YOKOHAMA MATHEMATICAL
JOURNAL VOL. 41, 1994

A NOTE ON THE SECTIONAL CURVATURE OF LEGENDRE FOLIATIONS

By

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(Received January 19, 1993; Revised June 10, 1993)

1. Introduction

A *Legendre foliation* is a foliation of a $(2n+1)$-dimensional contact manifold (M, η) by n-dimensional integral submanifolds of η. This paper investigates on a Legendre foliation the sectional curvature of non-degenerate plane sections containing ξ. §2 is devoted to the preliminaries on contact metric manifolds and Legendre foliations. In §3 a formula for the sectional curvature for any non-degenerate plane section containing ξ and its simplification for non-degenerate Legendre foliations on the canonical contact manifold are found. In §4, the major section of this paper, we investigate Legendre foliations on contact metric manifolds with constant ξ-sectional curvature, that is, the sectional curvature $K(W, \xi)$ is constant for all nonnull vector fields W. Contact metric manifolds where the curvature tensor R satisfies $R(W, W')\xi=\varpi(g(W', \xi)W-g(W, \xi)W')$ for some real number ϖ and any vector fields W, W' on M, have been studied by D. E. Blair and S. Tanno. It can be shown that these manifolds have constant ξ-sectional curvature, also there are naturally defined Legendre foliations on these contact manifolds. In Theorem 4.4 we prove that when $\varpi<1$ and $\varpi\neq 0$ the Legendre foliations naturally defined are non-degenerate and the contact metric structure is not the canonical one. In Theorem 4.6 we generalise this result to Legendre foliation on contact metric manifolds with constant ξ-sectional curvature with the added condition that $hX=\mu X$ where X is a tangential vector field. Finally in Theorem 4.7 we prove that for a non-degenerate Legendre foliation on the canonical contact metric manifold with constant ξ-sectional curvature, ξ-sectional curvature of zero or one.

2. Preliminaries

A *contact manifold* (M, η) has a *contact metric structure* (ϕ, ξ, η, g) where ϕ is a tensor of type (1, 1), ξ is a global vector field and g is a semi-Riemannian

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* 1991 Mathematics Subject Classification: 53C12, 53C15.
Key words and phrases: Contact (Manifolds and metric structures), Legendre foliations.

metric such that

(2.1) $\eta(\xi)=1$, $\phi^2=-I+\eta\otimes\xi$, $g(W, \phi W')=d\eta(W, W')$ $\forall W, W'\in\Gamma TM$.

A contact metric structure (ϕ, ξ, η, g) usually has g positive definite but we do not make this restriction and allow g to be any semi-Riemannian metric compatible with the contact structure. A *contact metric manifold* (M, ϕ, ξ, η, g) is a contact manifold (M, η) with (ϕ, ξ, η, g) as its contact metric structure. We will assume that ξ, the *characteristic vector field*, is spacelike, that is $g(\xi, \xi)=1$. From this and the properties of (2.1) it can be shown that

(2.2) (i) $\phi\xi=0$,

(ii) $\eta(W)=g(\xi, W)$ $\forall W, W'\in\Gamma TM$,

(iii) $g(\phi W, \phi W')=g(W, W')-\eta(W)\eta(W')$ $\forall W, W'\in\Gamma TM$,

(iv) $g(\phi W, W')+g(W, \phi W')=0$ $\forall W, W'\in\Gamma TM$.

Other properties of (ϕ, ξ, η, g) are

(2.3) $$\nabla_\xi\phi=0 \quad\text{and}\quad \nabla_\xi\xi=0.$$

An operator h is defined by $h=1/2(\mathcal{L}_\xi\phi)$, where $\mathcal{L}$ denotes Lie differentiation, it can be shown, see Blair [1] and [2], that h satisfies

(2.4) (i) $h\xi=0$,

(ii) $\nabla_W\xi=-\phi hW-\phi W$ $\forall W, W'\in\Gamma TM$,

(iii) h and ϕh are symmetric operators,

(iv) $\phi h+h\phi=0$,

where ∇ is the Levi-Civita connection of g. ξ is a Killing vector field if and only if h vanishes and a contact metric manifold with ξ a Killing vector field is called a *K-contact manifold*.

Since the leaves of a Legendre foliation $\mathcal{F}$ are integral submanifolds of η, the *tangent bundle*, L, of $\mathcal{F}$ is a subbundle of H, where $H=\text{ann}\{\eta\}$ is the *contact distribution* of (M, η). For a Legendre foliation $\mathcal{F}$ on a contact manifold we introduce a contact metric structure, (ϕ, ξ, η, g), and define $\mathcal{F}$ on the resultant contact metric manifold. This contact metric structure allows us to define $L^\perp=\{Y\in TM \mid g(Y, X)=0,\ \forall X\in L\}$ *the transverse bundle of* $\mathcal{F}$ and also $Q=H\cap L^\perp$ subbundle of H.

In Jayne [3] it is shown that the leaves of a Legendre foliation on a contact metric manifold are anti-invariant submanifolds which gives us

(2.5) (i) $\phi X\in\Gamma Q$ $\forall X\in\Gamma L$

(ii) $\phi Y\in\Gamma L$ $\forall Y\in\Gamma Q$.

Furthermore if $\{X_\alpha\}$ is a local orthonormal frame for L with respect to g then $\{\phi X_\alpha\}$ is a local orthonormal frame for Q with respect to g.

For a Legendre foliation, $\mathcal{F}$, on a contact manifold (M, η), Pang [4] introduced on L a symmetric, bilinear form Π which can be defined by, Jayne [3],

$$\Pi(X, X')=2g([\xi, X], \phi X') \qquad \forall X, X' \in \Gamma L \tag{2.6}$$

where (ϕ, ξ, η, g) is any contact metric structure on (M, η). When Π is non-degenerate $\mathcal{F}$ is a non-degenerate Legendre foliation and when $\Pi\equiv 0$ is $\mathcal{F}$ is a flat Legendre foliation. For a non-degenerate Legendre foliation on a contact manifold (M, η) there exists a family of contact metric structures for (M, η) such that $g|_L=(1/4)\Pi$, Jayne [3]. Furthermore there exists a unique member of this family, (ϕ^c, ξ, η, g^c), called *the canonical contact metric structure* defined by the properties

(2.7) (i) $g^c|_L=\dfrac{1}{4}\Pi$,

(ii) $h^c: L \longrightarrow L$.

$(M, \phi^c, \xi, \eta, g^c)$ is called *the canonical contact metric manifold*. We also have the following equivalence

$$g|_L=\frac{1}{4}\Pi \Longleftrightarrow p_Q([\xi, X])=2\phi X \qquad \forall X \in \Gamma L, \tag{2.8}$$

where p_Q is the natural projection $p_Q: TM \to Q$.

A Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) where g is a bundle-like metric, that is

$$\mathcal{L}_X g(W, W')=0 \qquad \forall X \in \Gamma L,\ \forall W, W' \in \Gamma Q \oplus \Gamma E$$

is a *semi-Riemannian Legendre foliation*. Finally if Q is completely integrable then Q also defines a Legendre foliation, we call this $\bar{\mathcal{F}}$ *the conjugate Legendre foliation of* $\mathcal{F}$.

3. Sectional Curvature

For any contact metric manifold the following theorem gives us the sectional curvature of non-degenerate plane sections containing ξ.

Theorem 3.1. *For any contact metric manifold (M, ϕ, ξ, η, g) we have*

$$K(W, \xi)=\frac{g(W, W)+\xi g(\phi hW, W)-2g(\nabla_\xi W, \phi hW)-g(hW, hW)}{g(W, W)}$$

for any $W \in H$ such that $g(W, W)\neq 0$.

Proof. Using the torison free property of ∇, (2.3) and (2.4) it is a straightforward calculation to express the sectional curvature

$$K(W, \xi)=\frac{g(R(W, \xi)\xi, W)}{g(W, W)}$$

without $\nabla_W \xi$, and $[\,,\,]$, to obtain the required result. □

Note that if (M, ϕ, ξ, η, g) is a K-contact manifold, that is $h=0$, we have Theorem 3.1, part (2), Yano and Kon [6], Chapter 5, that for a K-contact manifold the sectional curvature for non-degenerate plane sections containing ξ are equal to 1 at every point of M.

Lemma 3.2. *Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with $g|_L=(1/4)\Pi$. Then for any $X\in\Gamma L$ and $Y\in\Gamma Q$*

$$g(\nabla_\xi X, Y)=g(\phi X-\phi hX, Y).$$

Proof. For any $X\in\Gamma L$ and $Y\in\Gamma Q$ we have

$$2g(\phi X, Y)=g([\xi, X], Y) \qquad \text{by (2.8) as } g|_L=\frac{1}{4}\Pi$$

$$=g(\nabla_\xi X, Y)+g(\phi X, Y)+g(\phi hX, Y) \qquad \text{by (2.4) (ii).}$$ □

By using this lemma when $\mathcal{F}$ is a non-degenerate Legendre foliation on the canonical contact metric manifold we obtain the following simplification of Theorem 3.1.

Theorem 3.3. *Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$. Then*

$$K(X_\alpha, \xi)=\frac{g^c(X_\alpha-h^cX_\alpha, X_\alpha-h^cX_\alpha)}{g^c(X_\alpha, X_\alpha)}$$

and

$$K(\phi^cX_\alpha, \xi)=\frac{g^c(X_\alpha+3h^cX_\alpha, X_\alpha-h^cX_\alpha)}{g^c(X_\alpha, X_\alpha)}$$

where $\{X_\alpha, \phi^cX_\alpha, \xi\}\,\alpha=1, \cdots, n$ is a local orthonormal frame of TM with respect to g^c.

Proof. As $(M, \phi^c, \xi, \eta, g^c)$ is the canonical contact metric structure by (2.5) (i) and (2.7) (ii) for any α, $\phi^c h^c X_\alpha\in\Gamma Q$, this enables us to use Lemma 3.2 in the proof of Theorem 3.1 to obtain

$$g^c(R(X_\alpha, \xi)\xi, X_\alpha)=g^c(X_\alpha, X_\alpha)-g^c(h^cX_\alpha, h^cX_\alpha,)$$
$$-2g^c(\nabla_\xi X_\alpha, \phi^c h^c X_\alpha)+\xi g^c(\phi^c h^c X_\alpha, X_\alpha)$$

$$=g^c(X_\alpha, X_\alpha)-g^c(h^cX_\alpha, h^cX_\alpha)-2g^c(\phi^cX_\alpha, \phi^ch^cX_\alpha)$$
$$+2g^c(\phi^ch^cX_\alpha, \phi^ch^cX_\alpha)$$
$$=g^c(X_\alpha-h^cX_\alpha, X_\alpha-h^cX_\alpha) \qquad \text{by (2.2) (iii).}$$

Similarly for any α we have

$$g^c(R(\phi^cX_\alpha, \xi)\xi, \phi^cX_\alpha)=g^c(X_\alpha+3h^cX_\alpha, X_\alpha-h^cX_\alpha) \qquad \square$$

4. Constant sectional curvature for plane sections, containing ξ

The sectional curvature of a plane section spanned by ξ and a mon-null vector orthogonal to ξ is called a *ξ-sectional curvature.* In this section we consider Legendre foliations on contact metric manifolds with constant ξ-sectional curvature. That is there exists a real number ϖ such that

$$K(W, \xi)=\varpi \qquad \forall W\in\Gamma H \text{ such that } g(W, W)\neq 0.$$

Theorems 4.4 and 4.6 use the following Lemma in their proofs.

Lemma 4.1. *Let $\mathcal{F}$ be a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with constant ξ-sectional curvature then for any α*

$$g(\nabla_\xi X_\alpha, \phi hX_\alpha)=0$$

where $\{X_\alpha, \phi X_\alpha, \xi\}\alpha=1, \cdots, n$ is a local orthonormal frame of TM with respect to g.

Proof. As the non-degenerate plane sections containing ξ have constant sectional curvature for any α we have.

$$0=g(X_\alpha, X_\alpha)\{K(X_\alpha, \xi)-K(\phi X_\alpha, \xi)\}$$
$$=\xi g(\phi hX_\alpha, X_\alpha)-2g(\nabla_\xi X_\alpha, \phi hX_\alpha)-\xi g(hX_\alpha, \phi X_\alpha)$$
$$+2g(\nabla_\xi\phi X_\alpha, hX_\alpha) \qquad \text{by Theorem 3..1}$$
$$=-4g(\nabla_\xi X_\alpha, \phi hX_\alpha) \qquad \text{by (2.2) (iv), (2.3), (2.5) (i) and (2.4) (iv)} \qquad \square$$

Tanno [5] investigated contact metric manifolds where the curvature tensor satisfied

$$R(W, W')\xi=\varpi(g(W', \xi)W-g(W, \xi)W') \qquad \forall W, W'\in\Gamma TM.$$

We will first investigate Legendre foliations on contact metric manifolds of this class. It is easily seen that these manifolds have constant ξ-sectional curvature. On these contact metric manifolds a Legendre foliation and its conjugate are naturally defined as given by the following proposition, which is a rewording of [5] Proposition 5.1.

Proposition 4.2. *Let (M, ϕ, ξ, η, g) be a contact metric manifold where for some real number ϖ the curvature of ∇, the Levi-Civita connection of g, satisfies*

$$R(W, W')\xi=\varpi(g(W', \xi)W-g(W, \xi)W') \qquad \forall W, W'\in\Gamma TM.$$

Then $\varpi\leqq 1$, also if $\varpi<1$ then (M, ϕ, ξ, η, g) admits three mutually orthogonal and completely integrable distributions $D(0)$, $D(\mu)$ and $D(-\mu)$ defined by the eigenspaces of h where $\mu=\sqrt{1-\varpi}$.

Note: $D(\mu)=\{W\in TM \mid hW=\mu W\}$, $D(-\mu)=\{W\in TM \mid hW=-\mu W\}$, $D(0)=E$.

Let $\mathcal{F}$ be the Legendre foliation defined by $D(\mu)$ then $\bar{\mathcal{F}}$ the conjugate Legendre foliation of $\mathcal{F}$ is defined by $D(-\mu)$.

When $\varpi=0$ we get the following theorem from Jayne [3], the proof of which adapts the proof of Blair [2] Theorem B, to Legendre foliations.

Theorem 4.3. *Let (M, ϕ, ξ, η, g) be a contact metric manifold such that*

$$R(W, W')\xi=0 \qquad \forall W, W'\in\Gamma TM.$$

Then $D(1)$ defines a semi-Riemannian Legendre foliation $D(-1)$ defines its conjugate which is flat and totally geodesic. Furthermore (ϕ, ξ, η, g) is the canonical contact metric structure of the Legendre foliation defined by $D(1)$.

The last theorem of Blair [1] gives the flat Legendre foliation defined by $D(-1)$, but Theorem 4.2 has strengthened the condition

$$R(Y, Y')\xi=0 \qquad \forall Y, Y'\in\Gamma D(-1)$$

to

$$R(W, W')\xi=0 \qquad \forall W, W'\in\Gamma H$$

this is a sufficient condition for $D(1)$ to also be completely integrable, hence $D(1)$ defines a Legendre foliation transverse to the Legendre foliation defined by $D(-1)$. Furthermore this Legendre foliation can be shown to be semi-Riemannian.

When $\varpi<1$ and $\varpi\neq 0$ we have the following theorem.

Theorem 4.4. *Let (M, ϕ, ξ, η, g) be a contact metric manifold such that*

$$R(W, W')\xi=\varpi(g(W', \xi)W-g(W, \xi)W') \qquad \forall W, W'\in\Gamma TM,$$

for some real number $\varpi<1$ and $\varpi\neq 0$. Then the Legendre foliations $\mathcal{F}$ defined by $D(\mu)$ and $\bar{\mathcal{F}}$ defined by $D(-\mu)$ are non-degenerate and (ϕ, ξ, η, g) is not the canonical contact metric structure for either $\mathcal{F}$ or $\bar{\mathcal{F}}$.

Proof. Let $\{X_\alpha\}$ be a local orthonormal frame for L with respect to g where $L=D(\mu)$ is the tangent bundle of $\mathcal{F}$. Then $\{\phi X_\alpha\}$ is a local orthonormal frame for $\bar{L}$ with respect to g where $\bar{L}=Q=D(-\mu)$ is the tangent bundle of $\bar{\mathcal{F}}$.

Therefore

$$hX_\alpha=\mu X_\alpha \quad \text{and} \quad h\phi X_\alpha=-\mu\phi X_\alpha . \tag{4.5}$$

As $\varpi<1$ and $\varpi\neq 0$ we have $\mu>0$ and $\mu\neq 1$ where $\mu=\sqrt{1-\varpi}$.

For any α by substituting (4.5) into (2.6) we obtain

$$\begin{aligned}\Pi(X_\alpha, X_\alpha)&=\frac{2}{\mu}(g[\xi, X_\alpha], \phi hX_\alpha)\\ &=2g((1+\mu)X_\alpha, X_\alpha) \quad \text{by (2.2) (iii), (2.4) (ii), Lemmas 4.1 and 4.5.}\end{aligned}$$

Therefore as $\mu>0$ $\Pi(X_\alpha, X_\alpha)\neq 0$ for any α hence $\mathcal{F}$ is a non-degenerate Legendre foliation. Furthermore as $\mu\neq 1$, $g|_L\neq(1/4)\Pi$ thus by (2.7) (i) (ϕ, ξ, η, g) is not the canonical contact metric structure for $\mathcal{F}$.

Now we let Π be the symmetric, bilinear form on $\bar{L}$ defined by (2.6), thus for any α, substituting (4.5) into (2.6) we obtain

$$\begin{aligned}\Pi(\phi X_\alpha, \phi X_\alpha)&=-\frac{2}{\mu}g([\xi, \phi X_\alpha], hX_\alpha)\\ &=2g((1-\mu)X_\alpha, X_\alpha)\\ &\qquad \text{by (2.2) (iii) \& (iv), (2.3), (2.4) (ii), Lemmas 4.1 and 4.5.}\end{aligned}$$

Therefor as $\mu\neq 1$ $\Pi(\phi X_\alpha, \phi X_\alpha)\neq 0$ for any α hence $\bar{\mathcal{F}}$ is a non-degenerate Legendre foliation. Furthermore as $\mu>0$ $g|_L\neq(1/4)\Pi$ thus by (2.7) (i) (ϕ, ξ, η, g) is not the canonical contact metric structure for $\bar{\mathcal{F}}$. □

In the following theorem we generalise Theorems 4.3 and 4.4.

Theorem 4.6. *Let $\mathcal{F}$ be a Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with constant ξ-sectional curvature such that for any $X\in L$ $hX=\mu X$ for some real number μ, then*

(i) *(M, ϕ, ξ, η, g) has ξ-sectional curvature of $\varpi=1-\mu^2$,*

(ii) *If: $\mu=0$ then (M, ϕ, ξ, η, g) is a K-contact manifold,*
$\mu=-1$ then $\mathcal{F}$ is a flat Legendre foliation,
$\mu=1$ then $\mathcal{F}$ is a non-degenerate Legendre foliation and (ϕ, ξ, η, g) is the canonical contact metric structure,
otherwise $\mathcal{F}$ is a non-degenerate Legendre foliation and (ϕ, ξ, η, g) is not the canonical contact metric structure.

Proof.

(i) $K(X_\alpha, \xi)$ is calculated from Theorem 3.1 using $hX=\mu X$, (2.5) (i) and Lemma 4.1.

(ii) If $\mu=0$ then $h\equiv 0$ thus (M, ϕ, ξ, η, g) is a K-contact manifold.
Suppose $\mu\neq 0$, let $\{X_\alpha\}$ be a local orthonormal frame for L with respect

to g, then for any α we have $\Pi(X_\alpha, X_\alpha)=2g((1+\mu)X_\alpha, X_\alpha)$ and the result follows directly from this. □

The example in [1] gives a flat Legendre foliation with zero ξ-sectional curvature which satisfies the conditions of Theorem 4.6, with $\mu=-1$, but not those of Theorem 4.3, thus Theorem 4.6 is a generalization of Theorem 4.3.

For a non-degenerate Legendre foliation on the canonical contact metric manifold with constan ξ-sectional curvature we get the following theorem and corollary.

Theorem 4.7. *Let $\mathcal{F}$ be a non-degenerate Legendre foliation on the canonical contact metric manifold $(M, \phi^c, \xi, \eta, g^c)$ such that the sectional curvature for non-degenerate plane sections containing ξ is constant. Then either*

(i) *$(M, \phi^c, \xi, \eta, g^c)$ has constant ξ-sectional curvature of one and if g^c is a Riemannian metric then $(M, \phi^c, \xi, \eta, g^c)$ is a K-contact manifold, or*

(ii) *$(M, \phi^c, \xi, \eta, g^c)$ has zero ξ-sectional curvature.*

Proof. Let $\{X_\alpha\}$ be a local orthonormal frame for L with respect to g^c.

As the non-degenerate plane sections containing ξ have constant sectional curvature, for any α we have,

$$\begin{aligned}
0&=g^c(X_\alpha, X_\alpha)\{K(X_\alpha, \xi)-K(\phi^c X_\alpha, \xi)\}\\
&=g^c(X_\alpha-h^cX_\alpha, X_\alpha-h^cX_\alpha)-g^c(X_\alpha+3h^cX_\alpha, X_\alpha-h^cX_\alpha) \qquad \text{by Theorem 3.3}\\
&=-4g^c(h^cX_\alpha, X_\alpha-h^cX_\alpha).
\end{aligned}$$

Therefore as $h^c: L\rightarrow L$ by (2.7) (ii) this implies that $g^c(X_\alpha, h^cX_\alpha)=g^c(h^cX_\alpha, h^cX_\alpha)=0$ or $h^cX_\alpha=X_\alpha$.

If $g^c(X_\alpha, h^cX_\alpha)=g^c(h^cX_\alpha, h^cX_\alpha)=0$, by substituting this in Theorem 3.3 we obtain $K(X_\alpha, \xi)=K(\phi^cX_\alpha, \xi)=1$. Also if g^c is a Riemannian metric $g^c(h^cX_\alpha, h^cX_\alpha)=0$ if and only if $h^c=0$, that is $(M, \phi^c, \xi, \eta, g^c)$ is a K-contact manifold hence (i).

If $h^cX_\alpha=X_\alpha$, by substituting this in Theorem 3.3 we obtain $K(X_\alpha, \xi)=K(\phi^cX_\alpha, \xi)=0$ hence (ii). □

Corollary 4.8. *Let $\mathcal{F}$ be a non-degenerate Legendre foliation on a contact metric manifold (M, ϕ, ξ, η, g) with constant ξ-sectional curvature not equal to one or zero, then (ϕ, ξ, η, g) is not the canonical contact metric structure.*

Corollary 4.8 is illustrated by the following example.

Example. Let $M=\boldsymbol{R}^3=\{(x, y, z)\}$ and $\eta=(\cos z dx+\sin z dy)/2$, thus (M, η) is a contact manifold with $\xi=2(\cos z(\partial/\partial x)+\sin z(\partial/\partial y))$. Then define a contact metric structure (M, ϕ, ξ, η, g) on (M, η) by

$$\phi=\begin{bmatrix} -\sin^2 z & \sin z\cos z & \sin z \\ \sin z\cos z & -\cos^2 z & -\cos z \\ -2\sin z & 2\cos z & 1 \end{bmatrix}$$

and g is the Riemannian metric such that $\{X, \phi X, \xi\}$ is a orthonormal frame of TM where $X=2(\partial/\partial z)$.

It is a straightforward calculation to show that (M, ϕ, ξ, η, g) has constant ξ-sectional curvature of -3 and that $L=\mathrm{span}\{X\}$ defines a non-degenerate Legendre foliation $\mathcal{F}$. Thus by Corollary 4.8 (ϕ, ξ, η, g) is not the canonical contact metric structure for $\mathcal{F}$.

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