tx)P(X>x)dx=\int_0^a t\sin(tx)P(X>x)dx\,.$$

Since, for fixed $t\neq 0$, $\sin(tx)$ has period $2\pi/t$ and $\sin(tx+\pi)=-\sin(tx)$, we obtain on splitting up the range of integration

$$\int_0^\infty t\sin(tx)P(X>x)dx$$

$$=\int_0^{\pi/t} t\sin(tx)\sum_{n\geq 0}P\left(X>x+\frac{2n\pi}{t}\right)dx-\int_{\pi/t}^{2\pi/t} t\sin(tx)\sum_{n\geq 0}P\left(X>x+\frac{2n\pi}{t}\right)dx$$

$$=\int_0^{\pi/t} t\sin(tx)\sum_{n\geq 0}\left(P\left(X>x+\frac{2n\pi}{t}\right)-P\left(X>\frac{(2n+1)\pi}{t}\right)\right)dx$$

$$=\int_0^{\pi/t} t\sin(tx)\Gamma(X,\,x,\,t)dx,\qquad \text{i.e. (5.7)}.$$

If X is unbounded, then the same formula yields by using it for $X\wedge n$ and by then letting n tend to infinity. Since $\Gamma(\cdot,\,x,\,t)$ is always bounded by 1, the desired result follows by dominated convergence with majorant $t\sin(tx)$ on the right-hand side.

The proof of (5.8) goes very similar. Here we have for $t\neq 0$

$$Im(\varphi(t))=E(\sin(tX))=E\left(\int_0^X t\cos(tx)dx\right)=\int_0^\infty t\cos(tx)P(X>x)dx\,.$$

The remaining calculations are then done analogously, first for bounded X, and by splitting up the range of integration of the last integral above in an obvious manner. We do not supply the details again.

Lemma 5.4. *If Y_{N+1} is non-reducible and tight, then there are a, $T>0$ such that*

(5.9) $$1-\varphi_{\max}(t)\geqq at^2\qquad \textit{for all}\quad t\in(-T,\,T).$$

Proof. Let us consider the sequence Y^s_{N+1} whose associated F.t. are given by $|\varphi_n|^2$, $n\geqq 1$. As already mentioned, Y^s_{N+1} is also non-reducible and tight whence

$$F(t)\overset{\text{def}}{=}\sup_{n\geq 1}P(Y^s_n\leqq t)\qquad\text{and}\qquad G(t)\overset{\text{def}}{=}\inf_{n\geq 1}P(Y^s_n\leqq t)$$

are proper distribution functions (by tightness) and satisfy $F(0)<1$ (by non-reducibility and symmetry). Now use formula (5.7) to obtain

$$\frac{1-|\varphi_n(t)|^2}{t^2}=\int_0^{\tau/t}\frac{\sin(tx)}{t}\Gamma(|Y_n^s|,\ x,\ t)dx$$

$$\geqq\int_0^{\pi/t}\frac{\sin(tx)}{t}P\Big(x<Y_n^s\leqq x+\frac{\pi}{t}\Big)dx$$

$$\geqq\int_0^{\pi/t}\frac{\sin(tx)}{t}\Big(G\Big(x+\frac{\pi}{t}\Big)-F(x)\Big)^+dx \qquad \text{for all } n\geqq 1.$$

However, the latter integral does no longer depend on n and converges to

$$\mu_+(F)\overset{\text{def}}{=}\int_0^\infty x(1-F(x))dx, \qquad \text{as} \quad t\downarrow 0,$$

which is positive, possibly infinite, because $F(0)<1$. Consequently, by choosing any $a\in(0,\ \mu^+(F))$, we finally conclude

$$1-\varphi_{\max}(t)=\inf_{n\geq 1}(1-|\varphi_n(t)|)=\inf_{n\geq 1}\Big(\frac{1-|\varphi_n(t)|^2}{1+|\varphi_n(t)|}\Big)$$

$$\geqq\frac{1}{2}\inf_{n\geq 1}(1-|\varphi_n(t)|^2)\geqq at^2$$

for all sufficiently small t which proves the desired result.

Proof of Theorem 3.2 (b). Let X_N be of type CC and choose a, $T>0$ so small that (5.9) of the previous lemma holds for $\varphi_{\max}$ given here. The following estimation is similar to one given by Smith [13], p. 483. Recall from (5.5) that $\Psi_{abs}(t)\leqq\sum_{n\geq 0}E(\xi_{n+1}-E\xi_n)\varphi_{\max}(t)^n$. We obtain with suitable constants C_1, $C_2>0$

$$\int_{-T}^T t^2\Psi_{abs}(t)dt=\int_0^T 2t^2\Psi_{abs}(t)dt\leqq\sum_{n\geq 0}E(\xi_{n+1}-\xi_n)\int_0^T 2t^2\varphi_{\max}(t)^n dt$$

$$\leqq\sum_{n\geq 0}E(\xi_{n+1}-\xi_n)\int_0^T 2t^2(1-at^2)^n dt$$

$$\leqq C_1\sum_{n\geq 0}E(\xi_{n+1}-\xi_n)\int_0^1 u^{1/2}(1-u)^n du$$

$$\leqq C_2\sum_{n\geq 0}n^{-3/2}E(\xi_{n+1}-\xi_n),$$

the latter expression being finite by assumption (3.2). Note that we have used for it that

$$\int_0^1 u^{1/2}(1-u)^n du=\frac{\Gamma(3/2)\Gamma(n+1)}{\Gamma(n+(5/2))}=o(n^{-3/2}), \qquad \text{as } n\to\infty,$$

where the asymptotic behavior may be seen by an appeal to Stirling's formula.

If X_N is of type AC the same arguments apply with $\varphi_{\max}$ replaced $\alpha\varphi_{\max}$

$+(1-\alpha)$ which still satisfies (5.9) of Lemma 5.4 (with αa instead of a). We do not give the details again.

References

[1] G. Alsmeyer, On the moments of certain first passage times for linear growth processes, *Stoch. Proc. Appl.*, **25** (1987), 109-136.

[2] ———, Random walks with stochastically bounded increments: Foundations and characterization results, *Results in Math.*, **19** (1991), 22-45.

[3] ———, Random walks with stochastically bounded increments: Renewal theory, *to appear in Math. Nachr.* (1994).

[4] ———, On the Markov renewal theorem, *Stoch. Proc. Appl.*, **50** (1994), 37-56.

[5] K.B. Athreya and P. Ney, A new approach to the limit theory of recurrent Markov chains. *Trans. Amer. Math. Soc.*, **245** (1978), 493-501.

[6] L. Breiman, *Probability*. Addison-Wesley, Reading, Massachussets 1968.

[7] H. Carlsson, Remainder term estimates of the renewal function, *Ann. Probab.*, **11** (1983), 143-157.

[8] D. Cox and W.L. Smith, A direct proof of a fundamental theorem of renewal theory, *Skand. Aktuarietidskr.* **36** (1953), 139-150.

[9] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd ed. Wiley, New York 1971.

[10] S. Lalley and G. Lorden, A control problem arising in the sequential design of experiments, *Ann. Probab.*, **14** (1986), 136-172.

[11] M. Maejima, On local limit theorems and Blackwell's renewal theorem, *Ann. Inst. Statist. Math.*, **27** (1975), 507-520.

[12] V.M. Shurenkov, On the theory of Markov renewal, *Th. Probab. Appl.*, **29** (1984), 247-265.

[13] W. Smith, On some general renewal theorems for nonidentically distributed random variables, *Proc. 4th Berkeley Symp. on Math. Stat. and Prob., Vol. 2,* 467-514, (1961).

[14] J.A. Williamson, Some renewal theorems for non-negative random variables, *Trans. Amer. Math. Soc.*, **114** (1965), 417-445.

[15] M. Woodroofe, *Nonlinear Renewal Theory in Sequential Analysis*, Society for Industrial and Applied Mathematics, Philadelphia 1982.

Mathematisches Seminar
Universität Kiel
Ludewig-Meyn-Straße 4
D-24098 Kiel 1