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NONPARAMETRIC DENSITY ESTIMATIONS FOR A CLASS OF MARKED POINT PROCESSES

By

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Abstract. Let $\{X_z : z \in R^d\}$ be a strictly stationary real-valued random field and $\{N(A) : A \subset R^d\}$ a Poisson random measure which is independent of X. Assume that the distribution of X_z has a density function $f(x)$. Fix an observation-domain $V \subset R^d$. Suppose that we are to estimate the value $f(x)$ using the data observed on V. If the data are given as observations at counting points of $N(dz)$, then the following kernel density estimator is natural: $f_V(x) = (N(V)h_{N(V)})^{-1}\int_V K((x-X_z)/h_{N(V)})N(dz)$ where h_n is a band-width parameter. In this paper we discuss the central limit theorem and the convergence rate of the bias and the mean square error for $f_V(x)$ as the volume $|V|$ tends to ∞. In addition, we shall refer to the estimations of joint probability density functions.

1. Introduction

The marked point process is an important model for a wide variety of scientific disciplines. For the definition of marked point process the readers should refer to [1]. The aim of this paper is to estimate density functions related to a class of marked point processes. Among marked point processes a typical class can be described as follows:

$$\sum_i \delta(z_i, X_{z_i})$$

where z_i are realizations (locations) of the marginal point process on R^d, $\delta(A)$ the Dirac measure at the point $A \in R^d \times R^{d_0}$ and X_{z_i} random variables with values in R^{d_0} indexed by locations z_i. Consider the case where the random variables X_{z_i} are not independent. If we assume the invariance of joint probability distributions of $\{X_{z_i}\}$ under translation of locations, then the marked point process may be represented as randomly sampled data from a random field $X = \{X_z ; z \in R^d\}$ with strict stationarity.

Now let $f(x)$ (resp. $f(x, y : z_1, z_2)$) be the (resp. joint) probability density

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function (pdf) for X_z (resp. (X_{z_1}, X_{z_2}), $z_1 \neq z_2$)) with respect to the Lebesgue measure. Our purpose in this paper is to study some properties of kernel density estimators for the pdf $f(x)$ and joint pdf $f(x, y: z_1, z_2)$ which are formed from the observations based on a realization of the marked point process under the condition that the marginal point process is a Poisson one and is independent of marks (random field X).

Random sampling from a stochastic process ($d=1$) has been studied by several authors (for example, Stoyanov [13], Masry [7] and Takahata [14]). Among them, especially, Masry's work should be referred. Throughout the present paper we will treat only the case $d_0=1$ (the random variables X_z are real-valued). The extensions to multidimensional cases are easy. In this paper we discuss the central limit theorems and the convergence rates of bias and the mean square errors for estimators of $f(x)$ and $f(x, y: z_1, z_2)$. The main tool for proofs is the martingale theory. In the case $d \geqq 2$ it seems difficult to extend the results in this paper to non-Poisson cases. The difficulty of extensions is due to *randomness* of the band-width parameter $h_{N(V)}$ (see section 2). Ellis [2] introduced a kernel density estimator with non-random band-width parameters and a k-nearest neighbor estimator for a wide class of marked point processes, but these estimators can be applied only when the intensities of marginal point processes are constant and, are inconvenient to calculate when the observation-domain is not regular. We wish to emphasize the simplicity of our estimators.

2. Preliminaries

Let $X=\{X_z: z \in R^d\}$ and $N=\{N(A): A \in \mathcal{B}\}$ ($\mathcal{B}$ is the Borel field of R^d) be, respectively, a strictly stationary real-valued random field indexed by $z \in R^d$ and a Poisson point process on R^d with mean measure $m(dz)$, defined on a probability space $(\Omega, \mathcal{F}, P)$. Throughout this paper we assume that X and N are independent of each other. Let $f(x)$ be the probability density function of the distribution of X_z and $f(x, y: v)=f(x, y: z_1, z_2)$ with $v=z_2-z_1$ be the joint probability density function of the two-dimensional distribution of (X_{z_1}, X_{z_2}) w.r.t. the Lebesgue measure. For each finite set $A \subset R^d$ denote by $\mathcal{M}(A)$ the σ-field generated by the random variables X_z, $z \in A$ and by $\mathcal{H}(A)$ the Hilbert space of $\mathcal{M}(A)$-measurable random variables with second moment. For each finite sets $A_1, A_2 \subset R^d$, let us denote by $D(A_1, A_2)$ the usual euclidian distance between A_1 and A_2. When we discuss the problems related to the central limit theorem and the convergence rate of the mean square error for estimators of $f(x)$, the following mixing condition for X is assumed:

$$(2.1) \qquad \rho_1(r)=\sup_{A_1, A_2: D(A_1, A_2)=r} \ \sup_{X \in \mathcal{H}(A_1), Y \in \mathcal{H}(A_2)} |\mathrm{Corr}(X, Y)| \longrightarrow 0 \qquad \text{as } r \uparrow \infty,$$

where the cardinal number of A_i is two. On the other hand when we discuss the problems related to estimators of $f(x, y : v)$, another mixing condition is imposed. The mixing coefficient is defined as follows:

$$\rho_2(r) = \sup_{D(A_1, A_2)=r} \sup_{X \in \mathcal{H}(A_1), Y \in \mathcal{H}(A_2)} |\mathrm{Corr}(X, Y)| \longrightarrow 0 \qquad \text{as } r \uparrow \infty \tag{2.2}$$

where the cardinal number of A_i is four. Clearly $\rho_1(r) \leqq \rho_2(r)$.

Let $K(x)$ and $H(x, y)$ be probability density functions on R^1 and R^2, respectively, and $W(z)$ a probability density function on R^d with a *compact* support and h_n a band-width parameter with $h_n \downarrow 0$ and $nh_n \to \infty$. Suppose that we observe the quantities X_z at atoms of $N(dz)$ in a domain $V \subset R^d$. In considering the estimation of $f(x, y : v)$ we assume that $m(dz) = \lambda dz$ for a positive constant λ. Now define estimators for $f(x_0)$ and $f(x_1, x_2 : v)$.

(I) The estimator of $f(x_0)$:

$$f_V(x_0) = \frac{1}{N(V) h_{N(V)}} \int_V K\Big(\frac{X_z - x_0}{h_{N(V)}}\Big) N(dz), \qquad V \subset R^d. \tag{2.3}$$

(II) The estimator of $f(x_1, x_2 : v)$ (according to Masry [7]):

(2.4) $f_V(x_1, x_2 : v)$

$$= \frac{1}{\lambda N(V) h_{N(V)}^2} \int_V \int_V w_{N(V)}(v - (z_2 - z_1)) H\Big(\frac{X_{z_1} - x_1}{h_{N(V)}}, \frac{X_{z_2} - x_2}{h_{N(V)}}\Big) N(dz_1) N(dz_2),$$

where $w_n(z) = W(z/h_n)/h_n^d$. In these definitions, if $N(V)=0$ (resp. $N(V) \leqq 1$) then we define $f_V(x_0)=0$ (resp. $f_V(x_1, x_2 : v)=0$) for convenience. In the sequel the accuracies of $f_V(x_0)$ and $f_V(x_1, x_2 : v)$ as estimators of $f(x_0)$ and $f(x_1, x_2 : v)$ shall be discussed. Throughout this paper the following lemmas related to Poisson distributions are fundamental. The first two lemmas are well-known. For a positive number $x \in R^1$ denote by $[x]$ the integral part of x.

Lemma 2.1. *For a positive integer k and an observation-domain $V \in \mathcal{B}$ given, if $g(z)$ is a measurable function on R^d, then the distribution of $\int_V g(z) N(dz)$ under the condition $\{N(V)=k\}$ is the same as that of $\sum_{i=1}^k g(Z_i)$, where $\{Z_i, i=1, \cdots, k\}$ is a set of i.i.d. random variables with distribution $P(Z_1 \in A) = m(A)/m(V)$ for $A \in \mathcal{B}(\subset V)$.*

Lemma 2.2. *For a positive integer k and an observation-domain $V \in \mathcal{B}$ given, if $g(z_1, z_2)$ is a measurable function on $R^d \times R^d$ such that $g(z, z)=0$ for all $z \in R^d$, then the distribution of $\int_V \int_V g(z_1, z_2) N(dz_1) N(dz_2)$ under the condition $\{N(V)=k\}$ is the same as that of $\sum_{i \neq j}^k g(Z_i, Z_j)$, where $\{Z_i, i=1, \cdots, k\}$ is a set of i.i.d. random variables with distribution $P(Z_1 \in A) = m(A)/m(V)$ for $A \in \mathcal{B}(\subset V)$*

The following lemma is obtained easily from Theorem 1 in Chap. 8 of [9].

Lemma 2.3. *For a positive number k let $n_1=[k-\sqrt{k}\log k]$ and $n_2=[k+\sqrt{k}\log k]$. Then as $k\to\infty$*

$$e^{-k}\left(\sum_{n=0}^{n_1-1}\frac{k^n}{n!}+\sum_{n=n_2+0}^{\infty}\frac{k^n}{n!}\right)=O\left(\frac{1}{\log k}k^{-(\log k)/2}\right). \tag{2.5}$$

Corollary 2.4. *For a positive number k let $n_1=[k-\sqrt{k}\log k]$ and $n_2=[k+\sqrt{k}\log k]$. Then as $k\to\infty$*

$$e^{-k}\left(\sum_{n=0}^{n_1-1}\frac{k^n}{n!}+\sum_{n=n_2+1}^{\infty}\frac{k^n}{n!}\right)=o(k^{-c}) \tag{2.6}$$

for any $c>0$.

3. Main Results for $f_V(x)$

In what follows we always assume the following conditions.

A.1 $h_n\downarrow 0$ and $nh_m\to\infty$ as $n\to\infty$.

A.2 $h_n/h_{n(1+o(1))}\to 1$ as $n\to\infty$.

A.3 The mean measure $m(dz)$ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is bounded and bounded away from 0.

A.4 $K(x)$ is bounded and symmetric, i.e., $K(x)=K(-x)$, and $\int x^2K(x)dx<\infty$.

In all the statements of theorems below we fix a sequence $\{V_n\}$ of observation-domains ($V_n\in\mathcal{B}$ for all n) with $|V_n|\to\infty$ where $|V_n|$ denotes the volume of the domain V_n.

Theorem 3.1. (Bias) *Assume that $f(x)$ is bounded and continuous at x_0. Then*

$$\text{Bias}\, f_{V_n}(x_0)=|E\{f_{V_n}(x_0)\}-f(x_0)|=o(1). \tag{3.1}$$

Theorem 3.2. *Assume that $f(x)$ has a bounded second derivative. Then for each $x_0\in R^1$*

$$\text{Bias}\, f_{V_n}(x_0)=|E\{f_{V_n}(x_0)\}-f(x_0)|=O(h^2_{[m(V_n)]}). \tag{3.2}$$

Theorem 3.3. (Mean Square Error) *Assume that $f(x)$ has a bounded second derivative and that X satisfies the mixing condition* (2.1) *with*

$$\int_{R^d}\rho_1(|z|)dz<\infty. \tag{3.3}$$

Then for each $x_0\in R^1$

$$E\{(f_{V_n}(x_0)-f(x_0))^2\}=O\Big(\frac{1}{m(V_n)h_{[m(V_n)]}}\Big)+O(h^4_{[m(V_n)]}). \tag{3.4}$$

Remark. By Theorem 3.3 we see that the optimal form of h_n is $an^{-1/5}$ for some $a>0$.

Consider the following set of conditions.

C.1 $h_n=o(n^{-1/5})$ and $nh_n\to\infty$ as $n\to\infty$.

C.2 X satisfies the mixing condition (2.1).

C.3 (3.5) $$\int_{R^d}\rho_1(|z|)dz<\infty.$$

C.4 $f(x)$ has a bounded second derivative.

C.5 For each z_1, $z_2\in R^d$ $(z_1\neq z_2)$ there exists a joint probability density function $f(x, y:z_1, z_2)$ of the joint distribution of (X_{z_1}, X_{z_2}) w.r.t. the Lebesgue measure, and for each $\varepsilon>0$ there exists an absolute constant $M_\varepsilon>0$ such that $f(x, y:z_1, z_2)\leqq M_\varepsilon$ for all x, $y\in R^1$ and z_1, $z_2\in R^d$ with $|z_1-z_2|\geqq\varepsilon$.

C.6 For any x, $y\in R^1$ and given $r>0$ there exist positive numbers δ and $M_{xy}(r)$ such that

$$\int_{|z|>r}|f(x+a, y+b:0, z)-f(x+a)f(y+b)|dz\leqq M_{xy}(r) \tag{3.6}$$

for all a, $b(|a|, |b|\leqq\delta)$.

C.7 $\int_{|u|\geq r}K(u)=du=O(r^{-4})$ as $r\to\infty$.

Theorem 3.4. (Central Limit Theorem) *Assume that the conditions* (C.1)—(C.7) *are satisfied. Then for a fixed* $x_0\in R^1$

$$\sqrt{N(V_n)h_{N(V_n)}}\,\{f_{V_n}(x_0)-f(x_0)\}\xrightarrow{\mathscr{D}}N(0, \sigma^2) \tag{3.7}$$

where $\sigma^2=f(x_0)\int_{-\infty}^{\infty}K^2(u)du$.

Theorem 3.5. *Assume that the conditions* (C.1)—(C.7) *are satisfied. Then for any* x_0, $y_0\in R^1$ $(x_0\neq y_0)$

$$\sqrt{N(V_n)h_{N(V_n)}}\,\{f_{V_n}(x_0)-f(x_0)\}$$

and

$$\sqrt{N(V_n)h_{N(V_n)}}\,\{f_{V_n}(y_0)-f(y_0)\}$$

are asymptotically independent.

Remark. The condition (C.6) is rather restrictive, but is essential when we consider some continuous versions of density estimations based on dependent sample; see [6], [7] and [8]. We find a discrete version of the condition (3.6) in [3].

4. Proofs of Theorems 3.1—3.5

We can prove Theorem 3.1 by the same way as Theorem 3.2, so we prove Theorem 3.2 only.

Proof of Theorem 3.2. We know

$$E\{f_V(x_0)\}=\sum_{k=0}^{\infty}E\{f_V(x_0)\,|\,N(V)=k\}\,e^{-m(V)}\frac{m(V)^k}{k\,!}. \tag{4.1}$$

Let k be a positive integer. By Lemma 2.1 and the stationarity of X we have

$$\begin{aligned}E\{f_V(x_0)\,|\,N(V)=k\}&=\frac{1}{kh_k}\sum_{i=1}^{k}E\Big\{K\Big(\frac{X_{Z_i}-x_0}{h_k}\Big)\Big\}\\&=\frac{1}{h_k}E\Big\{\int_V K\Big(\frac{X_z-x_0}{h_k}\Big)\frac{m(dz)}{m(V)}\Big\}\\&=\frac{1}{h_k}\int_V E\Big\{K\Big(\frac{X_z-x_0}{h_k}\Big)\Big\}\frac{m(dz)}{m(V)}\\&=\frac{1}{h_k}\int_{-\infty}^{\infty}K\Big(\frac{x-x_0}{h_k}\Big)f(x)dx\\&=\int_{-\infty}^{\infty}K(w)f(x_0+wh_k)dw\\&=f(x_0)+\int_{-\infty}^{\infty}K(w)\{f(x_0+wh_k)-f(x_0)\}\,dw\,.\end{aligned}$$

Therefore by the smoothness condition for $f(x)$ and the symmetricity of $K(x)$ there exists a positive constant M such that

$$|E\{f_V(x_0)\,|\,N(V)=k\}-f(x_0)|\leqq Mh_k^2\int_{-\infty}^{\infty}K(w)w^2dw \qquad (k>0).$$

Put $n_1=[m(V)-\sqrt{m(V)}\log m(V)]$. Therefore by (4.1) and Corollary 2.4 for a sufficiently large M' we have

$$\begin{aligned}&|E\{f_V(x_0)\}-f(x_0)|\leqq M\int_{-\infty}^{\infty}K(w)w^2dw\sum_{k=1}^{\infty}h_k^2e^{-m(V)}\frac{m(V)^k}{k\,!}\\&\leqq M\int_{-\infty}^{\infty}K(w)w^2dw\Big\{h_0^2\sum_{k=1}^{n_1-1}e^{-m(V)}\frac{m(V)^k}{k\,!}+h_{n_1}^2\sum_{k=n_1}^{\infty}e^{-m(V)}\frac{m(V)^k}{k\,!}\Big\}\\&\leqq M'\Big\{o\Big(\frac{1}{m(V)^c}\Big)+h_{n_1}^2\Big\}=O(h^2_{[m(V)]})\end{aligned}$$

as $m(V)\to\infty$ for any $c>0$. Thus we have completed the proof of Theorem 3.1.

Proof of Theorem 3.3. The following lemma is well-known.

Lemma 4.1. *For any random variables X and Y defined on a probability space, define* $\mathrm{Var}[X|Y]=E\{(X-E(X|Y))^2|Y\}$. *Then we have*

$$\mathrm{Var}(X)=E\{\mathrm{Var}[X|Y]\}+\mathrm{Var}(E\{X|Y\}). \tag{4.2}$$

Now we proceed to prove Theorem 3.3.

$$\begin{aligned}\mathrm{MSE}(f_V(x_0))&=E\{(f_V(x_0)-f(x_0))^2\}\\&=E\{(f_V(x_0)-Ef_V(x_0))^2\}+(\mathrm{Bias}\,f_V(x_0))^2\\&=I_1+I_2 \qquad \text{(say)}.\end{aligned}$$

Let $N=N(V)$. By Lemma 4.1,

$$\begin{aligned}I_1&=E\{\mathrm{Var}[f_V(x_0)|N]\}+\mathrm{Var}(E\{f_V(x_0)|N)\})\\&=I_{11}+I_{12} \qquad \text{(say)}.\end{aligned}$$

Denote by p_k the probability $e^{-m(V)}m(V)^k/k!$. Then we have

$$I_{11}=\sum_{k=0}^{\infty}\mathrm{Var}(f_k(x_0)|N=k)p_k\,.$$

Here $f_k(x_0)$ denotes $1/kh_k\sum_{i=1}^{k}K((X_{Z_i}-x_0)/h_k)$ where $\{Z_i: i=1,\cdots,k\}$ are i.i.d. random variables distributed over V with $P(Z_i\in dz)=m(dz)/m(V)$ $(dz\subset V)$ and independent of X. Let us denote by N_k the event $\{N=k\}$ and by E_k the conditional expectation w.r.t. $P(\cdot|N_k)$. Then by the definition of E_k and stationarity of X we have

$$E_k\Big\{K\Big(\frac{X_{Z_i}-x_0}{h_k}\Big)\Big\}=\int_V E\Big\{K\Big(\frac{X_z-x_0}{h_k}\Big)\Big\}\frac{m(dz)}{m(V)}=\int_{-\infty}^{\infty}K\Big(\frac{x-x_0}{h_k}\Big)f(x)dx\,. \tag{4.3}$$

Now

$$\begin{aligned}I_{11}&=\sum_{k=1}^{\infty}E_k\Big\{\Big[\frac{1}{kh_k}\sum_{i=1}^{k}\Big(K\Big(\frac{X_{Z_i}-x_0}{h_k}\Big)-E_k\Big\{K\Big(\frac{X_{Z_i}-x_0}{h_k}\Big)\Big\}\Big)\Big]^2\Big\}p_k\\&=\sum_{k=1}^{\infty}\frac{1}{kh_k^2}E_k\Big\{\Big(K\Big(\frac{X_{Z_i}-x_0}{h_k}\Big)-E_k\Big\{K\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)\Big\}\Big)^2\Big\}p_k\\&\quad+\sum_{k=2}^{\infty}\frac{k(k-1)}{k^2h_k^2}E_k\Big\{\Big(K\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)-E_k\Big\{K\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)\Big\}\Big)\\&\qquad\qquad\times\Big(K\Big(\frac{X_{Z_2}-x_0}{h_k}\Big)-E_k\Big\{K\Big(\frac{X_{Z_2}-x_0}{h_k}\Big)\Big\}\Big)\Big\}p_k\\&=J_1+J_2 \qquad \text{(say)}.\end{aligned}$$

$$J_1=\sum_{k=1}^{\infty}\frac{1}{kh_k^2}\Big[E_k\Big\{K^2\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)\Big\}-\Big\{E_kK\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)\Big\}^2\Big]p_k$$

$$=\sum_{k=1}^{\infty}\frac{1}{kh_k^2}\Big[\int_{-\infty}^{\infty}K^2\Big(\frac{x-x_0}{h_k}\Big)f(x)dx-\Big(\int_{-\infty}^{\infty}K\Big(\frac{x-x_0}{h_k}\Big)f(x)dx\Big)^2\Big]p_k$$

$$=\sum_{k=1}^{\infty}\frac{1}{kh_k^2}\Big\{h_kf(x_0)\int_{-\infty}^{\infty}K^2(x)dx+O(h_k^2)\Big\}p_k$$

$$=\sum_{k=1}^{\infty}\frac{1}{kh_k}\Big\{f(x_0)\int_{-\infty}^{\infty}K^2(x)dx+O(h_k)\Big\}p_k$$

$$=\frac{1}{m(V)h_{[m(V)]}}\Big\{f(x_0)\int_{-\infty}^{\infty}K(x)dx+O(h_{[m(V)]})\Big\}$$

as $m(V)\to\infty$ by Corollary 2.4 and the condition **A.2**. Next consider the term J_2. The cross term is estimated as follows.

$$\Big|E_k\Big\{\Big(K\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)-E_k\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)\Big)\Big(K\Big(\frac{X_{Z_2}-x_0}{h_k}\Big)-E_kK\Big(\frac{X_{Z_2}-x_0}{h_k}\Big)\Big)\Big\}\Big|$$

$$\leqq E_k\Big\{\rho_1(|Z_1-Z_2|)\Big\|K\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)-E_kK\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)\Big\|^2\Big\}$$

$$\leqq Mh_k\int_V m(dz_1)\int_V\rho_1(|z_1-z_2|)\frac{m(dz_2)}{m(V)^2}\leqq M\frac{h_k}{m(V)}$$

for a sufficiently large M. Thus by Corollary 2.4 we have

$$I_{11}=J_1+J_2\leqq M\Big(\sum_{k=1}^{\infty}\frac{1}{kh_k}p_k+\sum_{k=1}^{\infty}\frac{1}{h_km(V)}p_k\Big)=O\Big(\frac{1}{m(V)h_{[m(V)]}}\Big)$$

for a sufficiently large M, because $[m(V)\pm m(V)^{1/2}\log m(V)]\sim m(V)$. Lastly we consider $I_{12}=\mathrm{Var}[E\{f_V(x_0)|N\}]$. By (4.3), as $m(V)\to\infty$,

$$I_{12}=\sum_{k=1}^{\infty}\Big\{\frac{1}{h_k}\int_{-\infty}^{\infty}K\Big(\frac{x-x_0}{h_k}\Big)f(x)dx-\sum_{j=1}^{\infty}\frac{1}{h_j}\int_{-\infty}^{\infty}K\Big(\frac{x-x_0}{h_j}\Big)f(x)dx\,p_j\Big\}^2p_k$$

$$=\sum_{k=1}^{\infty}\Big\{\frac{1}{h_k}\int_{-\infty}^{\infty}K\Big(\frac{x-x_0}{h_k}\Big)f(x)dx-f(x_0)\Big\}^2p_k+\{\mathrm{Bias}\,f_V(x_0)\}^2$$

$$=O\Big(\sum_{k=1}^{\infty}h_k^4p_k\Big)+\{\mathrm{Bias}\,f_V(x_0)\}^2$$

$$=O(h_{[m(V)]}^4)+\{\mathrm{Bias}\,f_V(x_0)\}^2=O(h_{[m(V)]}^4)$$

by Corollary 2.4 and Theorem 3.2. Thus we have proved Theorem 3.3.

Proof of Theorem 3.4. We use the following central limit theorem (Lemma 4.2) for martingales (a specialization of Theorem 3.6 in [5]).

Lemma 4.2. *For each* $n\geqq 1$ *let* $\{S_{ni},\mathcal{F}_{ni},0\leqq i\leqq k_n\}$ $(S_{n0}=0)$ *be a zero-mean,* *square-integrable martingale with differences* X_{ni}. *Suppose that* $\sup_{n,i} E(S_{ni}^2)<\infty$. *If*

$$\sup_i |X_{ni}| \xrightarrow{P} 0, \tag{4.4}$$

$$\sum_i X_{ni}^2 \xrightarrow{P} \sigma^2 \tag{4.5}$$

$$E(\sup_i X_{ni}^2) \text{ is bounded,} \tag{4.6}$$

then

$$S_{nk_n} \xrightarrow{\mathcal{D}} N(0, \sigma^2). \tag{4.7}$$

Recall that we have fixed a sequence $\{V_n\}$ of the observation-domains. Let $n_1=[m(V_n)-\sqrt{m(V_n)}\log m(V_n)]$ and $n_2=[m(V_n)+\sqrt{m(V_n)}\log m(V_n)]$. Denote by $\{k_n, n=1, 2, \cdots\}$ an arbitrary sequence of positive integers satisfying $n_1\leqq k_n\leqq n_2$ for all n. We consider the conditional central limit theorem on the event $\{N(V_n)=k_n\}$. Sometimes we denote by N the random variable $N(V_n)$ and write k for k_n. On the event $\{N=k\}$ we have

$$\begin{aligned}\sqrt{Nh_N}\{f_{V_n}(x_0)-f(x_0)\} &=\sqrt{kh_k}\{f_V(x_0)-E_k f_{V_n}(x_0)+E_k f_{V_n}(x_0)-f(x_0)\}\\ &=\frac{1}{\sqrt{kh_k}}\sum_{i=1}^{k} K_n(X_{Z_i})+\sqrt{\frac{k}{h_k}}\left\{\int_{V_n} \bar{K}_n(X_z)\frac{m(dz)}{m(V_n)}\right\}\\ &\quad+\sqrt{kh_k}\{E_k f_{V_n}(x_0)-f(x_0)\}\\ &=I_1+I_2+\sqrt{kh_k}\{E_k f_{V_n}(x_0)-f(x_0)\}\end{aligned}$$

where

$$K_n(X_{Z_i})=K\left(\frac{X_{Z_i}-x_0}{h_k}\right)-\int_{V_n} K\left(\frac{X_z-x_0}{h_k}\right)\frac{m(dz)}{m(V_n)}$$

and

$$\bar{K}_n(X_z)=K\left(\frac{X_z-x_0}{h_k}\right)-E\left\{K\left(\frac{X_z-x_0}{h_k}\right)\right\}.$$

By the condition (C.1) and Theorem 3.2, we have easily

$$\sqrt{kh_k}\,|E_k f_{V_n}(x_0)-f(x_0)|=O(\sqrt{kh_k^5})=o(1). \tag{4.8}$$

Lemma 4.3. *As* $n\to\infty$,

$$I_2=\sqrt{\frac{k_n}{h_{k_n}}}\left\{\int_{V_n} \bar{K}_{k_n}(X_z)\frac{m(dz)}{m(V_n)}\right\} \xrightarrow{P} 0. \tag{4.9}$$

Lemma 4.4. *As* $n\to\infty$,

$$I_1=\frac{1}{\sqrt{k_n h_{k_n}}}\sum_{i=1}^{k_n} K_{k_n}(X_{Z_i}) \xrightarrow{\mathcal{D}} N(0, \sigma^2) \tag{4.10}$$

where $\sigma^2=f(x_0)\int_{-\infty}^{\infty}K^2(x)dx.$

Proof of Lemma 4.3. Fix a positive number $\varepsilon>0$. Then there exist positive numbers A_ε and δ satisfying $\int_{|z|>A_\varepsilon}\rho_1(|z|)dz<\varepsilon$ and $m(U_\delta(z))<\varepsilon$ for all $z\in R^d$ by the conditions **A.3** and (3.3), where $U_\delta(z)$ is the ball at center $z\in R^d$ with radius δ. We may assume that $A_\varepsilon>\delta$. For notational simplicity we write k for k_n.

$$E\{I_2^2\}\sim\frac{1}{m(V_n)h_k}E\Big\{\Big(\int_{V_n}\bar{K}_k(X_z)m(dz)\Big)^2\Big\}$$
$$=\frac{1}{m(V_n)h_k}\Big[E\Big\{\int_{V_n}m(dz_1)\int_{U_\delta(z_2)}\bar{K}_k(X_{z_1})\bar{K}_k(X_{z_2})m(dz_2)\Big\}$$
$$+E\Big\{\int_{V_n}m(dz_1)\int_{V_n-U_{A_\varepsilon}(z_2)}\bar{K}_k(X_{z_1})\bar{K}_k(X_{z_2})m(dz_2)\Big\}$$
$$+E\Big\{\int_{V_n}m(dz_1)\int_{U_{A_\varepsilon}(z_1)-U_\delta(z_1)}\bar{K}_k(X_{z_1})\bar{K}_k(X_{z_2})m(dz_2)\Big\}\Big]$$
$$=\frac{1}{m(V_n)h_k}\{I_{21}+I_{22}+I_{23}\}\,. \qquad \text{(say)}$$

In below the capital letter M denotes some absolute constant which is not ne-necessarily identical in different ocurrences.

$$\frac{1}{m(V_n)h_k}|I_{21}|\leqq\frac{M}{m(V_n)h_k}h_k m(V_n)\sup_{z_1}\int_{U_\delta(z_1)}\rho_1(|z_1-z_2|)m(dz_2)$$
$$\leqq M\sup_{z_1}m(U_\delta(z_1))<M\varepsilon\,,$$

$$\frac{1}{m(V_n)h_k}|I_{22}|\leqq\frac{M}{m(V_n)h_k}h_k m(V_n)\sup_{z_1}\int_{U_{A_\varepsilon}(z_1)^c}\rho_1(|z_1-z_2|)m(z_2)<M\varepsilon\,.$$

By condition (C.6) there exist positive numbers γ and M such that

$$\sup_{z\in R^n}\int_{U_\delta(z)^c}|f(x_0+a,\ x_0+b:z,\ z_1)-f(x+a)f(x+b)|m(dz_1)\leqq M \tag{4.12}$$

for all a, $b(|a|,\ |b|\leqq\gamma)$. By this and the condition (C.7) we have

$$\frac{1}{m(V_n)h_k}|I_{23}|\leqq\frac{M}{m(V_n)h_k}h_k^2 m(V_n)\sup\int_{U_{A_\varepsilon}(z)-U_\delta(z)}m(dz_1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(u)K(w)$$
$$\times|f(x_0+uh_k,\ x_0+wh_k:z,z_1)-f(x_0+uh_k)f(x_0+wh_k)|dudw$$
$$\leqq Mh_n\sup_z\int_{U_{A_\varepsilon}(z)-U_\delta(z)}m(dz_1)$$
$$\times\Big\{\int_{|w|\leq\frac{\gamma}{h_k}}\int_{|u|\leq\frac{\gamma}{h_k}}+\int_{|w|\leq\frac{\gamma}{kh}}\int_{|u|\geq\frac{\gamma}{h_k}}$$

$$+\int_{|w|\geq\frac{\gamma}{h_k}}\int_{|u|\leq\frac{\gamma}{h_k}}+\int_{|w|\geq\frac{\gamma}{h_k}}\int_{|u|\geq\frac{\gamma}{h_k}}\Big\}K(u)K(w)$$

$$\times|f(x_0+uh_k,\ x_0+wh_k : z,\ z_1)$$

$$-f(x_0+uh_k)f(x_0+wh_k)|\,du\,dw$$

$$\leqq M\Big\{h_k+h_k m(V_n)\Big(\frac{\gamma}{h_k}\Big)^{-4}+h_k m(V_n)\Big(\frac{\gamma}{h_k}\Big)^{-8}\Big\}$$

$$=O\Big(h_k+m(V_n)\frac{h_k^5}{\gamma^4}+m(V_n)\frac{h_k^9}{\gamma^8}\Big)=o(1),$$

by the condition (C.1). We have proved Lemma 4.3.

Proof of Lemma 4.4. Here also we write k for k_n. We shall consider the conditional central limit theorem w.r.t. the event $N_{nk}=\{N(V_n)=k\}$. Let $\{\mathcal{F}_{nj}: j=1,\cdots,k\}$ be the sequence of the σ-fields $\mathcal{F}_{nj}$ generated by $\{X, Z_{n1}, Z_{n2},\cdots, Z_{nj}\}$ where Z_{ni}'s are i.i.d. random variables with $P(Z_{ni}\in dz)=m(dz)/m(V_n)$ $(dz\subset V_n)$. For convenience we write $\mathcal{F}_{n0}$ for the σ-field $\{\phi, \Omega\}$. Denote by $S_{kj}(n)$ the sum $\sum_{i=1}^{j}K_k(X_{Z_{ni}})$ $(S_{k0}(n)=0)$. Then it is easily shown that for each n $\{S_{kj}(n), \mathcal{F}_{kj}; j=0,\cdots,k\}$ forms a martingale sequence, that is, for each $j\leqq k-1$ $E\{S_{kj+1}(n)|\mathcal{F}_{nj}\}=S_{kj}(n)$ a.s.. This martingale property of $S_{kj}(n)$ plays an essential role in the proof of Lemma 4.4.

Lemma 4.5. *As* $n\to\infty$,

$$E\{I_1^2\}\longrightarrow f(x_0)\int_{-\infty}^{\infty}K^2(u)du. \tag{4.13}$$

Proof.

$$E\{I_1^2\}=\frac{1}{h_k}E\Big\{\Big(K\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)-\int_{V_n}K\Big(\frac{X_z-x_0}{h_k}\Big)\frac{m(dz)}{m(V_n)}\Big)^2\Big\}$$

$$=\frac{1}{h_k}\int_{-\infty}^{\infty}K^2\Big(\frac{x-x_0}{h_k}\Big)f(x)dx-\frac{1}{h_k}E\Big\{\Big(\int_{V_n}K\Big(\frac{X_z-x_0}{h_k}\Big)\frac{m(dz)}{m(V_n)}\Big)^2\Big\}$$

$$=\int_{-\infty}^{\infty}K^2(u)f(x_0+h_k u)du$$

$$-\frac{1}{m(V_n)^2h_k}\int_{V_n}m(dz)\int_{V_n}E\Big\{\Big[K\Big(\frac{X_z-x_0}{h_k}\Big)-EK\Big(\frac{X_z-x_0}{h_k}\Big)\Big]$$

$$\times\Big[K\Big(\frac{X_{z_1}-x_0}{h_k}\Big)-EK\Big(\frac{X_{z_1}-x_0}{h_k}\Big)\Big]\Big\}m(dz_1)+O(h_k)$$

$$=f(x_0)\int_{-\infty}^{\infty}K^2(u)du+O(h_k^2)\int_{-\infty}^{\infty}K^2(u)|u|^2du$$

$$+O(m(V_n)^{-1})\sup_z \int \rho_1(|z-z_1|)m(dz_1)+O(h_k)$$

$$=f(x_0)\int_{-\infty}^{\infty} K^2(u)du+O(h_k)+O(m(V_n)^{-1}).$$

Lemma 4.6. *As* $n\to\infty$,

$$\frac{1}{kh_k}\sum_{i=1}^{k}\left[K\left(\frac{X_{Z_i}-x_0}{h_k}\right)-\int_{V_n} K\left(\frac{X_z-x_0}{h_k}\right)\frac{m(dz)}{m(V_n)}\right]^2 \xrightarrow{P} f(x_0)\int_{-\infty}^{\infty} K^2(u)du. \tag{4.14}$$

Proof. Define

$$K_n(X, Z_i)=K\left(\frac{X_{Z_i}-x_0}{h_k}\right)-\int_{V_n} K\left(\frac{X_z-x_0}{h_k}\right)\frac{m(dz)}{m(V_n)}.$$

It suffices to show that as $n\to\infty$

$$T=\frac{1}{k^2h_k^2}E\left\{\left[\sum_{i=1}^{k}K_n(X, Z_i)^2-kh_kf(x_0)\int_{-\infty}^{\infty}K^2(u)du\right]^2\right\} \longrightarrow 0. \tag{4.15}$$

From the proof of Lemma 4.5 we know

$$E\{K_n(X, Z_i)^2\}=h_kf(x_0)\int_{-\infty}^{\infty}K^2(u)du+O(h_k^2)+o\left(\frac{h_k}{m(V_n)}\right)+O(h_k^2),$$

hence we have

$$T\leqq\frac{2}{kh_k^2}E\left\{\left[\sum_{i=1}^{k}K_n(X, Z_i)^2-\sum_{i=1}^{k}E\{K_n(X, Z_i)^2\}\right]^2\right\}$$

$$+\frac{1}{k^2h_k^2}\{O(k^2h_k^4)+o(h_k^2)\}$$

$$=\frac{2}{k^2h_k^2}E\left\{\left(\sum_{i=1}^{k}W_n(X, Z_i)\right)^2\right\}+O(h_k^2)+o(k^{-2}),$$

where

$$W_n(X, Z_i)=K_n(X, Z_i)^2-E\{K_n(X, Z_i)^2\}, \qquad i=1, \cdots, k.$$

It remains us to show

$$T'=E\left\{\left(\sum_{i=1}^{k}W_n(X, Z_i)\right)^2\right\}=o(k^2h_k^2). \tag{4.16}$$

We know that

$$E\{W_n(X, Z_i)\}=0 \quad\text{and}\quad E\{W_n(X, Z_i)^2\}=O(h_k).$$

Hence for a sufficiently large C

$$T'=kE\{W_n(X, Z_i)^2\}+k(k-1)E\{W_n(X, Z_1)W_n(X, Z_2)\}$$

$$\leqq O(kh_k)+\frac{Ck(k-1)}{m(V_n)}h_k\int_{V_n}\rho_1(|z|)m(dz)\leqq O(kh_k).$$

In fact, putting $K_n(x)=K(x/h_k)$ (rem. $k=k_n$ depends on n),

(4.17) $E\{W_n(X, Z_1)W_n(X, Z_2)\}$

$$=E\Big\{\Big[K_n^2(X_{Z_1}-x_0)-\Big(\int_{V_n}K_n(X_z-x_0)\frac{m(dz)}{m(V_n)}\Big)^2$$

$$-EK_n^2(X_{Z_1}-x_0)+E\Big\{\Big(\int_{V_n}K_n(X_z-x_0)\frac{m(dz)}{m(V_n)}\Big)^2\Big\}\Big]$$

$$\times\Big[K_n^2(X_{Z_2}-x_0)-\Big(\int_{V_n}K_n(X_z-x_0)\frac{m(dz)}{m(V_n)}\Big)^2$$

$$-EK_n^2(X_{Z_2}-x_0)+E\Big\{\Big(\int_{V_n}K_n(X_z-x_0)\frac{m(dz)}{m(V_n)}\Big)^2\Big\}\Big]\Big\}$$

$$=E\{[K_n^2(X_{Z_1}-x_0)-EK_n^2(X_{Z_1}-x_0)][K_n^2(X_{Z_2}-x_0)-EK_n^2(X_{Z_2}-x_0)]\}$$

$$-E\Big\{[K_n^2(X_{Z_1}-x_0)-EK_n^2(X_{Z_1}-x_0)+K_n^2(X_{Z_2}-x_0)-EK_n^2(X_{Z_2}-x_0)]$$

$$\times\Big[\Big(\int_{V_n}K_n(X_z-x_0)\frac{m(dz)}{m(V_n)}\Big)^2-E\Big(\int_{V_n}K_n(X_z-x_0)\frac{m(dz)}{m(V_n)}\Big)^2\Big]\Big\}$$

$$+E\Big\{\Big[\Big(\int_{V_n}K_n^2(X_z-x_0)\frac{m(dz)}{m(V_n)}\Big)^2-E\Big(\int_{V_n}K_n(X_z-x_0)\frac{m(dz)}{m(V_n)}\Big)^2\Big]^2\Big\}$$

$$=I_1-(I_2+I_3)+I_4. \quad \text{(say)}$$

Now let us estimate each I_i.

$$|I_1|\leqq C\frac{h_k}{m(V_n)}\int\rho_1(|z|)m(dz),$$

$$|I_2|\leqq\int_{V_n}\frac{m(dz)}{m(V_n)}\int_{V_n}\int_{V_n}|E\{[K_n^2(X_z-x_0)-EK_n^2(X_z-x_0)]$$

$$\times[K_n(X_{z_1}-x_0)K_n(X_{z_2}-x_0)-EK_n(X_{z_1}-x_0)K_n(X_{z_2}-x_0)]\}|\frac{m(dz_1)}{m(V_n)}\frac{m(dz_2)}{m(V_n)}$$

$$\leqq C\frac{h_k}{m(V_n)}\int\rho_1(|z|)m(dz),$$

and similarly

$$|I_3|\leqq C\frac{h_k}{m(V_n)}\int\rho_1(|z|)m(dz).$$

Lastly

$$|I_4|=\frac{1}{m(V_n)}\int_{V_n}\int_{V_n}\int_{V_n}\int_{V_n}\Big|EK_n(X_z-x_0)K_n(X_{z_1}-x_0)K_n(X_{z_2}-x_0)K_n(X_{z_3}-x_0)$$

$$-E\{K_n(X_z-x_0)K_n(X_{z_1}-x_0)\}E\{K_n(X_{z_2}-x_0)K_n(X_{z_3}-x_0)\}\Big|$$
$$\times m(dz)m(dz_1)m(dz_2)m(dz_3)$$
$$\leqq C\frac{h_k}{m(V_n)}\int\rho_1(|z|)m(dz).$$

Thus we have proved Lemma 4.6.

It is easy to check that our conditions imply the other conditions in Lemma 4.2 except for (4.4). In order to prove (4.4) it suffices to show that

$$D=\frac{1}{k^2h_k^2}\sum_{i=1}^{k}E\{K_n^4(X_{Z_i})\}\longrightarrow 0. \tag{4.18}$$

$$D\leqq\frac{1}{kh_k^2}\Big[E\Big\{K^4\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)\Big\}+6E\Big\{K^2\Big(\frac{X_{Z_1}-x_0}{h_k}\Big)\Big(\int_{V_n}K\Big(\frac{X_z-x_0}{h_k}\Big)\frac{m(dz)}{m(V_n)}\Big)^2\Big\}$$
$$+E\Big\{\Big(\int_{V_n}K\Big(\frac{X_z-x_0}{h_k}\Big)\frac{m(dz)}{m(V_n)}\Big)^4\Big\}\Big]$$
$$\leqq\frac{1}{kh_k^2}\Big[\int_{-\infty}^{\infty}K^4\Big(\frac{x-x_0}{h_k}\Big)f(x)dx+6E\Big\{\int_{V_n}K^2\Big(\frac{X_z-x_0}{h_k}\Big)\frac{m(dz)}{m(V_n)}\int_{V_n}K^2\Big(\frac{X_z-x_0}{h_k}\Big)\frac{m(dz)}{m(V_n)}\Big\}$$
$$+\int_{-\infty}^{\infty}K^4\Big(\frac{x-x_0}{h_k}\Big)f(x)dx\Big]\quad\text{by Jensen's inequality}$$
$$=O\Big(\frac{1}{kh_k}\int_{-\infty}^{\infty}K^4(u)du\Big)=o(1),$$

so we have (4.19).

Thus by Lemmas 4.2, 4.3, 4.5 and 4.6 we have proved Lemma 4.4.

Now let us turn to the proof of Theorem 3.4. Denote by $\Phi_\sigma(x)$ the normal distribution function with mean 0 and standard deviation σ where $\sigma^2=f(x_0)\int_{-\infty}^{\infty}K^2(u)du$, by $F_n(x)$ the d.f. of $\sqrt{N(V_n)h_{N(V_n)}}\{f_{V_n}(x_0)-f(x_0)\}$ and by $F_{n,k}(x)$ the conditional distribution function of $\sqrt{N(V_n)h_{N(V_n)}}\{f_{V_n}(x_0)-f(x_0)\}$ given the event $\{N(V_n)=k\}$. Then by Corollary 2.4 for any $c>0$

$$\Delta_n=\sup_x|F_n(x)-\Phi_\sigma(x)|$$
$$\leqq\sum_{k=0}^{\infty}\sup_x|F_{n,k}(x)-\Phi_\sigma(x)|e^{-m(V_n)}\frac{m(V_n)^k}{k!} \tag{4.19}$$
$$\leqq\sup_{n_1\leqq k\leqq n_2}\sup_x|F_{nk}(x)-\Phi_\sigma(x)|+O(m(V_n)^{-c})\longrightarrow 0$$

by Lemma 4.4, completing the proof.

Proof of Theorem 3.5. By Theorem 3.4 it suffices to show that for $x_0\neq y_0$

(4.20) $$\mathrm{Cov}\{\sqrt{N(V^u)h_{N(V_n)}}f_{V_n}(x_0),\ \sqrt{N(V_n)h_{N(V_n)}}f_{V_n}(y_0)\}=o(1).$$

The following lemma is similar to Lemma 4.1 and easily proved.

Lemma 4.7. *Let X, Y and Z be random variables defined on a probability space. Then*

(4.21) $$\mathrm{Cov}(X,Y)=E\{\mathrm{Cov}(X,Y\,|\,Z)\}+\mathrm{Cov}(E\{X\,|\,Z\},\ E\{Y\,|\,Z\})$$

where $\mathrm{Cov}(X,Y\,|\,Z)$ *denotes the covariance of X and Y w.r.t. the conditional expectation $E(\cdot\,|\,Z)$.*

For convenience we assume that $x_0>y_0$. As well as the preceding proofs let us denote by N the random variable $N(V_n)$, and by p_k the probability $P(N=k)$.

(4.22) $$\begin{aligned}&\mathrm{Cov}\{\sqrt{Nh_N}f_{V_n}(x_0),\ \sqrt{Nh_N}f_{V_n}(y_0)\}\\&=E\{\mathrm{Cov}(\sqrt{Nh_N}f_{V_n}(x_0),\ \sqrt{Nh_N}f_{V_n}(y_0)\,|\,N\}\\&\quad+\mathrm{Cov}(E\{\sqrt{Nh_N}f_{V_n}(x_0)\,|\,N\},\ E\{\sqrt{Nh_N}\ f_{V_n}(y_0)\,|\,N\})\\&=H_1+H_2.\qquad\text{(say)}\end{aligned}$$

We write $K_n(X_{Z_j}-x)$ for $K((X_{Z_j}-x)/h_k)-E\{K((X_{Z_j}-x)/h_k)\}$.

(4.23) $$\begin{aligned}H_1&=\sum_{k=1}^{\infty}\frac{1}{kh_k}E\Big\{\Big(\sum_{j=1}^{k}K_n(X_{Z_j}-x_0)\Big)\Big(\sum_{j=1}^{k}K_n(X_{Z_j}-y_0)\Big)\Big\}p_k\\&=\sum_{k=1}^{\infty}\frac{1}{h_k}E\{K_n(X_{Z_1}-x_0)K_n(X_{Z_1}-y_0)\}\,p_k\\&\quad+\sum_{k=1}^{\infty}\frac{k-1}{h_k}E\{K_n(X_{Z_1}-x_0)K_n(X_{Z_2}-y_0)\}\,p_k.\end{aligned}$$

Now we estimate each expectation in the summations.

(4.24) $$\begin{aligned}&|E\{K_n(X_{Z_1}-x_0)K_n(X_{Z_1}-y_0)\}|\\&=\int_{-\infty}^{\infty}K\Big(\frac{u-x_0}{h_k}\Big)K\Big(\frac{u-y_0}{h_k}\Big)f(u)du+O(h_k^2)\\&=h_k\int_{-\infty}^{\infty}K(w)K\Big(w+\frac{x_0-y_0}{h_k}\Big)f(x_0+h_kw)dw+O(h_k^2)\\&=O(h_k^2)\end{aligned}$$

by dividing the integral domain into three parts $w\geqq 0$, $-(h_0-y_0)/2h_k<w<0$ and $w\leqq-(x_0-y_0)/2h_k$ and considering the condition (C.7).

We estimate another expectation. For $\varepsilon>0$ given take positive numbers δ and A such that

$$\sup_z m(U_\delta(z))<\varepsilon \quad \text{and} \quad \sup_z \int_{U_A(z)^c} \rho_1(|z-z_1|)m(dz_1)<\varepsilon\,.$$

We may assume there exist r and $M<\infty$ such taht

$$\sup_z \int_{U_\delta(z)^c} |f(x_0+a,\ y_0+b:z,\ z_1)-f(x_0+a)f(y_0+b)|m(dz_1)\leqq M$$

for all a, $b(|a|, |b|\leqq r)$. In below the capital letter M denotes absolute constants.

$$\begin{aligned}
&|E\{K_n(X_{Z_1}-x_0)K_n(X_{Z_2}-y_0)\}| \\
&=\Big|\int_{V_n}\frac{m(dz)}{m(V_n)}\int_{V_n} E\Big\{\Big(K\Big(\frac{X_z-x_0}{h_k}\Big)-EK\Big(\frac{X_z-x_0}{h_k}\Big)\Big) \\
&\qquad \times\Big(K\Big(\frac{X_{z_1}-y_0}{h_k}\Big)-EK\Big(\frac{X_{z_1}-x_0}{h_k}\Big)\Big)\Big\}\frac{m(dz_1)}{m(V_n)}\Big|
\end{aligned}$$

$$\begin{aligned}
(4.25)\quad &\leqq \frac{Mh_k}{m(V_n)}\sup_z\int_{U_\delta(z)}\rho_1(|z-z_1|)m(dz_1) \\
&\quad +\frac{Mh_k^2}{m(V_n)}\sup_z\int_{U_\delta(z)^c} m(dz_1)\int_{-r/h_k}^{r/h_k}\int_{-r/h_k}^{r/h_k} K(u)K(v)|f(x_0+uh_k,\ y_0+vh_k:z,z_1) \\
&\quad -f(x_0+uh_k)f(y_0+vh_k)|\,dudv+O(h_k^6)+\frac{Mh_k}{m(V_n)}\sup_z\int_{U_A(z)}\rho_1(|z-z_1|)m(dz_1) \\
&\leqq M\Big(\frac{\varepsilon h_k}{m(V_n)}+\frac{h_k^2}{m(V_n)}+h_k^6+\frac{\varepsilon h_k}{m(V_n)}\Big).
\end{aligned}$$

By (4.23), (4.24) and (4.25) we have

$$(4.26)\qquad H_1\leqq M\sum_{k=1}^\infty\Big\{h_k+\frac{\varepsilon(k-1)}{m(V_n)}+\frac{h_k(k-1)}{m(V_n)}+(k-1)h_k^5+\frac{\varepsilon(k-1)}{m(V_n)}\Big\}p_k\,.$$

Since ε is arbitrary, by Corollary 2.4 we have

$$(4.27)\qquad H_1=o(1)\qquad \text{as } n\to\infty\,.$$

Lastly we show that $H_2=o(1)$ as $n\to\infty$. Since

$$E\{f_{V_n}(x_0)|N=k\}=\frac{1}{h_k}\int_{-\infty}^\infty K\Big(\frac{x-x_0}{h_k}\Big)f(x)dx\,,$$

we have

$$\begin{aligned}
H_2&=\sum_{k=1}^\infty kh_k\Big\{\frac{1}{h_k}\int_{-\infty}^\infty K\Big(\frac{x-x_0}{h_k}\Big)f(x)dx-Ef_{V_n}(x_0)\Big\} \\
&\qquad\times\Big\{\frac{1}{h_k}\int_{-\infty}^\infty K\Big(\frac{x-y_0}{h_k}\Big)f(x)dx-Ef_{V_n}(y_0)\Big\}p_k
\end{aligned}$$

$$(4.28)\qquad =\sum_{k=1}^\infty kh_k\Big\{\frac{1}{h_k}\int_{-\infty}^\infty K\Big(\frac{x-x_0}{h_k}\Big)f(x)dx-f(x_0)-\text{Bias}\,f_{V_n}(x_0)\Big\}$$

$$\times\left\{\frac{1}{h_k}\int_{-\infty}^{\infty}K\left(\frac{x-y_0}{h_k}\right)f(x)dx-f(y_0)-\text{Bias}\, f_{V_n}(y_0)\right\}p_k$$

$$=\sum_{k=1}^{\infty}\{kh_k(O(h_k^2)+O(h_{[m(V_n)]}^2))(O(h_k^2)+O(h_{[m(V_n)]}^2))\}\,p_k$$

$$=O(m(V_n)h_{[m(V_n)]}^5)=o(1)$$

by the condition (C.1), Theorem 3.2 and Corollary 2.4. Thus we have completed and proof.

5. Main Results for $f_V(x_1, x_2 : v)$

Let $H(x, y)$ and $W(z)$ be bouuded probability density functions on R^2 and R^d, respectively. In what follows we always assume the following conditions (**B.1 -B.4**) in addition to **A.1** and **A.2** in sect. 3.

B.1 The support of W is contained in the unit ball at center **O**.

B.2 $\iint H^2(x, y)dxdy<\infty$.

B.3 The Poisson point process $N(A)$ is homogeneous, that is, there exists a positive constant λ such that $m(V)=\lambda|V|$ for all $V\in\mathscr{B}$.

Remark. The condition **B.1** is rather restrictive, but this condition simplifies several proofs. To weaken the condition **B.3** is easy. See Remark in Section 5.

Let $f_V(x_1, x_2 : v)$ be the estimator for $f(x_1, x_2 : v)$, defined by (2.4). We shall discuss asymptotic properties of $f_V(x_1, x_2 : v)$ under some regularity conditions when $|V|\to\infty$. Fix a sequence $\{V_n\}$ of observation-domains with $V_n\subset V_{n+1}$ and $|V_n|\to\infty$ as $n\to\infty$. It is clear that $\{V_n\}$ cannot be chosen arbitrarily. In fact if V_n's are very thin, then for some vector v we can not estimate $f(x_1, x_2 : v)$ by $f_V\ (x_1, x_2 : v)$. For example ($d=2$) consider the case where $V_n=\{(x, y);\ 0<x<1, |y|\leqq n\}$ and $v=(0, 2)$. We assume the following condition.

B.4 For each vector $v\in R^d$ and $a>0$ as $n\to\infty$,

(5.1) $$|V_n\cap\{V_n\pm(1+a)v\}|/|V_n| \longrightarrow 1.$$

Theorem 5.1. *Fix a non-zero vector v_0. If $f(x, y : v)$ is continuous at $\{(x_1, x_2), v_0\}$ and bounded, then as $n\to\infty$*

(5.2) $$E\{f_{V_n}(x_1, x_2 : v_0)\} \longrightarrow f(x_1, x_2 : v_0).$$

In what follows we assume the following mixing condition.

(5.3) $$\int_{R^d}\rho_2(|z|)dz<\infty.$$

Theorem 5.2. *Fix a non-zero vector* v_0. *If* $f(x, y: v)$ *is continuous at* $\{(x_1, x_2), v_0\}$ *and bounded, then as* $n \to \infty$,

$$(5.4)\qquad \begin{aligned} &E\{N(V_n)h_{N(V_n)}^{d+2}[f_{V_n}(x_1, x_2: v_0)-Ef_{V_n}(x_1, x_2: v_0)]^2\} \\ &\longrightarrow \lambda^{-1}f(x_1, x_2: v_0)\int W^2(z)dz\iint H^2(x, y)dxdy. \end{aligned}$$

Theorem 5.3. *Fix a non-zero vector* v_0. *Suppose that* $kh_k^{d+2} \to \infty$ *as* $k \to \infty$. *If the function* $f(x, y: v)$ *is continuous at* $\{(x_1, x_2), v_0\}$ *and bounded, then as* $n \to \infty$,

$$(5.5)\qquad \sqrt{N(V_n)h_{N(V_n)}^{d+2}}\{f_{V_n}(x_1, x_2: v_0)-Ef_{V_n}(x_1, x_2: v_0)\} \xrightarrow{\mathcal{D}} N(0, \sigma^2)$$

where $\sigma^2=\lambda^{-1}f(x_1, x_2: v_0)\int W^2(z)dz\iint H^2(x, y)dxdy$.

Let $\mathcal{A}$ be the family of subsets A of R^d with $\#(A)=4$. For $A=\{z_1, z_2, z_3, z_4\} \in \mathcal{A}$ put $L(A)=\min\{|z_i-z_j|;\ 1 \leqq i<j \leqq 4\}$.

B.5 For each $A \in \mathcal{A}$ the random field X has a joint probability density function $f(\boldsymbol{x}: A)=f(x_1, x_2, x_3, x_4: z_1, z_2, z_3, z_4)$ $(z_i \in A)$ of random vector

$$(X_{z_1}, X_{z_2}, X_{z_3}, X_{z_4}),$$

and for any $\varepsilon>0$ there exists an absolute constant M_ε such that for any $A=\{z_1, z_2, z_3, z_4\} \in \mathcal{A}$ with $L(A) \geqq \varepsilon$

$$f(\boldsymbol{x}: A) \leqq M_\varepsilon.$$

Theorem 5.4. *Fix non-zero vectors* v_0, v_1. *Suppose that* $f(x, y: v)$ *is continuous at* $\{(x_1, x_2), v_0\}$ *and* $\{(x_3, x_4), v_1\}$, *that the condition* **B.5** *is satisfied, that* $kh_k^{d+2} \to \infty$ *as* $k \to \infty$, *and that there exists an absolute constant* $C>0$ *such that*

$$(5.6)\qquad \int_{|x|>u}\int_{|y|>v} H^2(x, y)dxdy \leqq \frac{C}{1+u^2+v^2}$$

for all $u, v>0$. *If* $\{(x_1, x_2), v_0\} \neq \{(x_3, x_4), v_1\}$ *then the random variables*

$$(5.7)\qquad \sqrt{N(V_n)h_{N(V_n)}^{d+2}}\{f_{V_n}(x_1, x_2: v_0)-Ef_{V_n}(x_1, x_2: v_0)\}$$

and

$$\sqrt{N(V_n)h_{N(V_n)}^{d+2}}\{f_{V_n}(x_3, x_4: v_1)-Ef_{V_n}(x_3, x_4: v_1)\}$$

are asymptotically independent.

6. Proofs of Theorem 5.1—5.4.

In what follows we write $H_k(x, y)$ for $H(x/h_k, y/h_k)$.

Proof of Theorem 5.1. We restrict ourselves on the event $\{N(V_n)=k\}$

where $k=k_n$ is a positive integer such that

$$[\lambda|V_n|-\sqrt{\lambda|V_n|}\log\lambda|V_n|]\leqq k\leqq[\lambda|V_n|+\sqrt{\lambda|V_n|}\log\lambda|V_n|] \tag{6.1}$$

By the condition **B.1** if n is sufficiently large, using the usual argument we have

$$\begin{aligned}
&E\{f_{V_n}(x_1, x_2: v_0)|N(V_n)=k\}\\
&=\frac{1}{\lambda k h_k^2}E\Big\{\sum_{i\neq j}^{k} w_k(v_0-(Z_j-Z_i))H_k(x_1-X_{Z_i}, x_2-X_{Z_j})\Big\}\\
&=\frac{k(k-1)}{\lambda k h_k^2}E\{w_k(v_0-(Z_2-Z_1))H_k(x_1-X_{Z_1}, x_2-X_{Z_2})\}\\
&=\frac{k-1}{\lambda h_k^2|V_n|^2}\int_{V_n}\int_{V_n}dz_1dz_2w_k(v_0-(z_2-z_1))\iint H_k(x_1-x, x_2-y)f(x, y: z_1, z_2)dxdy \qquad (6.2)\\
&=\frac{k-1}{\lambda|V_n|^2}\int_{V_n}\int_{V_n}dz_1dz_2w_k(v_0-(z_2-z_1))\iint H(u_1, u_2)\\
&\qquad\times f(x_1+u_1h_k, x_2+u_2h_k: z_1, z_2)du_1du_2\\
&=f(x_1, x_2: v_0)+o(1).
\end{aligned}$$

by **B.4** and the assumption of Theorem 5.1. Thus we have completed the proof.

In advance to prove Theorems 5.2-5.4 we need some preparations to apply martingale theory. We restrict ourselves on the event $\{N(V_n)=k\}$ where k satisfies (6.1). As in the proof of Theorem 5.1 we may assume that $w_k(2v_0)=0$.

For notational simplicity write $\bar{m}(dz)$ for $dz/|V_n|$. Put

$$\begin{aligned}
S_k=&\sum_{i\neq j}^{k} w_k(v_0-(Z_j-Z_i))H_k(x_1-X_{Z_j}, x_2-X_{Z_i})\\
&-E\Big\{\sum_{i\neq j}^{k} w_k(v_0-(Z_j-Z_i))H_k(x_1-X_{Z_i}, x_2-X_{Z_j})\Big\} \qquad (6.3)\\
&=\sum_{i<j}^{k} G_k(Z_i, Z_j: X_{Z_i}, X_{Z_j})+\sum_{i\neq j}\bar{G}_k(Z_i, Z_j: X_{Z_i}, X_{Z_j})+R_k,
\end{aligned}$$

where

$$\begin{aligned}
&G_k(Z_i, Z_j: X_{Z_i}, X_{Z_j})\\
&=w_k(v_0-(Z_j-Z_i))H_k(x_1-X_{Z_i}, x_2-X_{Z_j})\\
&\quad-\int_{V_n} w_k(v_0-(z_2-Z_i))H_k(x_1-X_{Z_i}, x_2-X_{z_2})\bar{m}(dz_2)\\
&\quad-\int_{V_n} w_k(v_0-(Z_j-z_1))H_k(x_1-X_{z_1}, x_2-X_{Z_j})\bar{m}(dz_1)\\
&\quad+\int_{V_n}\int_{V_n} w_k(v_0-(z_2-z_1))H_k(x_1-X_{z_1}, x_2-X_{z_2})\bar{m}(dz_1)\bar{m}(dz_2) \qquad (6.4)
\end{aligned}$$

$$+w_k(v_0-(Z_i-Z_j))H_k(x_1-X_{Z_j},\ x_2-X_{Z_i})$$

$$-\int_{V_n} w_k(v_0-(Z_i-z_2))H_k(x_1-X_{z_2},\ x_2-X_{Z_i})\bar{m}(dz_2)$$

$$-\int_{V_n} w_k(v_0-(z_1-Z_j))H_k(x_1-X_{Z_j},\ x_2-X_{Z_1})\bar{m}(dz_1)$$

$$+\int_{V_n}\int_{V_n} w_k(v_0-(z_1-z_2))H_k(x_1-X_{z_2},\ x_1-X_{z_1})\bar{m}(dz_1)\bar{m}(dz_2),$$

$$\bar{G}_k(Z_i,\ Z_j: X_{Z_i},\ X_{Z_j})$$

$$=\int_{V_n} w_k(v_0-(z_2-Z_i))H_k(x_1-X_{Z_i},\ x_2-X_{z_2})\bar{m}(dz_2)$$

$$(6.5)\qquad +\int_{V_n} w_k(v_0-(Z_j-z_1))H_k(x_1-X_{z_1},\ x_2-X_{Z_j})\bar{m}(dz_1)$$

$$-2\int_{V_n}\int_{V_n} w_k(v_0-(z_2-z_1))H_k(x_1-X_{z_1},\ x_2-X_{z_2})\bar{m}(dz_1)\bar{m}(dz_2)$$

and

$$(6.6)\qquad R_k=k(k-1)\int_{V_n}\int_{V_n} w_k(v_0-(z_2-z_1))\{H_k(x_1-X_{z_1},\ x_2-X_{z_2}) -EH_k(x_1-X_{z_1},\ x_2-X_{z_2})\}\bar{m}(dz_1)\bar{m}(dz_2).$$

Lemma 6.1. *As $n\to\infty$, we have*

$$(6.7)\qquad \frac{k_n h_{k_n}^{d+2}}{k_n^2 h_{k_n}^4} E\{R_{k_n}^2\} \longrightarrow 0.$$

Proof. In below the capital letter C denotes an absolute constant.

$$E\{R_k^2\}=k^2(k-1)^2E\left\{\left[\int_{V_n}\int_{V_n} w_k(v_0-(z_2-z_1))(H_k(x_1-X_{z_1},\ x_2-X_{z_2}) -EH_k(x_1-X_{z_1},\ x_2-X_{z_2}))\bar{m}(dz_1)\bar{m}(dz_2)\right]^2\right\}$$

$$(6.8)\qquad \leqq \frac{Ck^2(k-1)^2}{|V|^4}\int_{V_n}\int_{V_n}\int_{V_n}\int_{V_n} w_k(v_0-(z_2-z_1))w_k(v_0-(z_4-z_3)) \times h_k^2\rho_2(D((z_1,\ z_2),\ (z_3,\ z_4)))dz_1dz_2dz_3dz_4$$

$$\leqq \frac{Ck^2(k-1)^2h_k^2}{|V_n|^3}\int\rho_2(|z|)dz,$$

because w_k is a probability density. Thus we have

$$\frac{kh_k^{d+2}}{k^2h_k^4}E\{R_k^2\}=O(h_k^d)$$

by the condition on k, completing the proof.

Lemma 6.2. *As $n\to\infty$ we have*

$$\frac{k_n h_{k_n}^{d+2}}{k_n^2 h_{k_n}^4} E\left\{\left[\sum_{i\neq j}^{k_n} \bar{G}_{k_n}(Z_i, Z_j : X_{Z_i}, X_{Z_j})\right]^2\right\} \longrightarrow 0. \tag{6.9}$$

Proof. For notational simplicity we write $g_r(i)$ and $g_l(i)$ for

$$\int_{V_n} w_k(v_0-(z-Z_i))H_k(x_1-X_{Z_i}, x_2-X_z)\bar{m}(dz)$$

$$-\int_{V_n}\int_{V_n} w_k(v_0-(z_2-z_1))H_k(x_1-X_{z_1}, x_2-X_{z_2})\bar{m}(dz_1)\bar{m}(dz_2)$$

and

$$\int_{V_n} w_k(v_0-(Z_i-z))H_k(x_1-X_z, x_2-X_{Z_i})\bar{m}(dz)$$

$$-\int_{V_n}\int_{V_n} w_k(v_0-(z_2-z_1))H_k(x_1-X_{z_1}, x_2-X_{z_2})\bar{m}(dz_1)\bar{m}(dz_2)$$

respectively. By independence of Z_i's we have

$$\begin{aligned} E\left\{\left[\sum_{i\neq j}^{k} \bar{G}_k(Z_i, Z_j : X_{Z_j}, X_{Z_j})\right]^2\right\} &= k(k-1)^2\{Eg_r^2(1)+Eg_l^2(1)+2E[g_r(1)g_l(1)]\} \\ &\leqq \frac{C\,k(k-1)^2 h_k^2}{|V_n|^2}, \end{aligned} \tag{6.10}$$

where C is an absolute constant. Thus we have

$$\frac{k h_k^{d+2}}{k^2 h_k^4} E\left\{\left[\sum_{i\neq j}^{k} \bar{G}_k(Z_i, Z_j : X_{Z_i}, X_{Z_j})\right]^2\right\} = O(h_k^d), \tag{6.11}$$

completing the proof of the lemma.

Lemma 6.3. *As $n\to\infty$,*

$$\begin{aligned} &\frac{k_n h_{k_n}^{d+2}}{k_n^2 h_{k_n}^4} E\left\{\left[\sum_{i<j}^{k_n} G_{k_n}(Z_i, Z_j : X_{z_i}, X_{Z_j})\right]^2\right\} \\ &\quad\longrightarrow \lambda f(x_1, x_2 : v_0)\int W^2(z)dz\iint H^2(x, y)dxdy \end{aligned} \tag{6.12}$$

Proof. By the definition of G_k we know that for $i<j$

$$E\{G_k(Z_i, Z_j : X_{Z_i}, X_{Z_j})|Z_j\}=0 \qquad \text{a.s.}$$

and

$$E\{G_k(Z_i, Z_j : X_{Z_i}, X_{Z_j})|Z_j\}=0 \qquad \text{a.s..}$$

Hence by some elementary calculations we have

$$E\left\{\left[\sum_{i<j}^{k} G_k(Z_i, Z_j : X_{z_i}, X_{Z_j})\right]^2\right\}=\frac{k(k-1)}{2} E\{G_k^2(Z_1, Z_2 : X_{Z_1}, X_{Z_2})\}$$

$$(6.13) \quad \sim \frac{k(k-1)}{h_k^d} h_k^2 f(x_{11}, x_2 : v_0)\int W^2(z)dz\iint H^2(x, y)dxdy\cdot\frac{1}{|V_n|}$$

$$+O\left(\frac{k(k-1)h_k^2}{|V_n|^2}\right).$$

Thus we have

$$(6.14) \quad \frac{kh_k^{d+2}}{k^2h_k^4} E\left\{\left[\sum_{i<j}^{k} G_k(Z_i, Z_j : X_{Z_i}, X_{Z_j})\right]^2\right\}$$

$$\sim\lambda f(x_1, x_2 : v_0)\int W^2(z)dz\iint H(x, y)dxdy+O\left(\frac{h_k^d}{|V_n|}\right).$$

We have proved the lemma.

Proof of Theorem 5.2. Immediate from Lemmas 6.2-6.5.

Proof of Theorem 5.3. We omit some details. A detailed proof can be carried out in the same way as in [4]. From Theorem 5.2 we may assume that $f(x_1, x_2 : v_0)>0$. We use the notations in the proof of Theorem 5.2. Let $\{k_n\}$ be an arbitray sequence of positive integers satisfying the relation (6.1). Write k for k_n and put

$$(6.15) \quad \begin{aligned} T_{k,j}&=\textstyle\sum_{i<j} G_k(Z_i, Z_j : X_{Z_i}, X_{Z_j}),\\ S_{k,l}&=\textstyle\sum_{j=2}^{l} T_{k,j}, \qquad S_{k,1}=0,\\ \mathcal{F}_{k,l}&=\sigma(Z_1, Z_2, \cdots, Z_l, X_z(z\in V_n)) \qquad 2\leqq l\leqq k. \end{aligned}$$

For convenience we set $\mathcal{F}_{k,0}=\{\phi, \Omega\}$. Then $\{S_{k,l}, \mathcal{F}_{k,l} : 2\leqq l\leqq k\}$ forms a martingale sequence, that is, for any $l(1\leqq l\leqq k)$

$$(6.16) \quad E\{S_{k,l}|\mathcal{F}_{k,l-1}\}=S_{k,l-1} \qquad \text{a.s.}.$$

Now we proceed to prove Theorem 5.3. As in the proof of Theorem 3.4 we shall apply the martingale central limit theorem (Lemma 4.2) to the sequence $\{S_{k,l}\}$. Thus in view of Lemmas 6.1 and 6.2 and the argument in the finish of the proof of Theorem 3.4 it suffices to prove that as $n\to\infty$

$$(6.17) \quad s_k^{-4}\textstyle\sum_{j=2}^{k} E\{T_{kj}^4\} \longrightarrow 0,$$

where $s_k^2=E\{S_{k,k}^2\}$, and

$$(6.18) \quad s_k^{-2}\textstyle\sum_{j=2}^{k} T_{kj}^2 \longrightarrow 1 \qquad \text{in probability.}$$

Remark. Remark that

$$s_k^2 = \sum_{j=2}^{k} E\{T_{kj}^2\}$$
(6.19)
$$\sim \frac{kh_k^2}{h_k^d} f(x_1, x_2 : v_0) \int W^2(z)dz \iint H^2(x, y)dxdy.$$

Lemma 6.4. *As* $n \to \infty$,

(6.20) $$s_{k_n}^{-4} \sum_{j=2}^{k_n} E\{T_{k_n j}^4\} \longrightarrow 0.$$

Proof. By the definition of G_k we have

$$E\{T_{kj}^4\} = E\{(\sum_{i<j} G_k(Z_i, Z_j : X_{Z_i}, X_{Z_j}))^4\}$$
(6.21)
$$= (j-1)E\{G_k^4(Z_1, Z_2 : X_{Z_1}, X_{Z_2})\}$$
$$+3(j-1)(j-2)E\{G_k^2(Z_1, Z_2 : X_{Z_1}, X_{Z_2})G_k^2(Z_1, Z_3 : X_{Z_1}, X_{Z_3})\}.$$

Hence we have

$$s_k^{-4} \sum_{j=2}^{k} E\{T_{kj}^4\} \leqq \text{const}\, s_k^{-4} k^2 E\{G_k^4(Z_1, Z_2 : X_{Z_1}, X_{Z_2})\}$$
$$+\text{const}\, s_k^{-4} k^3 E\{G_k^2(Z_1, Z_2 : X_{Z_1}, X_{Z_2})G_k^2(Z_1, Z_3 : X_{Z_1}, X_{Z_3})\}$$
(6.22)
$$= O\Big(\frac{1}{kh_k^{d+2}}\Big) + O\Big(\frac{1}{kh_k^2}\Big).$$

It remains to show that as $n \to \infty$,

(6.23) $$T = s_k^{-4} E\{[\sum_{j=2}^{k} T_{kj}^2 - \sum_{j=1}^{k} E\{T_{kj}^2\}]^2\} \longrightarrow 0.$$

We write $\tilde{T}_{kj}$ for $T_{kj}^2 - E\{T_{kj}^2\}$.

$$T = s_k^{-4} E\{\sum_{j=2}^{k} \tilde{T}_{kj}^2\} + 2s_k^{-4} E\} \sum_{j_1<j_2} \tilde{T}_{kj_1} \tilde{T}_{kj_2}\}$$
(6.24)
$$= I_1 + 2I_2. \quad \text{(say)}$$

By Lemma 6.4 we have that as $n \to \infty$

(6.25) $$I_1 = s_k^{-4} \sum_{j=2}^{k} E\{\tilde{T}_{kj}^2\} \leqq s_k^{-4} \sum_{j=2}^{k} E\{T_{kj}^4\} \longrightarrow 0.$$

For $j_1 < j_2$, by the definitions of w_k and G_k we have

$$E\{\tilde{T}_{kj_1} \tilde{T}_{kj_2}\} \leqq E\{T_{kj_1}^2 T_{kj_2}^2\}$$
$$= E\Big\{\Big[\sum_{i=1}^{j_1-1} G_k^2(Z_i, Z_{j_1} : X_{Z_i}, X_{Z_{j_1}}) + 2\sum_{i<l}^{j_1-1} G_k(Z_i, Z_{j_1} : X_{Z_i}, X_{Z_{j_1}})$$
$$\times G_k(Z_l, Z_{j_1} : X_{Z_l}, X_{Z_{j_1}})\Big]$$
$$\times \Big[\sum_{i=1}^{j_2-1} G_k^2(Z_i, Z_{j_2} : X_{Z_i}, X_{Z_{j_2}}) + 2\sum_{i<l}^{j_2-1} G_k(Z_i, Z_{j_2} : X_{Z_i}, X_{Z_{j_2}})$$
(6.26)
$$\times G_k(Z_l, Z_{j_2} : X_{Z_l}, X_{Z_{j_2}})\Big]\Big\}$$

$$=(j_1-1)\int_{V_n}\bar{m}(dz)E\{G_k^2(z,\ Z_1:X_z,\ X_{Z_1})G_k^2(z,\ Z_2:X_z,\ X_{Z_2})\}$$

$$+(j_1-1)(j_2-1)\int_{V_n}\int_{V_n}\bar{m}(dz_1)\bar{m}(dz_2)$$

$$\times E\{G_k^2(z_1,\ Z_1:X_{z_1},\ X_{Z_1})G_k^2(z_2,\ Z_2:X_{z_2},\ X_{Z_2})\}$$

$$\leqq \frac{C_1(j_1-1)h_k^2}{h_k^{2d}|V_n|^2}+\frac{C_2(j_1-1)(j_2-1)h_k^2}{h_k^{2d}|V_n|^2}.$$

Here we used the Schwarz inequality. Thus using the relation $k\sim\lambda|V_n|$ we have

$$T=O\Big(\frac{h_k^{2d}}{k^2h_k^4}\frac{k^3h_k^2}{h_k^{2d}|V_n|^2}\Big)=O\Big(\frac{1}{kh_k^2}\Big)=o(1). \tag{6.27}$$

Thus we have completed the proof of Theorem 5.3.

Proof of Theorem 5.4. We treat with the case $(x_1,\ x_2)\neq(x_3,\ x_4)$ only. We can treat with the case $v_0\neq v_1$ similarly. In order to prove Theorem 5.4 it suffices to show that

$$T_k=E\Big\{\Big[\sum_{i<j}G_k(Z_i,Z_j:X_{Z_i},X_{Z_j})\Big]\Big[\sum_{s<l}G_k'(Z_s,Z_l:X_{Z_s},X_{Z_l})\Big]\Big\}=o\Big(\frac{k^2h_k^4}{kh_k^{d+2}}\Big), \tag{6.28}$$

where G_k' denotes the formula corresponding to G_k (see (6.4)) with parameters replaced by $(x_3,\ x_4)$. By **B.5** and the definitions of G_k, G_k', using the argument similar to the proof of Theorem 3.5 we have

$$T_k=\sum_{i<j}E\{G_k(Z_i,\ Z_j:X_{Z_i},\ X_{Z_j})G_k'(Z_i,\ Z_j:X_{Z_i},\ X_{Z_j})\}$$

$$=\frac{k(k-1)}{2}E\{G_k(Z_1,\ Z_2:X_{Z_1},\ X_{Z_2})G_k'(Z_1,\ Z_2:X_{Z_1},\ X_{Z_2})\}$$

$$\sim k(k-1)E\{w_k(v_1-(Z_2-Z_1))H_k(x_1-X_{Z_1},\ x_2-X_{Z_2}) \tag{6.29}$$

$$\times w_k(v_1-(Z_2-Z_1))H_k(x_3-X_{Z_1},\ x_2-Z_2)\}$$

$$=k(k-1)\int_{V_n}\int_{V_n}w_k^2(v_1-(z_2-z_1))$$

$$\times E\{H_k(x_1-X_{z_1},\ x_2-X_{z_2})H_k(x_3-X_{z_1},\ x_4-X_{z_2})\}\bar{m}(dz_1)\bar{m}(dz_2).$$

By the condition (5.6) we see that

$$E\{H_k(x_1-X_{z_1},\ x_2-X_{z_2})H_k(x_3-X_{z_1},\ x_4-X_{z_2})\}$$

$$=\iint H_k(x_1-x,\ x_2-y)H_k(x_3-x,\ x_4-y)f(x,\ y:z_1,\ z_2)dxdy \tag{6.30}$$

$$=\iint H(u, v)H(u+(x_3-x_1)/h_k, v+(x_4-x_2)/h_k)dudv$$

$$=O(h_k^4).$$

Therefore since w_k is a probability density we have

(6.31) $$T_k=O(k(k-1)h_k^{4-d}|V_n|^{-1})=o(kh_k^{4-d}),$$

proving (6.28). Thus we completed the proof of Theorem 5.4.

Remark. I. For $d\geqq 2$ the following estimate is obtained from the proof of Theorem 5.1.

(6.32) $$E\{f_V(x_1, x_2: v)\}-f(x_1, x_2: v)=O(|V|^{-1/d}).$$

which cannot be improved without some edge correction (c.f. [10]).

II. Several results in Section 5 can be extended to the case where N is a non-homogeneous Poisson process. Assume that there exists a positive bounded and continuous function $p(z)$ on R^d such that

(6.33) $$m(dz)=p(z)dz$$

and

(6.34) $$p(z)\geqq a \quad \text{for some } a>0.$$

Then an estimator $\bar{f}_V(x_1, x_2: v)$ of $f(x_1, x_2: v)$ is given by

$$\frac{1}{N(V)h_{N(V)}}\int_V\int_V \frac{w_{N(V)}(v-(z_2-z_1))}{p(z_1)p(z_2)} H_{N(V)}(x_1-X_{z_1}, x_2-X_{z_2})N(dz_1)N(dz_2).$$

If the sequence $\{V_n\}$ of observation-domains satisfies **B.4** and $\lim_{n\to\infty}|V_n|^{-1}m(V_n)=\alpha$ for some positive constant α, then we can prove the results in Section 5 without any other additional assumptions. For example we can prove that as $n\to\infty$

(6.35) $$E\{\bar{f}_{V_n}(x, y: v)\} \longrightarrow \alpha f(x, y: v).$$

However there is a substantial drawback in estimating joint densities using $\bar{f}_V(x_1, x_2: v)$, because we must have the complete knowledge of intensity $p(z)$ in advance to estimation.

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