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SOME LIMIT THEOREMS FOR NEGATIVELY DEPENDENT RANDOM VARIABLES

By

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Abstract. This paper is devoted to weak convergence of sums of negatively dependent random variables, we establish the central limit theorem and the rate of convergence in the CLT. An extension of the CLT to the invariance principle in $D[0,1]$ is presented as well as conditions for weak relative compactness and weak invariance principle in $L^2[0,1]$. We also prove some new properties of negatively dependent random variables.

1. Introduction

At the begining we shall recall some negative dependence concepts. Let $(X_n)_{n\in N}$ be a sequence of random variables defined on some probability space $(\Omega, \mathfrak{F}, P)$. Two random variables X_1 and X_2 are said to be negative quadrant dependent (NQD) if

$$(1.1)\qquad P[X_1>x_1,\ X_2>x_2]-P[X_1>x_1]\,P[X_2>x_2]\leqq 0, \qquad \text{for all } x_1,\ x_2\in \boldsymbol{R}.$$

We define random variables $(X_n)_{n\in N}$ to be linearly negative quadrant dependent (LNQD) if for any finite, disjoint subsets $A, B\subset N$ and positive numbers $(\lambda_k)_{k\in A\cup B}$, $\sum_{k\in A}\lambda_k X_k$ and $\sum_{k\in B}\lambda_k X_k$ are NQD.

If $\mathrm{Cov}(f(X_k,\ k\in A),\ g(X_k,\ k\in B))\leqq 0$ for any coordinatewise nondecreasing functions $f: \boldsymbol{R}^A\to\boldsymbol{R}$, $g: \boldsymbol{R}^B\to\boldsymbol{R}$, such that this covariance exists, with A and B as above, then we say that random variables $(X_n)_{n\in N}$ are negatively associated (NA). It is easy to see, that NA random variables are LNQD and LNQD are pairwise NQD. For more details about positive and negative dependence see Newman [7] where further references are given, we only mention that if $X_1, \cdots, X_n$ are LNQD, then

$$(1.2)\qquad \left|E\left(\exp\left(i\sum_{k=1}^{n}t_kX_k\right)\right)-\prod_{k=1}^{n}E(\exp(it_kX_k)\right|\leqq \sum_{1\leq k<m\leq n}|t_kt_m|\,|\mathrm{Cov}(X_k,\ X_m)|\,,$$

for any real $t_1, \cdots, t_n$.

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Now we shall prove that weak limit of LNQD (respectively NA) random variables is also LNQD (NA), these properties have not been stated anywhere, so we formulate them as propositions.

Proposition 1.1. *Let $X_1^n, \cdots, X_k^n$ be LNQD for every $n \in \boldsymbol{N}$ and $[X_1^n, \cdots, X_k^n] \xrightarrow[n\to\infty]{d} [X_1, \cdots, X_k]$, then $X_1, \cdots X_k$ are LNQD.*

Proof. Let $A, B \subset \{1, \cdots, k\}$, $A \cap B = \emptyset$, $\lambda_j \geqq 0$ for $j \in A \cup B$. Let us put $Y_A^n = \sum_{j\in A} \lambda_j X_j^n$, $Y_B^n = \sum_{j\in B} \lambda_j X_j^n$, $Y_A = \sum_{j\in A} \lambda_j X_j$, $Y_B = \sum_{j\in B} \lambda_j X_j$. By definition of LNQD, we have

$$P[Y_A^n > x_1, Y_B^n > x_2] - P[Y_B^n > x_1] P[Y_B^n > x_2] \leqq 0, \qquad \text{for all } x_1, x_2 \in \boldsymbol{R}.$$

If $n \to \infty$, then from this inequality, assumed weak convergence and Crammer-Wald theorem we get

$$P[Y_A > x_1, Y_B > x_2] - P[Y_A > x_1] P[Y_B > x_2] \leqq 0,$$

for all continuity points $x_1, x_2 \in \boldsymbol{R}$ and by standard argument for every $x_1, x_2 \in \boldsymbol{R}$, thus (1.1) is valid and the proof is completed.

From this proposition and inequality (1.2) we derive the following remark.

Remark 1.1. If $X_1^n, \cdots, X_k^n$ are LNQD for every $n \in \boldsymbol{N}$, $[X_1^n, \cdots, X_k^n] \xrightarrow[n\downarrow\infty]{d} [X_1, \cdots, X_k]$, $\mathrm{Cov}(X_i, X_j) = 0$ for $1 \leqq i < j \leqq k$, then $X_1, \cdots, X_k$ are jointly independent. In particular this statement remains valid if $X_1^n, \cdots, X_k^n$ are NA for every $n \in \boldsymbol{N}$.

This remark remark is sufficient for our further purposes, but for the sake of completness we present a very important analogue of Proposition 1.1 for NA random variables, it will be preceded by some auxiliary statements.

Proposition 1.2. *Random variables $X_1, \cdots, X_n$ are NA if and only if for any disjoint subsets $A, B \subset \{1, \cdots, n\}$ and nondecreasing, binary functions $u: \boldsymbol{R}^A \to \boldsymbol{R}$, $v: \boldsymbol{R}^B \to \boldsymbol{R}$,*

$$\mathrm{Cov}(u(X_j, j \in A), v(X_j, j \in B)) \leqq 0, \tag{1.3}$$

where a binary function is a function, which takes only two values 0 *and* 1.

Proposition 1.3. *If for every disjoint subsets $A, B \subset \{1, \cdots, n\}$ and bounded, continuous and nondecreasing function $f: \boldsymbol{R}^A \to \boldsymbol{R}$, $g: \boldsymbol{R}^B \to \boldsymbol{R}$*

$$\mathrm{Cov}(f(X_j, j \in A), g(X_j, j \in B)) \leqq 0,$$

then

$$\mathrm{Cov}(u(X_j, j \in A), \nu(X_j, j \in B)) \leqq 0,$$

for every binary, upper continuous and nondecreasing functions $u: \boldsymbol{R}^A \to \boldsymbol{R}$, $v: \boldsymbol{R}^B \to \boldsymbol{R}$.

Propositions 1.2 and 1.3 may be proved in the same manner as Theorem 3.1 and Lemma 3.2 of [4], so we omit details.

Proposition 1.4. *Assume that for any disjoint subsets* A, $B\subset\{1, \cdots, n\}$ *and bounded, continuous and nondecreasing functions* $f: \boldsymbol{R}^A\to\boldsymbol{R}$, $g: \boldsymbol{R}^B\to\boldsymbol{R}$

$$\operatorname{Cov}(f(X_j,\ j\in A),\ g(X_j,\ j\in B))\leqq 0\,,$$

then $X_1, \cdots, X_n$ *are NA.*

Proof. Our proof is similar to the proof of Theorem 3.3 of [4], so we only sketch it. Let A, B be such as above, by virtue of Proposition 1.2 it suffices to prove (1.3) for every binary, nondecreasing functions $u: \boldsymbol{R}^A\to\boldsymbol{R}$, $v: \boldsymbol{R}^B\to\boldsymbol{R}$. Let us put $U=\{x\in\boldsymbol{R}^A: u(x)=1\}$, $V=\{x\in\boldsymbol{R}^B: v(x)=1\}$. For any $\varepsilon>0$ there exist compact sets $K_u\subset U$, $K_v\subset V$, such that $P(U\setminus K_u)\leqq\varepsilon$, $P(V\setminus K_v)\leqq\varepsilon$. Now take

$$K_u'=\{x+t: x\in K_u,\ t\in\boldsymbol{R}^A,\ t\geqq 0\}\,,$$

$$K_v'=\{x+t: x\in K_v,\ t\in\boldsymbol{R}^B,\ t\geqq 0\}\,.$$

Functions

$$u'(x)=\begin{cases} 1, & \text{if } x\in K_u' \\ 0, & \text{if } x\notin K_u' \end{cases}$$

$$v'(x)=\begin{cases} 1, & \text{if } x\in K_v' \\ 0, & \text{if } x\notin K_v' \end{cases}$$

are binary, nondecreasing, upper continuous on $\boldsymbol{R}^A$ and $\boldsymbol{R}^B$ respectively, thus from our assumptions and Proposition 1.3 follows that

$$\operatorname{Cov}(u'(X_j,\ j\in A),\ v'(X_j,\ j\in B))\leqq 0\,. \tag{1.4}$$

We have

$$\begin{aligned} &|\operatorname{Cov}(u(X_j,\ j\in A),\ v(X_j,\ j\in B))-\operatorname{Cov}(u'(X_j,\ j\in A),\ v'(X_j,\ j\in B))| \\ &\quad=|Euv-EuEv-Eu'v'+Eu'Ev'| \\ &\quad=|Ev(u-u')+Eu'(v-v')+EuE(v'-v)+Ev'E(u'-u)| \\ &\quad\leqq 2E|u-u'|+2E|v-v'|\leqq 2P(U\setminus K_u)+2P(V\setminus K_v)\leqq 4\varepsilon\,. \end{aligned} \tag{1.5}$$

Combining (1.4) with (1.5) we get

$$\operatorname{Cov}(u,\ v)\leqq\operatorname{Cov}(u',\ v')+4\varepsilon\leqq 4\varepsilon\,,$$

since ε is arbitrarily close to 0, this proves (1.3).

From Proposition 1.4 and the Helly-Bray theorem we get the following property of negatively associated random variables.

Proposition 1.5. *Let* $X_1^n, \cdots, X_k^n$ *be NA for every* $n\in\boldsymbol{N}$ *and* $[X_1^n, \cdots, X_k^n]$

$\xrightarrow[n\to\infty]{d} [X_1, \cdots, X_k]$, *then* $X_1, \cdots, X_k$ *are NA.*

As far as we know there are only two results concerning weak convergence of sums of negatively dependent r.v.'s—central limit theorems for strictly stationary sequences of LNQD or NA r.v.'s (cf. [7], Theorems 12 and 17). The aim of this paper is to extend these results to nonstationary case as well as to the invariance principle. In Section 2 we study the central limit problem for nonstationary sequences of LNQD r.v.'s and investigate the rate of convergence in the CLT. For strictly stationary sequences of NA r.v.'s we establish sufficient conditions for the invariance principle in the space $D[0, 1]$ of all real functions on $[0, 1]$, which have left hand side limits and are continuous from the right, endowed with the Skorohod topology. In Section 3 we are concerned with the invariance principles in $L^2[0, 1]$, developing the approach of Oliveira [8]. The proofs of our main theorems from Sections 2 and 3 are given in Section 4.

2. CLT, rate of convergence in CLT, invariance principle in $D[0,1]$

Let $EX_k=0$, $EX_k^2<\infty$, $k\in N$ and for $n\in N$ let us put $S_n=\sum_{k=1}^n X_k$, $\sigma_n^2=ES_n^2$. Similarly as for positively associated r.v.'s we introduce the following coefficient (cf. [2])

$$u(n)=\sup_{k\in N}\sum_{j:|j-k|\geq n}|\mathrm{Cov}(X_j, X_k)|\,, \qquad \text{for } n\in N\cup\{0\}\,.$$

We remark that for wide sense stationary sequence of negatively correlated r.v.'s $u(n)=2\sum_{k=n+1}^{\infty}|\mathrm{Cov}(X_1, X_k)|$ and by Lemma 8 of [7]

$$u(0)<\infty \quad \text{and} \quad \lim_{n\to\infty} u(n)=0\,. \tag{2.1}$$

In the stationary case define $\sigma^2=EX_1^2+2\sum_{k=2}^{\infty}\mathrm{Cov}(X_1, X_k)$, notice that $\sigma^2\in[0, EX_1^2]$, for $t\in[0, 1]$ define $W_n(t)=S_{[nt]}/(\sigma n^{1/2})$, where $S_0=0$.

Theorem 2.1. *Let* $(X_n)_{n\in N}$ *be a sequence of LNQD r.v.'s with* $EX_k=0$. $EX_k^2<\infty$, $k\in N$. *If*

$$\lim_{n\to\infty} u(n)=0\,, \qquad u(1)<\infty\,, \tag{2.2}$$

$$\inf_{n\in N} n^{-2}\sigma_n^2>0\,. \tag{2.3}$$

$$\lim_{n\to\infty}\sigma_n^{-2}\sum_{k=1}^{n}E(X_k^2 I[|X_k|\geq\varepsilon\sigma_n])=0\,, \qquad \textit{for any } \varepsilon>0\,, \tag{2.4}$$

then $S_n/\sigma_n \xrightarrow[n\to\infty]{d} N(0, 1)$.

When moments of higher than second order exsist we may deduce from Theorem 2.1 the following Corollary, which extends Theorem 12 of [7] from strictly to wide sense stationary sequences.

Corollary 2.1. *Let $(X_n)_{n\in N}$ be a wide sense stationary sequence of LNQD r.v.'s with $EX_1=0$, $E|X_1|^{2+\delta}<\infty$ for some $\delta>0$. If $\sigma>0$, then $S_n/\sigma_n \xrightarrow[n\to\infty]{d} N(0, 1)$.*

Proof. (2.2) follows from (2.1), we also have $\lim_{n\to\infty}\sigma_n^2/n=\sigma^2$, thus (2.3) holds, under given assumptions Lapunov condition is fulfilled implying the Lindeberg condition (2.4).

Theorem 2.2. *Let $(X_n)_{n\in N}$ be a sequence of LNQD r.v,'s such that $EX_k=0$, $E|X_k|^3<\infty$, for $k\in \boldsymbol{N}$. For $n=kp$, with k, $p\in \boldsymbol{N}$ we have:*

(2.5) $$\sup_{x\in R}|P[S_n/\sigma_n\leqq x]-\Phi(x)|$$

$$\leqq C_1\frac{n\left(\sum_{l=1}^{k}\operatorname{Var}\xi_l\right)^2}{\left(\sum_{l=1}^{k}E|\xi_l|^3\right)^2}\frac{1}{p}\sum_{j=1}^{p}u(j)+C_2\frac{\sum_{l=1}^{k}E|\xi_l|^3}{\left(\sum_{l=1}^{k}\operatorname{Var}\xi_l\right)^{3/2}}+C_3\frac{\sum_{l=1}^{k}E|\xi_l|^3}{\sigma_n\sum_{l=1}^{k}\operatorname{Var}\xi_l}$$

where $\xi_l=\sum_{j=n(l-1)+1}^{pl}X_j$, $l=1, 2, \cdots, k$.

Positive constants C_1, C_2, C_3 do not depend on n, k, p and $\Phi(x)$ denotes the standard normal distributio.

Imposing some regularity conditions on moments $E(S_{n+m}-S_m)^2$, $E|S_{n+m}-S_m|^3$ we may obtain more clear bounds on the rate of convergence, but they are still far from optimal estimate $O(n^{-1/2})$ and the best convergence rate which may be obtained from Theorem 2.2 is $O(n^{-1/5})$ (compare with [3], [10]).

Corollary 2.2. *Let $(X_n)_{n\in N}$ be a sequence of LNQD r.v.'s such that $EX_k=0$, $E|X_k|^3<\infty$, for $k\in \boldsymbol{N}$. If*

(2.6) $$\inf_{m\in N\cup\{0\}} E(S_{n+m}-S_m)^2\geqq B_1 n\,,$$

(2.7) $$\sup_{m\in N\cup\{0\}} E|S_{n+m}-S_m|^3\leqq B_2 n^{3/2}, \qquad \text{for positive } B_1,\ B_2,$$

(2.8) $$\textstyle\sum_{j=1}^{\infty}u(j)<\infty\,,$$

then

(2.9) $$\sup_{x\in R}\left|P[S_n/\sigma_n\leqq x]-\Phi(x)\right|=O(n^{-1/5})\,.$$

Proof. Under conditions (2.6), (2.7) and (2.8) from Theorem 2.2 we get

(2.10) $$\sup_{x\in R}\left|P[S_n/\sigma_n\leqq x]-\Phi(x)\right|\leqq C_1'\frac{k}{p}+C_2'\frac{1}{k^{1/2}}\,,$$

where C_1', C_2' are absolute constants. It is easy to see that the best estimation in (2.10) we have for $k(n)=n^{2/5}$ and $p(n)=n^{3/5}$, thus (2.9) is proved.

In the next theorem we present an extension of the CLT to the invariance principle.

Theorem 2.3. *Let $(X_n)_{n\in N}$ be a strictly stationary sequence of NA r.v.'s such that $EX_1=0$, $E|X_1|^{2+\delta}<\infty$ for some $\delta>0$. If moreover $\sigma>0$, then $W_n \xrightarrow[n\to\infty]{d} W$ in $D[0, 1]$, where W denotes the Wiener measure on $D[0, 1]$.*

3. Tightness and invariance principles in $L^2[0,1]$

In this paragraph we study weak convergence in $L^2[0, 1]$ of processes $W_n(t)=S_{[nt]}/(\sigma n^{1/2})$ and $\overline{W}_n(t)=S_{m_n(t)}/k_n^{1/2}$, where $m_n(t)=\max(i: t\geqq k_i/k_n)$ and $(k_n)_{n\in N\cup\{0\}}$ fulfills the following condition

$$\text{(3.1)} \qquad 0=k_0\leqq k_1\leqq \cdots,\ k_n \longrightarrow \infty,\ \max_{1\leq i\leq n}(k_i-k_{i-1})/k_n \longrightarrow 0, \qquad \text{as } n\to\infty.$$

In the sequel we follow the idea of Oliveira [8]. Let us remind some fundamental concepts (for more details see [8] and references given therein). Introduce an auto-reproducing Hibert space H_R, defined by the kernel $R(s, t)=1-\max(s, t)$. Functions $f\in H_R$ are of the form $f(u)=\int_u^1 g(t)dt$ for some $g\in L^2[0, 1]$. scalar product in H_R is given by $\langle f_1, f_2\rangle_R=\int_0^1 g_1(t)g_2(t)dt$, where $f_i(u)=\int_u^1 g_i(t)dt$, $i=1, 2$. H_R is isometrically isomorph to $L^2[0, 1]$ by the isomorphism

$$\text{(3.2)} \qquad \Psi: L^2[0, 1] \longrightarrow H_R,\ g \longrightarrow \int_u^1 g(t)dt.$$

The space of Borel, bounded, signed measures μ on $[0, 1]$ may be embedded in H_R by the function $\varphi(\mu)(s)=\int_0^1 R(s, t)d\mu=\int_s^1 \mu[0, u]du$, φ has the following property:

$$\text{(3.3)} \qquad \langle f, \varphi(\mu)\rangle_R=\int_0^1 g(u)\mu[0, u]du=\int_0^1 f(u)d\mu(u), \qquad \text{where } f(u)=\int_u^1 g(t)dt.$$

Define random measures

$$\mu_n=\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^n X_i\delta(i/n), \qquad \text{then } \mu[0, u]=\frac{1}{\sigma\sqrt{n}}S_{[nu]},$$

$$\bar{\mu}_n=\frac{1}{\sqrt{k_n}}\sum_{i=1}^n X_i\delta(k_i/k_n), \qquad \text{then } \bar{\mu}_n[0, u]=\frac{1}{\sqrt{k_n}}S_{m_n(u)}.$$

In view of isometry Ψ and embedding φ, we interpret μ_n or $\bar{\mu}_n$ as elements of H_R. Let us also take an orthonormal basis $G_k=\sqrt{\lambda_k}g_k$, where $\lambda_k=((k+1/2)\pi)^{-2}$, $g_k(t)=\cos(k+1/2)t\pi$.

We have the following tightness criterion in $L^2[0, 1]$.

Theorem 3.1. *Suppose that* $(X)_{n\in N}$ *is a sequence of r.v.'s, such that* $EX_n=0$, $EX_n^2<\infty$, $n\in \boldsymbol{N}$, *and satisfying condition*

$$\sup_{n\in N} k_n^{-1}\Big(\sum_{k=1}^{n} EX_k^2+\sum_{1\leq i\neq j\leq n}|EX_iX_j|\Big)\leqq B\,, \tag{3.4}$$

for some nonnegative constant B. Then the set $\{\overline{W}_n(t),\ n\in\boldsymbol{N}\}$ *is weakly relatively compact in* $L^2[0, 1]$.

When $k_n=\sigma^2 n$. then the above theorem reduces to Theorem 3.1 of [8]. We remark, that processes $W_n(t)$ correspond to the stationary case, while it is more appropriated to study $\overline{W}_n(t)$ in nonstationary cases (see [6], where further references are given). In the stationary case we have the following invariance principle in $L^2[0, 1]$.

Theorem 3.2. *Let* $(X_n)_{n\in N}$ *be a strictly stationary sequence of LNQD r.v.'s with* $EX_1=0$ *and* $EX_1^2<\infty$, *if* $\sigma>0$, *then* $W_n\xrightarrow[n\to\infty]{d}W$ *in* $L^2[0, 1]$.

In nonstationary case we get the following extension of Theorem 2.1 to weak invariance principle.

Theorem 3.3. *Let* $(X_n)_{n\in N}$ *be a sequence of LNQD r.v.'s with* $EX_k=0$, $EX_k^2<\infty$, $k\in\boldsymbol{N}$, *assume that conditions* (2.2), (2.3) *and* (2.4) *are satisfied, then* $\overline{W}_n\xrightarrow[n\to\infty]{d}W$ *in* $L^2[0, 1]$ *with* $k_n=\sigma_n^2$, *provided that* (3.1) *is fulfilled for such chosen* k_n.

Using the representation of characteristic functionals in $L^2[0, 1]$ we may derive the following corollary concerning stochastic integrals (see also Corollary 4.5 of [8]).

Corollary 3.1. *Let* $F(u)=\int_u^1 f(t)dt$ *for some* $f\in L^2[0, 1]$.

(i) *If assumptions of Theorem* 3.2 *are satisfied, then*

$$\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n}X_iF(i/n)\xrightarrow[n\to\infty]{d}\int_0^1F(u)dW(u).$$

(ii) *Under assumptions of Theorem* 3.3

$$\sigma_n^{-1}\sum_{i=1}^{n}X_iF(\sigma_i^2/\sigma_n^2)\xrightarrow[n\to\infty]{d}\int_0^1F(u)dW(u).$$

4. Proofs

Proof of Theorem 2.1. Since the inequality (1.2) holds and r.v.'s are negatively correlated, we may follow the proof ofTheorem 3 of [2], so we omit it.

Proof of Theorem 2.2. We shall modify the idea of [10]. One can easily get

$$(4.1)\qquad |P[S_n/\sigma_n \leqq x]-\Phi(x)| \leqq \pi^{-1}(I_1+I_2)+C/T,$$

for every $x\in \boldsymbol{R}$, $T>0$ and some constant $C>0$, where

$$I_1=\int_{-T}^{T}\Big|Ee^{itS_n/\sigma_n}-\prod_{l=1}^{k}Ee^{it\xi_l/\sigma_n}\Big|\Big/|t|\,dt,$$

$$I_2=\int_{-T}^{T}\Big|\prod_{l=1}^{k}Ee^{it\xi_l/\sigma_n}-e^{-t^2/2}\Big|\Big/|t|\,dt.$$

We have

$$(4.2)\qquad \Big|1-\sigma_n^{-2}\sum_{l=1}^{k}\mathrm{Var}(\xi_l)\Big|=2\sigma_n^{-2}\Big|\sum_{l=1}^{k-1}\mathrm{Cov}\Big(\xi_l,\sum_{m=l+1}^{k}\xi_m\Big)\Big|$$

$$\leqq 2\sigma_n^{-2}k\sum_{j=1}^{p}u(j)\leqq 2\sigma_n^{-2}np^{-1}\sum_{j=1}^{p}u(j).$$

Using inequality (1.2) and (4.2) we get the following estimation of I_1.

$$(4.3)\qquad I_1=\sigma_n^{-2}\int_{-T}^{T}\sum_{1\leq i\neq j\leq k}|\mathrm{Cov}(\xi_i,\xi_j)|t^2/|t|\,dt$$

$$=\frac{T^2}{2}\sigma_n^{-2}\Big|\sigma_n^2-\sum_{l=1}^{k}\mathrm{Cov}(\xi_l)\Big|\leqq T^2\sigma_n^{-2}np^{-1}\sum_{j=1}^{p}u(j).$$

In order to estimate I_2 we set $t=s\sigma_n\bar{\sigma}_k^{-1}$, where $\bar{\sigma}_k=\Big(\sum_{l=1}^{k}\mathrm{Var}\,\xi_l\Big)^{1/2}$, then we obtain

$$(4.4)\qquad I_2=\int_{-T\bar{\sigma}_k/\sigma_n}^{T\bar{\sigma}_k/\sigma_n}\Big|\prod_{l=1}^{k}E\exp(is\xi_l\bar{\sigma}_k^{-1})-\exp\Big(-\frac{s^2}{2}\sigma_n^2\bar{\sigma}_k^{-2}\Big)\Big|\Big/|s|\,ds$$

$$\leqq I_3+I_4,$$

where

$$I_3=\int_{-T\bar{\sigma}_k/\sigma_n}^{T\bar{\sigma}_k/\sigma_n}\Big|\exp\Big(-\frac{s^2}{2}\Big)-\exp\Big(-\frac{s^2}{2}\sigma_n^2\bar{\sigma}_k^{-2}\Big)\Big|\Big/|s|\,ds.$$

$$I_4=\int_{-T\bar{\sigma}_k/\sigma_n}^{T\bar{\sigma}_k/\sigma_n}\Big|\prod_{l=1}^{k}E\exp(is\xi_l\bar{\sigma}_k^{-1})-\exp\Big(-\frac{s^2}{2}\Big)\Big|\Big/|s|\,ds.$$

Standard inequality $|e^{-x}-e^{-y}|\leqq|x-y|$, which holds for every x, $y\geqq 0$ and (4.2) yields

$$(4.5)\qquad I_3\leqq\int_0^{T\bar{\sigma}_k/\sigma_n}s^2|1-\sigma_n^2\bar{\sigma}_k^{-2}|/|s|\,ds$$

$$=\frac{T^2}{2}\bar{\sigma}_k^2\sigma_n^{-2}|1-\sigma_n^2\bar{\sigma}_k^{-2}|\leqq T^2\sigma_n^{-2}np^{-1}\sum_{j=1}^{p}u(j).$$

Applying Lemma 1 from Section 5.2 of [9] we get

$$(4.6)\qquad \left|\prod_{l=1}^{k} E\exp(is\xi_l\bar{\sigma}_k^{-1})-\exp\left(-\frac{s^2}{2}\right)\right|\leqq 16\left(\sum_{l=1}^{k}\operatorname{Var}\xi_l\right)^{-3/2}\sum_{l=1}^{k}E|\xi_l|^3|s|^3e^{-s^2/3},$$

for

$$|s|\leqq\frac{1}{4}\left(\sum_{l=1}^{k}\operatorname{Var}\xi_l\right)^{3/2}\left(\sum_{l=1}^{k}E|\xi_l|^3\right)^{-1},$$

hence, if

$$(4.7)\qquad T\leqq\frac{1}{4}\sigma_n\sum_{l=1}^{k}\operatorname{Var}\xi_l\left(\sum_{l=1}^{k}E|\xi_l|^3\right)^{-1},$$

then from (4.6) we obtain

$$(4.8)\qquad I_4\leqq C\left(\sum_{l=1}^{k}\operatorname{Var}\xi_l\right)^{-3/2}\sum_{l=1}^{k}E|\xi_l|^3,$$

for some constant C, which do not depend on n, p, k. Now, from (4.3), (4.4), (4.5), (4.7) and (4.8) we get the desired conclusion.

Proof of Theorem 2.3. Under our conditions, from the Corollary to the Theorem 4 of [5], follows that

$$(4.9)\qquad \sup_{m\in N\cup\{0\}} E|S_{n+m}-S_m|^{2+\delta}\leqq Cn^{1+\delta/2}.$$

Thus by inequality (12.42) and Theorem 12.2 of [1] with $\gamma=2+\delta$, $\alpha=1+\delta/2$ and $u_1=\cdots=u_n=C^{-\delta/2}$, we get

$$(4.10)\qquad P[\max_{0\leq k\leq n}|S_k|\geqq\lambda]\leqq(K_1/\lambda^{2+\delta})Cn^{1+\delta/2},$$

where K_1 is positive and depends only on δ, λ is an arbitrary positive real number. Replacing λ by $\lambda\sigma\sqrt{n}$ we obtain

$$(4.11)\qquad P[\max_{0\leq k\leq n}|S_k|\geqq\lambda\sigma\sqrt{n}]\leqq K_2/\lambda^{2+\delta}.$$

where K_2 is a positive constant. From stationarity and (4.11) follows that the assumptions of Theorem 8.4 of [1] are satisfied, thus $(W_n)_{n\in N}$ is tight. Let us take a sequence $0\leqq t_0\leqq t_1\leqq\cdots\leqq t_r\leqq 1$, then using stationarity and the central limit theorem, which is satisfied under our assumptions, we obtain

$$(4.12)\qquad Y_{n,i}:=W_n(t_{i+1})-W_n(t_i)\xrightarrow[n\to\infty]{d}N(0,\, t_{i+1}-t_i)\qquad \text{for any } i=0,\cdots,r-1.$$

We also easily verify that $\operatorname{Cov}(Y_{n,i}, Y_{n,j})\to 0$ as $n\to\infty$ for $i\neq j$, if $[Y_1,\cdots,Y_n]$ is a weak limit of any subsequence of $[Y_{n,1},\cdots,Y_{n,r}]$, then by Remark 1.1 from Section 1, $Y_1,\cdots,Y_n$ are independnt. We may conclude that finite dimensional distributions of W_n converge to those of W and the proof is completed.

Proof of Theorem 3.1. It suffices to prove that

$$\sup_{n\in N}\int_{H_R}\sum_{i=N}^{\infty}\langle F, G_j\rangle_R^2 P_n(dF)\longrightarrow 0, \qquad \text{as } N\to\infty, \tag{4.13}$$

where P_n is a distribution of $\bar{\mu}_n$ on H_R.

Applying (3.2), (3.3) and (3.4) we get:

$$\begin{aligned}(4.14)\qquad &\left|\int_{H_R}\sum_{i=N}^{\infty}\langle F, G_i\rangle_R P_n(dF)\right| \\ &=\left|E\Big(\sum_{i=N}^{\infty}\langle\varphi(\bar{\mu}_n), G_i\rangle_R^2\Big)\right|=\left|E\Big(\sum_{i=N}^{\infty}\Big(\int_0^1 G_i(u)d\bar{\mu}_n(u)\Big)^2\Big)\right| \\ &=\left|\sum_{i=N}^{\infty}\lambda_i k_n^{-1}\Big(\sum_{j,m=1}^{n}g_i(k_j/k_n)g_i(k_m/k_n)EX_jX_m\Big)\right| \\ &\leqq\left|\sum_{i=N}^{\infty}\lambda_i k_n^{-1}\Big(\sum_{j,m=1}^{n}|g_i(k_j/k_n)||g_i(k_m/k_n)||EX_jX_m|\Big)\right|\leqq\sum_{i=N}^{\infty}\lambda_i B,\end{aligned}$$

using $|g_i(t)|\leqq\|g_i\|_\infty=1$, for every $t\in[0, 1]$.

The convergence of $\sum_{i=1}^{\infty}\lambda_i$ and (4.14) yield (4.13).

Proof of Theorem 3.2. Under given conditions (2.1) holds, thus Corollary 3.2 of [8] implies tightness. To end the proof we establish stronger result, namely the convergence of marginal distributions

$$[W_n(t_1), \cdots, W_n(t_k)]\xrightarrow[n\to\infty]{d}[W(t_1), \cdots, W(t_k)],$$

but it may be similarly done as in the end of the proof of Theorem 2.3 so we omit details.

Proof of Theorem 3.3. Random variables are negatively correlated, so have

$$\sigma_n^{-2}\Big(\sum_{k=1}^{n}EX_k^2+\sum_{1\leq i\neq j\leq n}|EX_iX_j|\Big)=1+2\sigma_n^{-2}\sum_{1\leq i\neq j\leq n}|EX_iX_j|\leqq 1+2\sigma_n^{-2}nu(1) \tag{4.15}$$

applying (2.2) and (2.3) we easily see that from (4.15) follows (3.4) and tightness is proved. Now we shall use similar argument as in the proof of Theorem 3.2. For any $0\leqq t_1\leqq t_2\leqq t_3\leqq t_4\leqq 1$, using (2.1) and (2.2), we obtain

$$|\sigma_n^{-2}E(S_{m_n(t_4)}-S_{m_n(t_3)})(S_{m_n(t_2)}-S_{m_n(t_1)})|\leqq\sigma_n^{-2}\sum_{i=1}^{n}u(i)\longrightarrow 0, \qquad \text{as } n\to\infty. \tag{4.16}$$

We easily check that

$$\sigma_{m_n(t)}^2/\sigma_n^2\longrightarrow t, \qquad \text{as } n\to\infty, \quad \text{for every } t\in[0, 1]. \tag{4.17}$$

From (4.17) and Theorem 2.1 follows that

(4.18) $$\sigma_n^{-1} S_{m_n(t)} \xrightarrow[n\to\infty]{d} N(0, t) \text{ and moreover}$$

(4.19) the sets $\{\sigma_n^{-1} S_{m_n(t)},\ n \in \boldsymbol{N}\}$, $\{\sigma_n^{-1} S_{m_n(t)} S_{m_n(s)},\ n \in \boldsymbol{N}\}$ are uniformly integrable for any t, $s \in \boldsymbol{N}$. We shall prove

(4.20) $$\sigma_n^{-1}(S_{m_n(t)} - S_{m_n(s)}) \xrightarrow[n\to\infty]{d} N(0, t-s), \qquad \text{for } 0 < s < t \leqq 1,$$

by (4.18) $\{(\sigma_n^{-1} S_{m_n(s)}, \sigma_n^{-1} S_{m_n(t)}),\ n \in \boldsymbol{N}\}$ is tight, thus $(\sigma_n^{-1} S_{m_n(s)}, \sigma_n^{-1} S_{m_n(t)}) \xrightarrow[n\to\infty]{d} (X, Y)$, for subsequence, of course X and Y have normal distributions with mean 0 and variance s and t, respectively. Furthermore $(\sigma_n^{-1} S_{m_n(s)}, \sigma_n^{-1}(S_{m_n(t)} - S_{m_n(s)})) \xrightarrow[n\to\infty]{d} (X, Y-X)$. $\sigma_n^{-1} S_{m_n(s)}$ and $\sigma_n^{-1}(S_{m_n(t)} - S_{m_n(s)})$ are LNQD, thus X and $Y-X$ are LNQD. By (4.16) and (4.19) $\mathrm{Cov}(X, Y-X) = \lim_{n\to\infty} \mathrm{Cov}(\sigma_n^{-1} S_{m_n(s)}, \sigma_n^{-1}(S_{m_n(t)} - S_{m_n(s)})) = 0$, thus X and $Y-X$ are independent, so $Y-X$ has normal distribution with mean 0 and variance $t-s$, and (4.20) is proved. (4.20) together with (4.16) completes the proof.

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