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THEORY OF HILBERT TRIPLE SYSTEMS[1]

Dedicated to the patience of my wife

By

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Summary. Hilbert triple systems were introduced into mathematics because of the role they play in a certain part of infinite-dimensional geometry. In this paper we describe the bases for the theory of nonassociative Hilbert triple systems from the algebraic and functional analytic point of view. In the last section we give a new elementary proof that every simple associative Hilbert triple system is isomorphic either to the triple of Hilbert-Schmidt operators or to its negative.

0. Introduction and definitions

The theory of H^*-algebras was initiated by Ambrose in [1]. His motivation was to obtain an abstract characterization of the class of Hilbert-Schmidt operators.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{A}=\mathbf{HS}(\mathcal{H})$ the algebra of all Hilbert-Schmidt operators acting on $\mathcal{H}$. If we introduce the inner product in $\mathcal{A}$ with $\langle A, B\rangle=\mathrm{trace}(AB^*)$, we obtain an associative model for the following structure:

Definition 1. Let $\mathcal{A}$ be a complex Hilbert space with the inner product $\langle\ ,\ \rangle$. If $\mathcal{A}$ is also a (nonassociative) algebra with the involution $*$, then $\mathcal{A}$ is called an *H^*-algebra* if

$$\langle xy, z\rangle=\langle x, zy^*\rangle=\langle y, x^*z\rangle$$

holds for all $x, y, z\in\mathcal{A}$.

In [1] Ambrose proved that every (topologically) simple associative H^*-algebra is isomorphic to the algebra $\mathbf{HS}(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$. Later this theory was successfully extended to some classical nonassociative algebras such as Jordan (see [16], [17], [18] and [24]), noncommutative Jordan (see [16]), Lie (see [10], [11], [12], [14], [20], [21], [22] and [23]), alternative

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(see [19] and [24]) and Mal'cev (see [13] and [23]). Some important results on general nonassociative H^*-algebras were obtained in [15] and [16].

The concept of a Hilbert triple system can be regarded as a generalization of the H^*-algebra one. The motivation for this concept comes from the very important role that this triple systems play in geometry. They have been used by Kaup to give a classification of symmetric hermitian Banach manifolds. The following example is perhaps interesting also from a functional analytic point of view.

Example 1. Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hibert spaces and let $\mathcal{W}=\mathbf{HS}(\mathcal{H}, \mathcal{K})$ denote the class of all Hilbert-Schmidt operators from $\mathcal{H}$ into $\mathcal{K}$. Define the inner product on $\mathcal{W}$ with $\langle A, B\rangle=\mathrm{trace}(AB^*)$ and the triple product $[\cdots]:\mathcal{W}\times\mathcal{W}\times\mathcal{W}\to\mathcal{W}$ with $[ABC]=AB^*C$. It is easy to verify that $\mathcal{W}$ is a Hilbert triple system according to the following:

Definition 2. Let $\mathcal{W}\neq(0)$ be a complex Hilbert space with the inner product $\langle\,,\rangle$. Let $[\cdots]:\mathcal{W}\times\mathcal{W}\times\mathcal{W}\to\mathcal{W}$ be a mapping which is linear in the first and third variable and conjugate-linear in the second. Then $\mathcal{W}$ is called a *Hilbert triple system* if

$$\langle[xyz], w\rangle=\langle x, [wzy]\rangle=\langle z, [yxw]\rangle$$

holds for all $x, y, z, w\in\mathcal{W}$.

The Hilbert triple system in Example 1 belongs to the class of so called associative triple systems.

Definition 3. Let $\mathcal{W}$ be a Hilbert triple system and $\mathcal{I}$ a subspace of $\mathcal{W}$. $\mathcal{I}$ is called a *left* (*right*) *ideal* of $\mathcal{W}$ if $[\mathcal{W}\mathcal{W}\mathcal{I}]\subset\mathcal{I}$ ($[\mathcal{I}\mathcal{W}\mathcal{W}]\subset\mathcal{I}$) holds. If $\mathcal{I}$ is both right and left ideal, then $\mathcal{I}$ is called an *outsided ideal*. If $[\mathcal{I}\mathcal{W}\mathcal{W}]+[\mathcal{W}\mathcal{I}\mathcal{W}]+[\mathcal{W}\mathcal{W}\mathcal{I}]\subset\mathcal{I}$ holds, then $\mathcal{I}$ is an *ideal* of $\mathcal{W}$. A Hilbert triple system is called *simple* if $[\mathcal{W}\mathcal{W}\mathcal{W}]\neq(0)$ and if (0) and $\mathcal{W}$ are the only closed ideals of $\mathcal{W}$.

Example 2. Let $\mathcal{A}$ be a (nonassociative) H^*-algebra with an inner product $\langle\,,\rangle$ and involution $*$. Define a triple product on $\mathcal{A}$ with $[xyz]=xy^*\cdot z$. Then $(\mathcal{A}, \langle\,,\rangle, [\cdots])$ is a Hilbert triple system.

In the view of this example we can say that the theory of Hilbert triple systems is in some sense a generalization of the theory of H^*-algebras.

Example 3. Let $\mathcal{W}$ be a Hilbert space and $J:\mathcal{W}\to\mathcal{W}$ a conjugate-linear mapping such that $J^2(x)=-x$ and $\langle J(x), J(y)\rangle=\langle y, x\rangle$ hold for all $x, y\in\mathcal{W}$. Define the triple product on $\mathcal{W}$ with $[xyz]=\langle z, y\rangle x-\langle x, J(z)\rangle J(y)$. Then $\mathcal{W}$ becomes a Hilbert triple system which belongs to the class of so called alternative triple systems.

The theory of H^*-algebras is often a source of inspiration for papers on Hilbert triple systems. This leads every author to revisit some pieces of the theory of H^*-algebras and try to transfer the methods used there into the concept of Hilbert triple systems. We feel that it makes sense to select all important results which hold for general nonassociative H^*-algebras and present their generalizations to the concept of Hilbert triple systems (with complete proofs) in one paper which could then serve as a reference to the future papers on this subject.

Following this idea, in the present survey we first provide (sections 1, 2, 3 and 4) a basis for the theory of general nonassociative Hilbert triple systems.

In the last section we describe the structure of associative Hilbert triple systems. It seems that the main result of this section is not new. In [36] the authors independently obtained the same result using the theory of Hilbert modules given in [43] and the theory of induced representation of C^*-algebras. Our proof is entirely different and more elementary.

Our intention was to give this exposition as self-contained as possible. We assume only that the reader is familiar with the basic concepts of the functional analysis such as inner product spaces, the Hahn-Banach theorem, the closed graph theorem and the uniform boundedness principle.

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1. Continuity of multiplication and first structure theorem

Proposition 1. *Let $\mathcal{W}$ be a Hilbert triple system. Then the multiplication $[\cdots]: \mathcal{W}^3 \to \mathcal{W}$ is continuous.*

Proof. Denote the left multiplication operators with $L(a, b)(x)=[abx]$. The definition of the Hilbert triple system yields $L(a, b)^*=L(b, a)$. Since these operators are everywhere defined, it follows that they are continuous for all $a, b \in \mathcal{W}$.

Take any $a \in \mathcal{W}$ and define a mapping $\phi_a: \mathcal{W} \to B(\mathcal{W})$ with $\phi_a(x)=L(a, x)$. Here we use a rather standard notation $B(\mathcal{W})$ for the algebra of all bounded linear operators acting on $\mathcal{W}$. Similarly we define a mapping $\psi_a: \mathcal{W} \to B(\mathcal{W})$ with $\psi_a(x)=L(x, a)$. Now we shall prove, using the closed graph theorem, that ϕ_a and ψ_a are continuous mappings for all $a \in \mathcal{W}$.

Suppose that

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} \phi_a(x_n) = T.$$

Then we have

$$\begin{aligned}\langle T(x), y\rangle &= \lim_{n\to\infty}\langle \phi_a(x_n)(x), y\rangle = \lim_{n\to\infty}\langle L(a, x_n)(x), y\rangle \\ &= \lim_{n\to\infty}\langle [a x_n x], y\rangle = \lim_{n\to\infty}\langle a, [y x x_n]\rangle \\ &= \lim_{n\to\infty}\langle [x y a], x_n\rangle = 0.\end{aligned}$$

This means that $T=0$ and therefore ϕ_a is continuous. In a similar way we can prove that ψ_a is continuous.

Now we define a set $\mathcal{S}=\{\phi_a ; \|a\|\leqq 1\}$. Take any $\phi_a\in\mathcal{S}$. Then we have

$$\begin{aligned}\|\phi_a(x)\| &= \|L(a, x)\| = \|\psi_x(a)\| \\ &\leqq \|\psi_x\|\|a\| \leqq \|\psi_x\| < \infty\end{aligned}$$

for all $x\in\mathcal{W}$. By the uniform boundedness principle, there exists some constant $M>0$ such that $\phi_a\in\mathcal{S}$ implies $\|\phi_a\|\leqq M$. This gives us, for all $x\in\mathcal{W}$,

$$\|\phi_{x/\|x\|}\|\leqq M \qquad \text{or equivalently} \qquad \|\phi_x\|\leqq M\|x\|.$$

This futher implies

$$\|L(x, y)\| = \|\phi_x(y)\| \leqq \|\phi_x\|\|y\| \leqq M\|x\|\|y\|,$$

which finally gives us

$$\|[xyz]\| = \|L(x, y)(z)\| \leqq \|L(x, y)\|\|z\| \leqq M\|x\|\|y\|\|z\|$$

for all $x, y, z\in\mathcal{W}$, which means that $[\cdots]$ is a continuous mapping. □

Definition 4. Let $\mathcal{W}$ be a triple system. Define the *left, middle* and *right annihilator* of the triple system $\mathcal{W}$ by

$$\begin{aligned}\mathrm{Lann}(\mathcal{W}) &= \{a\in\mathcal{W} : [a\mathcal{W}\mathcal{W}]=(0)\}, \\ \mathrm{Mann}(\mathcal{W}) &= \{a\in\mathcal{W} ; [\mathcal{W}a\mathcal{W}]=(0)\}, \\ \mathrm{Rann}(\mathcal{W}) &= \{a\in\mathcal{W} ; [\mathcal{W}\mathcal{W}a]=(0)\}.\end{aligned}$$

Lemma 1. *Let $\mathcal{W}$ be a Hilbert triple system. Then*

$$\mathrm{Lann}(\mathcal{W}) = \mathrm{Mann}(\mathcal{W}) = \mathrm{Rann}(\mathcal{W})$$

holds.

Proof. Take any $a\in\mathrm{Lann}(\mathcal{W})$. Then we have

$$\langle [\mathcal{W}a\mathcal{W}], \mathcal{W}\rangle = \langle \mathcal{W}, [a\mathcal{W}\mathcal{W}]\rangle = (0),$$

which implies $a\in\mathrm{Mann}(\mathcal{W})$. Therefore $\mathrm{Lann}(\mathcal{W})\subset\mathrm{Mann}(\mathcal{W})$. Other inclusions can be proved in a similar way. □

In the view of the above lemma we shall speak about the *annihilator* of the Hilbert triple system.

Theorem 1. *Let $\mathcal{W}$ be a Hilbert triple system. Then $\mathcal{W}$ can be decomposed into an orthogonal sum $\mathcal{W}=\mathcal{W}_0\oplus\mathcal{W}_1$, where $\mathcal{W}_0$ is a Hilbert triple system with a trivial multiplication and $\mathcal{W}_1$ a Hilbert triple system with zero annihilator.*

Proof. Let $\mathcal{W}_0$ be the annihilator of $\mathcal{W}$. It is obvious that $\mathcal{W}_0$ is a linear subspace with the trivial multiplication. Since the multiplication of $\mathcal{W}$ is continuous, it is also easy to see that $\mathcal{W}_0$ is closed. Therefore we have $\mathcal{W}=\mathcal{W}_0\oplus\mathcal{W}_1$ with $\mathcal{W}_1=\mathcal{W}_0^{\perp}$. It remains to prove that $\mathcal{W}_1$ is a subtriple of $\mathcal{W}$ and that the annihilator of $\mathcal{W}_1$ is zero.

First we have

$$\langle[\mathcal{W}_1\mathcal{W}_1\mathcal{W}_1],\ \mathcal{W}_0\rangle=\langle\mathcal{W}_1,\ [\mathcal{W}_0\mathcal{W}_1\mathcal{W}_1]\rangle=(0),$$

which implies $[\mathcal{W}_1\mathcal{W}_1\mathcal{W}_1]\subset\mathcal{W}_0^{\perp}=\mathcal{W}_1$. Now suppose that $a\in\mathcal{W}_1$ belongs to the annihilator of $\mathcal{W}_1$. Next we see that

$$[a\mathcal{W}\mathcal{W}]=[a\mathcal{W}_0\mathcal{W}_0]+[a\mathcal{W}_0\mathcal{W}_1]+[a\mathcal{W}_1\mathcal{W}_0]+[a\mathcal{W}_1\mathcal{W}_1]=(0)$$

holds. The first three terms on the right side of this equality are zero because $\mathcal{W}_0$ is the annihilator of $\mathcal{W}$ and Lemma 1. The last term is zero because $\mathcal{W}_1$ is also a Hilbert triple system and we can use Lemma 1 for $\mathcal{W}_1$. This implies that a belongs to the annihilator of $\mathcal{W}$ and therefore $a\in\mathcal{W}_0\cap\mathcal{W}_1=(0)$. □

2. Ideals and second structure theorem

The purpose of this section is to prove that every Hilbert triple system with zero annihilator is a direct sum of simple Hilbert triple systems. First we need some facts about ideals in Hilbert triple systems which we can summarize in the following:

Proposition 2. *Let $\mathcal{W}$ be a Hilbert triple system with zero annihilator. Then the following statements hold:*

(i) *If $\mathcal{L}$ is a left ideal of $\mathcal{W}$, then $\mathcal{L}^{\perp}$ is also a left ideal of $\mathcal{W}$.*

(ii) *If $\mathcal{R}$ is a right ideal of $\mathcal{W}$, then $\mathcal{R}^{\perp}$ is also a right ideal of $\mathcal{W}$.*

(iii) *Every closed outsided ideal of $\mathcal{W}$ is also an ideal of $\mathcal{W}$.*

(iv) *If $\mathcal{I}$ is a closed ideal of $\mathcal{W}$, then $\{x\in\mathcal{W};\ [x\mathcal{W}\mathcal{I}]=(0)\}=\{x\in\mathcal{W};\ [\mathcal{I}\mathcal{W}x]=(0)\}=\mathcal{I}^{\perp}$ holds.*

(v) *Let $\mathcal{I}$ be a closed ideal of $\mathcal{W}$ and $\mathcal{K}$ a closed ideal of $\mathcal{I}$. Then $\mathcal{K}$ is also a closed ideal of $\mathcal{W}$.*

(vi) *Let $\mathcal{I}$ and $\mathcal{K}$ be minimal closed ideals of $\mathcal{W}$. Then either $\mathcal{I}=\mathcal{K}$ or $\mathcal{I}\perp\mathcal{K}$.*

Proof. (i) From

$$\langle[\mathcal{W}\mathcal{W}\mathcal{L}^{\perp}], \mathcal{L}\rangle=\langle\mathcal{L}^{\perp}, [\mathcal{W}\mathcal{W}\mathcal{L}]\rangle\subset\langle\mathcal{L}^{\perp}, \mathcal{L}\rangle=(0)$$

it easily follows that $\mathcal{L}^{\perp}$ is a left ideal. A similar observation shows that (ii) also holds.

(iii) Let $\mathcal{I}$ be an outsided ideal of $\mathcal{W}$. By (i) and (ii) $\mathcal{I}^{\perp}$ is also an outsided ideal of $\mathcal{W}$. This implies that $[\mathcal{I}^{\perp}\mathcal{W}\mathcal{I}]\subset\mathcal{I}\cap\mathcal{I}^{\perp}=(0)$ and hence

$$\langle[\mathcal{W}\mathcal{I}\mathcal{W}], \mathcal{I}^{\perp}\rangle=\langle\mathcal{W}, [\mathcal{I}^{\perp}\mathcal{W}\mathcal{I}]\rangle\subset\langle\mathcal{W}, (0)\rangle=(0),$$

which finally gives us $[\mathcal{W}\mathcal{I}\mathcal{W}]\subset\mathcal{I}^{\perp\perp}=\mathcal{I}$.

(iv) Denote by

$$\mathrm{Lann}_{\mathcal{W}}(\mathcal{I})=\{x\in\mathcal{W};\ [x\mathcal{W}\mathcal{I}]=(0)\}$$

and

$$\mathrm{Rann}_{\mathcal{W}}(\mathcal{I})=\{x\in\mathcal{W};\ [\mathcal{I}\mathcal{W}x]=(0)\}.$$

From (ii) and (iii) it is obvious that

$$\mathcal{I}^{\perp}\subset\mathrm{Lann}_{\mathcal{W}}(\mathcal{I})\cap\mathrm{Rann}_{\mathcal{W}}(\mathcal{I})$$

holds.

In order to prove the converse take any $x\in\mathrm{Lann}_{\mathcal{W}}(\mathcal{I})$ and decompose it into a sum $x=y+z$, where $y\in\mathcal{I}$ and $z\in\mathcal{I}^{\perp}$ hold. Take also elements $a, b\in\mathcal{W}$. Decompose b into a sum $b=b_1+b_2$, where $b_1\in\mathcal{I}$ and $b_2\in\mathcal{I}^{\perp}$ hold. Then we have

$$\begin{aligned}[yab]&=[xab]-[zab]=[xab_1]+[xab_2]-[zab]=\\&=[xab_2]-[zab_1]-[zab_2]=[xab_2]-[zab_2]=[yab_2]=0.\end{aligned}$$

This means that y belongs to the annihilator of the triple system $\mathcal{W}$, which is zero by assumption. Hence $x=z\in\mathcal{I}^{\perp}$. So $\mathrm{Lann}_{\mathcal{W}}(\mathcal{I})=\mathcal{I}^{\perp}$ holds. In a similar way we can prove $\mathrm{Rann}_{\mathcal{W}}(\mathcal{I})=\mathcal{I}^{\perp}$.

(v) This follows from

$$\begin{aligned}[\mathcal{W}\mathcal{W}\mathcal{K}]&=[\mathcal{I}\mathcal{W}\mathcal{K}]+[\mathcal{I}^{\perp}\mathcal{W}\mathcal{K}]=[\mathcal{I}\mathcal{W}\mathcal{K}]\\&=[\mathcal{I}\mathcal{I}\mathcal{K}]+[\mathcal{I}\mathcal{I}^{\perp}\mathcal{K}]=[\mathcal{I}\mathcal{I}\mathcal{K}]\subset\mathcal{K},\\[\mathcal{K}\mathcal{W}\mathcal{W}]&=[\mathcal{K}\mathcal{W}\mathcal{I}]+[\mathcal{K}\mathcal{W}\mathcal{I}^{\perp}]=[\mathcal{K}\mathcal{W}\mathcal{I}]\\&=[\mathcal{K}\mathcal{I}\mathcal{I}]+[\mathcal{K}\mathcal{I}^{\perp}\mathcal{I}]=[\mathcal{K}\mathcal{I}\mathcal{I}]\subset\mathcal{K}\end{aligned}$$

and (iii).

(vi) Obviously $\mathcal{I}\cap\mathcal{K}$ is also an ideal of $\mathcal{W}$, which is contained in $\mathcal{I}$ and $\mathcal{K}$. Because of the minimality of those ideals we have two possibilities: either $\mathcal{I}\cap\mathcal{K}=(0)$ or $\mathcal{I}\cap\mathcal{K}=\mathcal{I}=\mathcal{K}$. In the first case we get $[\mathcal{I}\mathcal{W}\mathcal{K}]\subset\mathcal{I}\cap\mathcal{K}=(0)$, which implies $\mathcal{I}\subset\mathrm{Lann}_{\mathcal{W}}(\mathcal{K})=\mathcal{K}^{\perp}$. In the second case there is nothing to prove. □

Our next goal is to prove that minimal closed ideals exist. Let $\mathcal{W}$ be a Hilbert triple system. A nonzero element $x\in\mathcal{W}$ will be called *minimal*, if for every closed ideal $\mathcal{I}$ of $\mathcal{W}$ either $x\in\mathcal{I}$ or $x\in\mathcal{I}^{\perp}$ holds.

Let x be a minimal element and $\mathcal{I}(x)$ the closed ideal generated by x. Let $\mathcal{K}\subset\mathcal{I}(x)$ hold for some closed ideal $\mathcal{K}$. Since x is minimal, either $x\in\mathcal{K}$ or $x\in\mathcal{K}^{\perp}$ holds. In the first case we immediately obtain $\mathcal{I}(x)=\mathcal{K}$. In the second case we get first $\mathcal{I}(x)\subset\mathcal{K}^{\perp}$, which implies $\mathcal{K}\subset\mathcal{K}^{\perp}$ and finally $\mathcal{K}=(0)$.

Therefore the existence of minimal elements in $\mathcal{W}$ implies that $\mathcal{W}$ contains minimal closed ideals.

Proposition 3. *Let $\mathcal{W}$ be a Hilbert triple system with zero annihilator. Then $\mathcal{W}$ contains a minimal element.*

Proof. Define $\phi_a:\mathcal{W}\to B(\mathcal{W})$ in the same way as in the proof of Proposition 1. Define also a new norm on $\mathcal{W}$ by $|a|=\|\phi_a\|$. This is a norm because $\phi_a=0$ if and only if $a=0$. $(\mathcal{W}, |\,|)$ is not necessarily a Banach space, since it is not necessarily complete.

Despite this fact, the unit ball $\mathcal{B}$ of the dual $(\mathcal{W}, |\,|)^{*}$ has, according to the theorems Alaoglu-Bourbaki and Krein-Milman, an extreme point f_0. We shall prove that $f_0(x)=\langle x, a\rangle$ holds and that a is a minimal element.

First we must observe that the functional f_0 is also continuous in the Hilbert space topology. Inequality $\|\phi_x\|\leqq M\|x\|$ gives us $|x|\leqq M\|x\|$. Therefore $\|f_0\|\leqq M|f_0|$ holds and so $f_0\in\mathcal{W}^{*}$. It is well-known that this implies $f_0(x)=\langle x, a\rangle$ for some $a\in\mathcal{W}$. Since f_0 is an extreme point of the set $\mathcal{B}$, it lies in the boundary of $\mathcal{B}$ and therefore $|f_0|=1$.

Take a closed ideal $\mathcal{I}$ of the triple system $\mathcal{W}$. As we already know, $\mathcal{W}$ can be decomposed into an orthogonal sum of two ideals $\mathcal{W}=\mathcal{I}\oplus\mathcal{I}^{\perp}$. We shall prove that the dual $(\mathcal{W}, |\,|)^{*}$ can also be decomposed into the direct sum of two suitable subspaces.

1. *step.* Take $x\in\mathcal{I}$ and $y\in\mathcal{I}^{\perp}$. We shall prove that the equality $|x+y|=$ $=\max\{|x|, |y|\}$ holds.

Take $z, w\in\mathcal{W}$ and compute $\|\phi_{x+y}(z)(w)\|$:

$$\|\phi_{x+y}(z)(w)\|^2=\|[xzw]+[yzw]\|^2=\|[xzw]\|^2+\|[yzw]\|^2=$$
$$=\|\phi_x(z)(w)\|^2+\|\phi_y(z)(w)\|^2.$$

This obviously implies that the following two inequalities hold:

$$\|\phi_{x+y}(z)(w)\|\geqq\|\phi_x(z)(w)\|,$$
$$\|\phi_{x+y}(z)(w)\|\geqq\|\phi_y(z)(w)\|$$

and finally

$$|x+y|=\|\phi_{x+y}\|\geqq\max\{\|\phi_x\|, \|\phi_y\|\}=\max\{|x|, |y|\}.$$

Now denote $\alpha=\max\{|x|, |y|\}$. We shall decompose the element w, in the sense of the above decomposition of $\mathcal{W}$, into a sum $w=u+v$. Then we have

$$\begin{aligned}\|\phi_{x+y}(z)(w)\|^2&=\|[xzw]+[yzw]\|^2=\|[xzu]+[yzv]\|^2\\&=\|[xzu]\|^2+\|[yzv]\|^2=\|\phi_x(z)(u)\|^2+\|\phi_y(z)(v)\|^2\\&\leqq\|\phi_x(z)\|^2\|u\|^2+\|\phi_y(z)\|^2\|v\|^2\\&\leqq\alpha^2\|z\|^2(\|u\|^2+\|v\|^2)=\alpha^2\|z\|^2\|w\|^2,\end{aligned}$$

and so

$$|x+y|=\|\phi_{x+y}\|\leqq\alpha=\max\{|x|, |y|\}.$$

2. *step*. Define $\mathcal{U}=\{g\in(\mathcal{W}, |\,|)^{\#};\ g(\mathcal{I}^{\perp})=(0)\}$ and $\mathcal{V}=\{g\in(\mathcal{W}, |\,|)^{\#};\ g(\mathcal{I})=(0)\}$. Take $g\in\mathcal{U}$ and $h\in\mathcal{V}$. Then the identity $|g+h|=|g|+|h|$ holds.

Take any $\varepsilon>0$. There exist such elements $x\in\mathcal{I}$ and $y\in\mathcal{I}^{\perp}$ that

$$|x|=|y|=1,\quad g(x)>|g|-\frac{\varepsilon}{2},\quad h(y)>|h|-\frac{\varepsilon}{2}$$

holds. This gives us

$$\begin{aligned}|g+h|&\geqq\frac{|(g+h)(x+y)|}{|x+y|}=\frac{g(x)+h(y)}{\max\{|x|, |y|\}}\\&=g(x)+h(y)>|g|+|h|-\varepsilon.\end{aligned}$$

Since ε was arbitrary, we obtain $|g+h|\geqq|g|+|h|$. Since the reverse inequality is trivial, the proof of this step is completed.

3. *step*. The dual of $(\mathcal{W}, |\,|)$ can be decomposed into a Banach space direct sum $(\mathcal{W}, |\,|)^{\#}=\mathcal{U}\oplus\mathcal{V}$. This time the notation $\oplus$ is used exceptionally for a sum which is not orthogonal.

Let P and Q be projections on $\mathcal{I}$ and $\mathcal{I}^{\perp}$ respectively. We must observe that they are continuous in the topology generated on $\mathcal{W}$ by the norm $|\,|$. To this end take $z\in\mathcal{W}$ and decompose $z=x+y$ where $x\in\mathcal{I}$ and $y\in\mathcal{I}^{\perp}$. The inequality

$$|P(z)|=|x|\leqq\max\{|x|, |y|\}=|z|$$

already gives us desired information for P. In a similar way we can observe that Q is also continuous in this topology.

Now take $f\in(\mathcal{W}, |\,|)^{\#}$. If we define $g=f\circ P$ and $h=f\circ Q$, it is obvious that $h\in\mathcal{U}$, $g\in\mathcal{V}$ and $f=g+h$. From the previous step we also know that $|f|=|g|+|h|$ holds. Now it only remains to prove that a is minimal.

Since $\mathcal{I}$ was an arbitrary closed ideal, we must see that either $a\in\mathcal{I}$ or $a\in\mathcal{I}^{\perp}$ hold. Decompose $f_0=g+h$ in the sense of the previous step. Suppose for a moment that g and h are both nonzero. Then we can write $f_0=|g|(g/|g|)$

$+|h|(h/|h|)$. From $|f_0|=1$ and from the second step of this proof, it follows that this is a convex combination of the elements from $\mathcal{B}$. Since f_0 in an extreme point of $\mathcal{B}$, it follows $f_0=g/|g|=h/|h|$, which is a contradiction. Therefore either $g=0$ or $h=0$ holds.

In the second case we get $(0)=f_0(\mathcal{I}^\perp)=\langle\mathcal{I}^\perp, a\rangle$. This implies that $a\in\mathcal{I}^{\perp\perp}=\mathcal{I}$. In the first case we get $a\in\mathcal{I}^\perp$. □

Theorem 2. *Let $\mathcal{W}$ be a Hilbert triple system with zero annihilator. Then $\mathcal{W}$ can be decomposed into an orthogonal sum of simple Hilbert triple systems.*

Proof. Using the previous proposition, we obtain the existence of some minimal closed ideal. Let $\{\mathcal{I}_\alpha\,;\,\alpha\in\Lambda\}$ be the family of all minimal closed ideals of $\mathcal{W}$. By Proposition 2(vi) we can form an orthogonal sum in $\mathcal{W}$ by $\mathcal{I}=\bigoplus_{\alpha\in\Lambda}\mathcal{I}_\alpha$. It is obvious that $\mathcal{I}$ is also a closed ideal of $\mathcal{W}$. Therefore by Proposition 2(i, ii, iii) $\mathcal{I}^\perp$ is also an ideal of $\mathcal{W}$. Suppose for a moment that $\mathcal{I}^\perp\neq(0)$. By Proposition 3 $\mathcal{I}^\perp$ contains some minimal closed ideal $\mathcal{K}$. By Proposition 2(v) $\mathcal{K}$ is also an ideal of $\mathcal{W}$ and by the same proposition it is a minimal closed ideal of $\mathcal{W}$. Therefore we obtain $\mathcal{K}=\mathcal{I}_\alpha$ for some $\alpha\in\Lambda$, which is obviously a contradiction. This tells us that $\mathcal{I}^\perp=0$ and finally $\mathcal{W}=\mathcal{I}$. From Proposition 2(v) it also follows that each $\mathcal{I}_\alpha$ is a simple Hilbert triple system.

3. Weak radical and uniqueness of topology

This section is devoted to the automatic continuity of isomorphisms between Hilbert triple systems. The concept of a weak radical developed by A. Rodriguez for nonassociative Banach algebras can easily be adapted to Banach triple systems and the main Rodriguez' result about automatic continuity can also be generalized.

Let $\mathcal{W}$ be a Banach space together with the triple product $[\cdots]$, which is continuous. Then $\mathcal{W}$ is called a *Banach triple system*. Define the left and right multiplication operators by $L(a,b)(x)=[abx]$ and $R(a,b)(x)=[xab]$. Denote by $\mathcal{M}_\mathcal{W}$ the subspace of all bounded linear operators $B(\mathcal{W})$ spanned by

$$\{L(a,b)\,;\,a,b\in\mathcal{W}\}\cup\{R(a,b)\,;\,a,b\in\mathcal{W}\}\,.$$

A (possibly unbounded) linear operator A is called *quasiinvertible* if there exists a linear operator B such that $AB=BA=A+B$ holds. The operator B is called *quasiinverse* of A. Let $\mathcal{A}$ be a subalgebra of the algebra of all (unbounded) linear operators acting on $\mathcal{W}$. Then $\mathcal{A}$ is called *full*, if quasiinverse of every quasiinvertible element, which belongs to $\mathcal{A}$, also belongs to $\mathcal{A}$. It is easy to see that intersection of full algebras is also a full algebra. This implies that there exists the smallest full algebra, which contains the subspace $\mathcal{M}_\mathcal{W}$. This algebra will be denoted by $\mathcal{F}_\mathcal{W}$. The algebra of all bounded linear operators on

$\mathcal{W}$ is full. This easily follows from the open mapping theorem. Therefore $\mathcal{F}_{\mathcal{W}} \subset B(\mathcal{W})$ holds.

Now define a subspace of $\mathcal{W}$ by

$$\mathcal{N}_{\mathcal{W}} = \{a \in \mathcal{W};\ \forall x \in \mathcal{W}\ L(a, x),\ R(x, a) \text{ are in Jacobson radical of } \mathcal{F}_{\mathcal{W}}\}.$$

The *weak radical* of the triple system $\mathcal{W}$ is the largest $\mathcal{F}_{\mathcal{W}}$-invariant subspace which is contained in $\mathcal{N}_{\mathcal{W}}$. The weak radical is an outsided ideal of the triple system $\mathcal{W}$ since $\mathcal{F}_{\mathcal{W}}$ contains all left and right multiplication operators. In the rest of this section we shall use the above notation without further comments.

Proposition 4. *Let $\mathcal{W}$ be a Hilbert triple system. Then the weak radical of $\mathcal{W}$ and the annihilator of $\mathcal{W}$ coincide.*

Proof. Define a subalgebra of $B(\mathcal{W})$ by

$$\mathcal{L} = \{T \in B(\mathcal{W});\ T(\mathrm{Ann}(\mathcal{W})) = (0)\}.$$

Here $\mathrm{Ann}(\mathcal{W})$ denotes the annihilator of a Hilbert triple system (see Lemma 1 and the comment after its proof). Let $A \in \mathcal{L}$ be quasiinvertible and B its quasiinverse. From

$$B(\mathrm{Ann}(\mathcal{W})) = (BA - A)(\mathrm{Ann}(\mathcal{W})) = B((0)) - (0) = (0)$$

we obtain that $\mathcal{L}$ is a full algebra. It is obvious that $L(a, b)$ and $R(a, b)$ belongs to $\mathcal{L}$ for all $a, b \in \mathcal{W}$. Therefore $\mathcal{F}_{\mathcal{W}} \subset \mathcal{L}$ holds. Hence $\mathcal{F}_{\mathcal{W}}(\mathrm{Ann}(\mathcal{W})) = (0) \subset \mathrm{Ann}(\mathcal{W})$, which shows that the annihilator of $\mathcal{W}$ is invariant subspace for the algebra $\mathcal{F}_{\mathcal{W}}$. If a belongs to the annihilator of $\mathcal{W}$, then operators $L(a, x) = R(x, a) = 0$ obviously belong to the Jacobson radical of $\mathcal{F}_{\mathcal{W}}$ and therefore the annihilator of $\mathcal{W}$ is contained in the weak radical of $\mathcal{W}$.

In order to prove the converse, observe first that in a Hibert triple system $L(a, b)^* = L(b, a)$ and $R(a, b)^* = R(b, a)$ holds. This means that the subspace $\mathcal{M}_{\mathcal{W}}$ is selfadjoint.

Let $\mathcal{A} \subset B(\mathcal{W})$ be a full algebra. Take some quasiinvertible $A^* \in \mathcal{A}^*$ and let B be its quasiinverse. From the equality $A^*B = BA^* = B + A^*$ we obtain $B^*A = AB^* = B^* + A$, which means that B^* is quasiinverse of A. Since $\mathcal{A}$ is full, B^* also belongs to $\mathcal{A}$. Therefore $B = B^{**}$ belongs to $\mathcal{A}^*$. This tells us that the full algebra generated by a selfadjoint subspace is itself selfadjoint. This means that $\mathcal{F}_{\mathcal{W}}$ is a full pre-C^*-algebra (it is not necessarily closed in $B(\mathcal{W})$). Such algebras have zero Jacobson radical becaus $\lim_{n\to\infty} \|(T^*T)^n\|^{1/n} = 0$ implies $T = 0$. This means that $a \in \mathcal{N}_{\mathcal{W}}$ if and only if $L(a, x) = R(x, a) = 0$ for all $x \in \mathcal{W}$. Since the latter statement implies that a belongs to the annihilator of $\mathcal{W}$, we see that the weak radical is contained in the annihilator of $\mathcal{W}$. □

Let $\mathcal{V}$ and $\mathcal{W}$ be Banach triple systems. A pair of linear operators (A_+, A_-), which map from $\mathcal{V}$ into $\mathcal{W}$, is called an *isomorphism pair*, if A_+ and A_- are invertible and

$$A_\sigma([xyz])=[A_\sigma(x)A_{-\sigma}(y)A_\sigma(z)]$$

holds for all $x, y, z\in\mathcal{V}$ and $\sigma\in\{+,-\}$. If $\mathcal{W}$ is a Hilbert triple system and λ a complex number, then $A_+(x)=\lambda x$, $A_-(x)=\bar{\lambda}^{-1}x$ form an automorphism pair. If $A^+=A_-=A$ holds, then A is called an *isomorphism*. Two triple systems are called *isomorphic*, if there exists some isomorphism between them.

Recall that the *separating subspace* of an operator T between Banach spaces is defined as subspace of those y which can be expressed in the form $y=\lim_{n\to\infty}T(x_n)$ where $\lim_{n\to\infty}x_n=0$. From the closed graph theorem it follows that T is continuous if and only if its separating subspace is zero.

Theorem 3. *Let (A_+, A_-) be an isomorphism pair between Banach triple systems $\mathcal{W}$ and $\mathcal{V}$. Then the separating subspaces of the operators A_+ and A_- are contained in the weak radical of the triple system $\mathcal{V}$.*

Proof. Denote with $L(\mathcal{W})$ the linear space of all (unbounded) operator acting on $\mathcal{W}$. Define mappings ϕ_σ: $L(\mathcal{W})\to L(\mathcal{V})$ with $\phi_\sigma(T)=A_\sigma TA_\sigma^{-1}$, where $\sigma\in\{+,-\}$. It is easy to verify that the mappings ϕ_+ and ϕ_- are isomorphisms.

1. *step.* Every isomorphism ψ maps full algebras into full algebras.

Let $\mathcal{A}$ be a full algebra and let $T\in\psi(\mathcal{A})$ be quasiinvertible. This means that there exist $S\in\mathcal{A}$ and a linear operator X, acting on $\mathcal{W}$, such that $T=\psi(S)$ and $\psi(S)X=X\psi(S)=X+\psi(S)$ hold. This immediately gives us $S\psi^{-1}(X)=\psi^{-1}(X)S=S+\psi^{-1}(X)$. Since $\mathcal{A}$ is full, we get $\psi^{-1}(X)\in\mathcal{A}$ and therefore $X=\psi\psi^{-1}(X)\in\psi(\mathcal{A})$. Hence $\psi(\mathcal{A})$ is full.

2. *step.* $\phi_\sigma(\mathcal{M}_\mathcal{W})=\mathcal{M}_\mathcal{V}$

Take $x, y\in\mathcal{W}$. Then we have

$$\begin{aligned}\phi_\sigma(L(x, y))(v)&=A_\sigma L(x, y)A_\sigma^{-1}(v)=A_\sigma([xyA_\sigma^{-1}(v)])\\&=[A_\sigma(x)A_{-\sigma}(y)v]=L(A_\sigma(x), A_{-\sigma}(y))(v),\\ \phi_\sigma(R(x, y))(v)&=A_\sigma R(x, y)A_\sigma^{-1}(v)=A_\sigma([A_\sigma^{-1}(v)xy])\\&=[vA_{-\sigma}(x)A_\sigma(y)]=R(A_{-\sigma}(x), A_\sigma(y))(v).\end{aligned}$$

This tells us that $\phi_\sigma(\mathcal{M}_\mathcal{W})\subset\mathcal{M}_\mathcal{V}$. Since A_σ, $A_{-\sigma}$ are invertible, we obtain $\phi_\sigma(\mathcal{M}_\mathcal{W})=\mathcal{M}_\mathcal{V}$ if we interchange the role of the triple systems $\mathcal{W}$ and $\mathcal{V}$.

3. *step.* $\phi_\sigma(\mathcal{F}_\mathcal{W})=\mathcal{F}_\mathcal{V}$.

Since $\mathcal{M}_\mathcal{W}\subset\mathcal{F}_\mathcal{W}$, using the previous step, we get $\mathcal{M}_\mathcal{V}\subset\phi_\sigma(\mathcal{F}_\mathcal{W})$. Using the first step we see that the algebra on the right-hand side of the above inclusion is full. This means, by the definition of weak radical, that $\mathcal{F}_\mathcal{V}\subset\phi_\sigma(\mathcal{F}_\mathcal{W})$ holds.

If we interchange the role of $\mathcal{V}$ and $\mathcal{W}$ we get $\mathcal{F}_{\mathcal{W}}\subset\phi_\sigma^{-1}(\mathcal{F}_{\mathcal{V}})$ and hence

$$\phi_\sigma(\mathcal{F}_{\mathcal{W}})\subset\phi_\sigma\phi_\sigma^{-1}(\mathcal{F}_{\mathcal{V}})=\mathcal{F}_{\mathcal{V}}.$$

From now on let ϕ_σ denote the restriction on the subspace $\mathcal{F}_{\mathcal{W}}$, which maps onto $\mathcal{F}_{\mathcal{V}}$.

4. *step.* The separating subspace of the operator ϕ_σ is contained in the Jacobson radical of the algebra $\mathcal{F}_{\mathcal{V}}$.

Since the algebras $\mathcal{F}_{\mathcal{W}}$ and $\mathcal{F}_{\mathcal{V}}$ are full, we can now apply the Aupetit lemma (see [46, Theorem 1 and Remark 2] and [49, Proposition 1.3]), which yields the desired conclusion.

5. *step.* The separating subspace of the operator A_σ is contained in the weak radical of the triple $\mathcal{V}$.

Take element y from the separating subspace of the operator A_σ. This means that there exists a sequence $x_n\in\mathcal{W}$ such that

$$\lim_{n\to\infty} x_n=0, \qquad \lim_{n\to\infty} A_\sigma(x_n)=y.$$

Take also $v\in\mathcal{V}$. Since the multiplications of $\mathcal{W}$ and $\mathcal{V}$ are continuous, we also get

$$\lim_{n\to\infty} L(x_n, A_{-\sigma}^{-1}(v))=0, \qquad \lim_{n\to\infty} R(A_{-\sigma}^{-1}(v), x_n)=0.$$

Besides we have

$$\lim_{n\to\infty}\phi_\sigma(L(x_n, A_{-\sigma}^{-1}(v)))=\lim_{n\to\infty} A_\sigma L(x_n, A_{-\sigma}^{-1}(v))A_\sigma^{-1}=\lim_{n\to\infty} L(A_\sigma(x_n), v)=L(y, v),$$

$$\lim_{n\to\infty}\phi_\sigma(R(A_{-\sigma}^{-1}(v), x_n))=\lim_{n\to\infty} A_\sigma R(A_{-\sigma}^{-1}(v), x_n)A_\sigma^{-1}=\lim_{n\to\infty} R(v, A_\sigma(x_n))=R(v, y).$$

This means that for every $v\in\mathcal{V}$

$$L(y, v),\ R(v, y)\in\mathcal{S}(\phi_\sigma)\subset \mathrm{JRad}(\mathcal{F}_{\mathcal{V}})$$

holds, where by JRad we have denoted the Jacobson radical. We have obtained that the separating subspace of the operator A_σ is contained in $\mathcal{N}_{\mathcal{V}}$.

It remains to see that the separating subspace of A_σ is an invariant subspace for the algebra $\mathcal{F}_{\mathcal{V}}$. Take an operator $T\in\mathcal{F}_{\mathcal{V}}$. From Step 3 we know that T is of the form $T=A_\sigma SA_\sigma^{-1}$, where $S\in\mathcal{F}_{\mathcal{W}}$. All we actually need is the fact that S is continuous. Let y be from the separating subspace of the operator A_σ. There exists a sequence $x_n\in\mathcal{W}$ such that

$$\lim_{n\to\infty} x_n=0, \qquad \lim_{n\to\infty} A_\sigma(x_n)=y$$

holds. Since T and S are continuous, we obtain

$$T(y)=\lim_{n\to\infty} T(A_\sigma(x_n))=\lim_{n\to\infty} A_\sigma S(x_n).$$

The continuity of S also implies that the sequence $S(x_n)$ tends to zero. Thus $T(y)$ belongs to the separating subspace of A_σ. □

Theorem 4. *Let $\mathcal{W}$ and $\mathcal{V}$ be Hilbert triple systems with zero annihilator. Then every isomorphism pair between them is automatically continuous. Every Hilbert triple system with zero annihilator has a unique complete norm topology.*

This follows directly from Theorem 3 and Proposition 4.

4. Centralizers, derivations and isomorphisms

In the previous section we have already defined the concept of an isomorphism pair. A concept of a derivation pair and centralizer can be defined in a similar way.

Let $\mathcal{W}$ be a Hilbert triple system with zero annihilator. A mapping $C:\mathcal{W}\to\mathcal{W}$ is called a *centralizer* of $\mathcal{W}$ if

$$C([xyz])=[C(x)yz]=[xyC(z)]$$

holds for all $x, y, z\in\mathcal{W}$. A pair (D_+, D_-) of linear operators acting on $\mathcal{W}$ is called a *derivation pair,* if

$$D_\sigma([xyz])=[D_\sigma(x)yz]+[xD_{-\sigma}(y)z]+[xyD_\sigma(z)]$$

holds for all $\sigma\in\{+,-\}$ and $x, y, z\in\mathcal{W}$. If $D=D_+=D_-$, then D is called a *derivation* of $\mathcal{W}$.

First we describe the structure of centralizers in Hilbert triple systems.

Lemma 2. *Let $\mathcal{W}$ be a Hilbert triple system with zero annihilator. Then every centralizer of $\mathcal{W}$ is a bounded linear operator.*

Proof. Let C be a centralizer of $\mathcal{W}$. Take $x, y, a, b\in\mathcal{W}$ and $\lambda\in\boldsymbol{C}$. From

$$[(\lambda C(x)-C(\lambda x))ab]=\lambda[C(x)ab]-[(\lambda x)aC(b)]=\lambda[C(x)ab]-\lambda[xaC(b)]=0,$$

$$[(C(x+y)-C(x)-C(y))ab]=[(x+y)aC(b)]-[xaC(b)]-[yaC(b)]=0$$

and the fact that $\mathcal{W}$ has zero annihilator, linearity of C easily follows.

Now suppose that for the sequence $x_n\in\mathcal{W}$, which tends to zero, the sequence $C(x_n)$ tends to $y\in\mathcal{W}$. Then we get for arbitrary $a, b\in\mathcal{W}$

$$[yab]=\lim_{n\to\infty}[C(x_n)ab]=\lim_{n\to\infty}[x_naC(b)]=0.$$

We used the continuity of the triple product in $\mathcal{W}$. Since the annihilator $\mathcal{W}$ is zero, it follows that $y=0$ and by the closed graph theorem C is bounded. □

Proposition 5. *Let $\mathcal{W}$ be a Hilbert triple system system with zero annihilator and let $C(\mathcal{W})$ denote the algebra of all centralizers of $\mathcal{W}$ equipped with the usual operator norm. Then $C(\mathcal{W})$ is a von Neumann algebra.*

Proof. It is easy to see that $C(\mathcal{W})$ is in fact an algebra with identity. Let C be a centralizer. Take $a, x, y, z \in \mathcal{W}$ and compute

$$\begin{aligned}\langle a, C^*([xyz])\rangle &= \langle C(a), [xyz]\rangle = \langle [yxC(a)], z\rangle \\ &= \langle C([yxa]), z\rangle = \langle [yxa], C^*(z)\rangle = \langle a, [xyC^*(z)]\rangle .\end{aligned}$$

In a similar way we can prove that $C^*([xyz]) = [C^*(x)yz]$ holds and therefore C^* is also a centralizer.

Now suppose that $\{C_\alpha\}$ converges to C in the strong operator topology and that all C_α are centralizers. Then we have

$$C_\alpha([xyz]) \longrightarrow C([xyz]), \quad [C_\alpha(x)yz] \longrightarrow [C(x)yz], \quad [xyC_\alpha(z)] \longrightarrow [xyC(z)].$$

We used the continuity of the triple product. This immediately gives us $C([xyz]) = [xyC(z)] = [C(x)yz]$. □

Theorem 5. *Let $\mathcal{W}$ be a Hilbert triple system with zero annihilator and let $\mathcal{W} = \bigoplus_{\alpha \in \Lambda} \mathcal{I}_\alpha$ be the decomposition of $\mathcal{W}$ into an orthogonal sum of its minimal closed ideals (see Theorem 2). Then every centralizer C is of the form $C = \sum \lambda_\alpha I_\alpha$ where λ_α is some complex number and I_α is the identity operator on $\mathcal{I}_\alpha$. Also $|\lambda_\alpha| \leq \|C\|$ holds for each $\alpha \in \Lambda$.*

Proof. Suppose first that $\mathcal{W}$ is simple. Let P be a projection and a centralizer. Take $x \in \mathrm{Ker}(P)$ and $a, b \in \mathcal{W}$. From

$$P([abx]) = [abP(x)] = [ab0] = 0,$$

$$P([xab]) = [P(x)ab] = [0ab] = 0$$

we see that Ker(P) is a (closed) outsided ideal. From Proposition 2(iii) it follows that Ker(P) is also a closed ideal. Therefore we have two possibilities: Ker$(P) = (0)$ or Ker$(P) = \mathcal{W}$. In the first case it follows $P = I_{\mathcal{W}}$. In the second case it follows $P = 0$. It is well-known that every von Neumann algebra is generated by its projections. Therefore $C(\mathcal{W}) = \boldsymbol{C}$ (see Proposition 5).

Now let $\mathcal{W}$ be arbitrary Hilbert triple system with zero annihilator and $\mathcal{I} \subset \mathcal{W}$ a closed ideal. Let $\mathcal{S}$ be a linear subspace of $\mathcal{W}$ spanned by $[\mathcal{I}\mathcal{I}\mathcal{I}]$. Let a be orthogonal to $\mathcal{S}$. Then we have

$$\langle [a\mathcal{W}\mathcal{I}], \mathcal{W}\rangle = \langle a, [\mathcal{W}\mathcal{I}\mathcal{W}]\rangle = \langle a, [\mathcal{I}\mathcal{I}\mathcal{I}]\rangle = (0),$$

which implies, by Proposition 2(iv), that $a \in \mathcal{I}^\perp$ holds. Thus $\mathcal{I} = \mathcal{I}^{\perp\perp} \subset \mathcal{S}^{\perp\perp} =$ closure$(\mathcal{S}) \subset \mathcal{I}$ tells us that the closure of $\mathcal{S}$ is equal to $\mathcal{I}$.

Let C be a centralizer of $\mathcal{W}$. From $C([\mathcal{I}\mathcal{I}\mathcal{I}])=[C(\mathcal{I})\mathcal{I}\mathcal{I}]\subset\mathcal{I}$ it follows that $C(\mathcal{S})\subset\mathcal{I}$ holds. From the continuity of C and the above paragraph it follows that $C(\mathcal{I})\subset\mathcal{I}$ holds. It is also easy to see that the restriction of C on the subspace $\mathcal{I}$, which is also a Hilbert triple system, is a centralizer of $\mathcal{I}$.

Let $\mathcal{W}=\bigoplus\mathcal{I}_\alpha$ be the decomposition of $\mathcal{W}$ into a sum of its minimal closed ideals. Let C be some centralizer of $\mathcal{W}$. If $C_\alpha:\mathcal{I}_\alpha\to\mathcal{I}_\alpha$ is a restriction of C, then from the first paragraph of this proof follows that $C_\alpha(x)=\lambda_\alpha x$ for all $x\in\mathcal{I}_\alpha$. The rest is easy. □

Now we give some results concerning derivation pairs. In the sequel we need the following construction:

Let $T_{\mathcal{W}}$ be a (complex) vector space of those mappings $f:\mathcal{W}\times\mathcal{W}\times\mathcal{W}\to\mathcal{W}$ which are linear in first and third variable and conjugate linear in second. We can introduce the norm in this vector space by $\|f\|=\sup\{\|f(x, y, z)\|;\|x\|, \|y\|, \|z\|\leqq 1\}$ thus turning $T_{\mathcal{W}}$ into a Banach space. The details are left to the reader. The following lemma is inspired by the results in [47].

Lemma 3. *Let $\mathcal{W}$ be a Hilbert triple system with zero annihilator. Let (D_+, D_-) be a derivation pair of $\mathcal{W}$. Suppose that D_+ and D_- are bounded. Then there exist complex numbers α, λ, ν from the spectrum of D_+ and μ from the spectrum of D_- such that $\alpha=\lambda+\bar{\mu}+\nu$ holds.*

Proof. Define operators D_1, D_2, D_3 and D_4 acting on the Banach space $T_{\mathcal{W}}$ by

$$(D_1f)(x, y, z)=D_+(f(x, y, z)),$$

$$(D_2f)(x, y, z)=f(D_+(x), y, z),$$

$$(D_3f)(x, y, z)=f(x, D_-(y), z),$$

$$(D_4f)(x, y, z)=f(x, y, D_+(z)).$$

It is easy to verify that this operators are continuous and mutually commuting. Define also

$$(Ef)(x, y, z)=D_+(f(x, y, z))-f(D_+(x), y, z)-f(x, D_-(y), z)-f(x, y, D_+(z)).$$

It is also easy to see that inclusions

$$\mathrm{sp}(E)\subset\mathrm{sp}(D_1)-\mathrm{sp}(D_2)-\mathrm{sp}(D_3)-\mathrm{sp}(D_4),$$

$$\mathrm{sp}(D_1),\ \mathrm{sp}(D_2),\ \mathrm{sp}(D_4)\subset\mathrm{sp}(D_+),\qquad \mathrm{sp}(D_3)\subset\overline{\mathrm{sp}}(D_-)$$

hold, where by $\mathrm{sp}(D)$ we denote the spectrum of the operator D.

Now take $f_0(x, y, z)=[xyz]$. From Proposition 1 it follows that f_0 belongs to $T_{\mathcal{W}}$ and since the annihilator of $\mathcal{W}$ is zero it follows $f_0\neq 0$. From the definition of the derivation pair it is obvious that $E(f_0)=0$. Hence

$$0 \in \mathrm{sp}(E) \subset \mathrm{sp}(D_+) - \mathrm{sp}(D_+) - \overline{\mathrm{sp}}(D_-) - \mathrm{sp}(D_+).$$

Therefore $0=\alpha-\lambda-\bar{\mu}-\nu$ holds with $\alpha, \lambda, \nu \in \mathrm{sp}(D_+)$ and $\mu \in \mathrm{sp}(D_-)$. □

Theorem 6. *Let $\mathcal{W}$ be a Hilbert triple system with zero annihilator and (D_+, D_-) a derivation pair of $\mathcal{W}$ such that D_+ and D_- are bounded. Then $D_+^* = -D_-$ holds.*

Proof. In the same way as in the proof of Theorem 5 we can prove that operators D_+ and D_- map every closed ideal $\mathcal{I}$ of $\mathcal{W}$ into $\mathcal{I}$ and thus we can restrict our considerations only to the simple Hilbert triple systems.

Now suppose that $D=D_+=D_-$ is a derivation. We shall prove that the operator $D+D^*$ is a centralizer. First we have

$$\begin{aligned}
&\langle (D+D_*)([xyz]), a\rangle = \langle D([xyz]), a\rangle + \langle D^*([xyz]), a\rangle \\
&= \langle D([xyz]), a\rangle + \langle [xyz], D(a)\rangle = \langle D([xyz]), a\rangle + \langle z, [yxD(a)]\rangle \\
&= \langle D([xyz]), a\rangle + \langle z, D([yxa])\rangle - \langle z, [D(y)xa]\rangle - \langle z, [yD(x)a]\rangle \\
&= \langle D([xyz]), a\rangle + \langle D^*(z), [yxa]\rangle - \langle [xD(y)z], a\rangle - \langle [D(x)yz], a\rangle \\
&= \langle [xyD(z)], a\rangle + \langle [xyD^*(z)], a\rangle = \langle [xy(D+D^*)(z)], a\rangle.
\end{aligned}$$

Therefore $(D+D^*)([xyz])=[xy(D+D^*)(z)]$ holds. In a similar way we obtain $(D+D^*)([xyz])=[(D+D^*)(x)yz]$. Using Theorem 5 we get $D+D^*=\lambda I$ for some complex number λ. Since this operator is selfadjoint, it follows that λ is real. Therefore $\mathrm{sp}(D) \subset \lambda/2+i\boldsymbol{R}$ holds.

Accoding to Lemma 3 the spectrum of the operator D contains four numbers $\lambda/2+i\alpha_1$, $\lambda/2+i\alpha_2$, $\lambda/2+i\alpha_3$ and $\lambda/2+i\alpha_4$ where $\alpha_j \in \boldsymbol{R}$ holds for $j=1, 2, 3, 4$, such that

$$\frac{\lambda}{2}+i\alpha_1 = \frac{\lambda}{2}+i\alpha_2 + \frac{\lambda}{2}+i\alpha_3 + \frac{\lambda}{2}+i\alpha_4.$$

This implies $3\lambda=\lambda$ and thus $\lambda=0$, which means that $D^*=-D$.

If (D_+, D_-) is a derivation pair we can decompose $D_+=D+iG$ and $D_-=D-iG$. A straightforward verification shows that $D=(D_++D_-)/2$ and $G=(D_+-D_-)/2i$ are derivations of $\mathcal{W}$. Therefore, by the above paragraph, we get

$$D_+^* = (D+iG)^* = D^* - iG^* = -D+iG = -(D-iG) = -D_-. \quad \square$$

Open problem 1. *Let $\mathcal{W}$ be a Hilbert triple system with zero annihilator and (D_+, D_-) a derivation pair of $\mathcal{W}$. Are the operators D_+ and D_- automatically continuous?*

For partial results see [44]. If $\mathcal{W}_0$ is an infinite dimensional Hilbert space, then we can define the trivial product $[xyz]=0$ for all $x, y, z \in \mathcal{W}_0$. It is easy to see that every pair of (unbounded) linear operators acting on $\mathscr{W}_0$ is a derivation pair of this Hilbert triple system.

The main results of this sections are about isomorphism between simple Hilbert triple systems. From Theorem 4 we know that they are automatically continuous. We shall prove that they are also scalar multiples of isometries.

This results cannot be generalized to the Hilbert triple systems with zero annihilator because the automorphisms do not necessarily preserve closed ideals. The following lemma is inspired by the results in [47].

Lemma 4. *Let $\mathscr{W}$ be a Hilbert triple system with zero annihilator and let (A_+, A_-) be an automorphism pair on $\mathscr{W}$. Then there exist complex numbers $\alpha, \lambda, \nu \in \mathrm{sp}(A_+)$ and $\mu \in \mathrm{sp}(A_-)$ such that $\alpha = \lambda\bar{\mu}\nu$ holds.*

Proof. Take the Banach space $T_{\mathscr{W}}$ and define the operators

$$(Bf)(x, y, z) = A_+(f(A_+^{-1}(x), A_-^{-1}(y), A_+^{-1}(z))),$$

$$(A_1 f)(x, y, z) = A_+(f(x, y, z)),$$

$$(A_2 f)(x, y, z) = f(A_+^{-1}(x), y, z),$$

$$(A_2 f)(x, y, z) = f(x, A_-^{-1}(y), z),$$

$$(A_4 f)(x, y, z) = f(x, y, A_+^{-1}(z)),$$

In a similar way as in the proof of Lemma 3 we obtain

$$1 \in \mathrm{sp}(B) \subset \mathrm{sp}(A_+) \frac{1}{\mathrm{sp}(A_+)\overline{\mathrm{sp}}(A_-)\mathrm{sp}(A_+)}.$$

Therefore $1 = \alpha(1/\lambda\bar{\mu}\nu)$, where $\alpha, \lambda, \nu \in \mathrm{sp}(A_+)$ and $\mu \in \mathrm{sp}(A_-)$. □

The following lemma is from [48].

Lemma 5. *Let A be some bounded operator acting on a Banach space and let $B = \exp(A)$. Suppose that $B(x) = x$ holds for some x. If the spectrum of the operator A does not contain any of the following numbers $\pm 2\pi i$, $\pm 4\pi i$, $\pm 6\pi i \cdots$, then $A(x) = 0$ holds.*

Proof. Define a complex mapping $f(\lambda) = (e^{\lambda} - 1)/\lambda$. This mapping is holomorphic on the entire complex plain and $f(0) = 1$ holds. From the spectral mapping theorem it follows that the operator $f(A)$ is invertible, since 0 does not belong to its spectrum. From $f(\lambda)\lambda = e^{\lambda} - \lambda$ it also follows that $f(A)A(x) = B(x) - x = 0$. Therefore $A(x) = f(A)^{-1} f(A)A(x) = 0$. □

Proposition 6. *Let $\mathscr{W}$ be a Hilbert triple system with zero annihilator. Suppose that (A_+, A_-) is an automorphism pair of $\mathscr{W}$ such that both operators A_+ and*

A_- have positive spectrum. Then they can be expressed in the form $A_+=\exp(D_+)$ and $A_-=\exp(D_-)$ where (D_+, D_-) is some derivation pair of $\mathcal{W}$.

Proof. From the spectral mapping theorem we know that the operators $D_+=\log(A_+)$ and $D_-=\log(A_-)$ have a real spectrum. Define the operators D_1 D_2, D_3, D_4, E and A_1, A_2, A_3, A_4, B as in the proofs of Lemma 3 and Lemma 4. It is easy to verify that $\exp(D_1)=A_1$ and $\exp(-D_i)=A_i$ hold for $i=2, 3, 4$. Since the operators D_1, D_1, D_3 and D_4 are mutually commuting, we obtain

$$\begin{aligned}\exp(E)=\exp(D_1-D_2-D_3-D_4)=&\\ =\exp(D_1)\exp(-D_2)\exp(-D_3)\exp(-D_4)=&\\ =A_1A_2A_3A_4=B\,.&\end{aligned}$$

If we define $f(x, y, z)=[xyz]$, we get $B(f)=f$. From the proof of Lemma 3, we also see that the spectrum of the operator E is real. Using Lemma 5 we obtain $E(f)=0$. This reads

$$D_+([xyz])=[D_+(x)yz]+[xD_-(y)z]+[xyD_+(z)]\,.$$

If we interchange the role of A_+ and A_-, we obtain

$$D_-([xyz])=[D_-(x)yz]+[xD_+(y)z]+[xyD_-(z)]$$

and hence (D_+, D_-) is a derivation pair of $\mathcal{W}$. □

Proposition 7. *Let $\mathcal{W}$ and $\mathcal{V}$ be simple Hilbert triple systems and let (A_+, A_-) be an isomorphism pair from $\mathcal{W}$ into $\mathcal{V}$. Then there exists some real number λ such that $A_-A_+^*=\lambda I_v$ holds.*

Proof. Take $a, b, c\in\mathcal{W}$, $z\in\mathcal{V}$ and compute

$$\langle A_+([bac]), z\rangle=\langle [bac], A_+^*(z)\rangle=\langle c, [abA_+^*(z)]\rangle,$$
$$\langle [A_+(b)A_-(a)A_+(c)], z\rangle=\langle A_+(c), [A_-(a)A_+(b)z]\rangle=\langle c, A_+^*([A_-(a)A_+(b)z])\rangle\,.$$

If we compare the above expressions, we obtain

$$A_+^*([A_-(a)A_+(b)z])=[abA_+^*(z)]\,.$$

Now take $x, y\in\mathcal{V}$. Since the operators A_+ and A_- are bijective, we can write $x=A_-(a)$ and $y=A_+(b)$. Then we get

$$\begin{aligned}A_-A_+^*([xyz])&=A_-A_+^*([A_-(a)A_+(b)z])=A_-([abA_+^*(z)])\\ &=[A_-(a)A_+(b)A_-A_+^*(z)]=[xyA_-A_+^*(z)]\,.\end{aligned}$$

In a similar way we can prove that

$$A_-A_+^*([xyz])=[A_-A_+^*(x)yz]$$

holds for all $x, y, z \in \mathcal{V}$. This means that the operator $A_- A_+^*$ is a centralizer of $\mathcal{V}$. Therefore by Theorem 5 we have

$$A_- A_+^* = \lambda I, \qquad A_+ A_-^* = \bar{\lambda} I.$$

Now define operators

$$B_+ = A_+^{-1} A_-, \qquad B_- = A_-^{-1} A_+.$$

A trivial verification tells us that (B_+, B_-) is an automorphism pair of the triple system $\mathcal{W}$. From

$$A_+^{-1} = (1/\bar{\lambda}) A_-^*, \qquad A_-^{-1} = (1/\lambda) A_+^*$$

we obtain

$$\mathrm{sp}(B_+) \subset (1/\bar{\lambda}) \boldsymbol{R}^+, \qquad \mathrm{sp}(B_-) \subset (1/\lambda) \boldsymbol{R}^+.$$

According to Lemma 4, there exist four positive numbers p_1, p_2, p_3 and p_4 such that

$$p_1/\lambda = (p_2/\lambda)\overline{(p_3/\bar{\lambda})}(p_4/\lambda).$$

This further implies

$$\lambda^2 = \frac{p_2 p_3 p_4}{p_1} > 0,$$

which means that λ must be real. □

Before we state the main result of this section, we introduce the notation of a negative of a triple system. Let $(\mathcal{W}, [\cdots])$ be a Hilbert triple system. Let $\mathcal{W}_-$ be the same set as $\mathcal{W}$ with the following structure:

(i) The inner product of $\mathcal{W}_-$ is the same as the inner product of $\mathcal{W}$.

(ii) The triple product $[xyz]_- = -[xyz]$.

It is easy to verify that $\mathcal{W}_-$ is also a Hilbert triple system which will be called a *negative of* $\mathcal{W}$.

Theorem 7. *Let $\mathcal{W}$ and $\mathcal{V}$ be simple Hilbert triple systems. Then the following holds:*

(i) *Suppose that there exists some isomorphism pair between the triple systems $\mathcal{W}$ and $\mathcal{V}$ such that λ from Proposition 7 is positive. Then the triple systems $\mathcal{W}$ and $\mathcal{V}$ are isomorphic.*

(ii) *Suppose that there exists some isomorphism pair between the triple systems $\mathcal{W}$ and $\mathcal{V}$ such that λ from Proposition 7 is negative. Then $\mathcal{W}$ is isomorphic to the negative of $\mathcal{V}$.*

(iii) *Every isomorphism between the triple systems $\mathcal{W}$ and $\mathcal{V}$ is a scalar multiple of isometry.*

(iv) *Every automorphism of $\mathcal{W}$ is isometric.*

Proof. (i) From the proof of Proposition 7, we can see that the operators

$B_+=A_+^{-1}A_-$ and $B_-=A_-^{-1}A_+$ have a positive spectrum. According to Proposition 6, there exists some derivation pair (D_+, D_-) of $\mathcal{W}$ such that $B_+=\exp(D_+)$ and $B_-=\exp(D_-)$ holds. If we define

$$F_+=\exp\left(\frac{1}{2}D_+\right), \qquad F_-=\exp\left(\frac{1}{2}D_-\right),$$

then (F_+, F_-) is an automorphism pair of $\mathcal{W}$ and $F_+^2=B_+$, $F_-^2=B_-$ hold. It is also easy to verify

$$A_+B_+=A_+F_+^2=A_-,$$

$$A_-B_-=A_-F_-^2=A_+.$$

Since $B_+B_-=B_-B_+=I$ holds, we get $F_+F_-=F_-F_+=I$, which together with the above equalities gives us $A_+F_+=A_-F_-$.

Denote this operator by G. It maps from $\mathcal{W}$ into $\mathcal{V}$. It is obvious that (G, G) is an isomorphism pair, since it is a compositum of the isomorphism pair (A_+, A_-) and automorphism pair (F_+, F_-). This means that the triple systems $\mathcal{W}$ and $\mathcal{V}$ are isomorphic.

(ii) If we replace the pair (A_+, A_-) with a pair $(A_+, -A_-)$ and triple system $\mathcal{V}$ with its negative, we arrive in the situation of (i).

(iii) Using the previous proposition, we obtain $AA^*=\lambda I$ for some real λ. This λ must be positive since the spectrum of the operator AA^* is contained in $\boldsymbol{R}^+$. Since A is invertible, it follows $A^*A=\lambda I$. Thus

$$\|A(w)\|^2=\langle A(w), A(w)\rangle=\langle A^*A(w), w\rangle=\langle \lambda w, w\rangle,$$

and finally $\|A(w)\|=\sqrt{\lambda}\|w\|$.

(iv) From (iii) it follows that the spectrum of the operator A is contained in the set

$$\{z\in \boldsymbol{C}\,;\ |z|=\sqrt{\lambda}\}.$$

According to Lemma 4 there exist complex numbers z_1, z_2, z_3 and z_4, whose absolute value is equal to 1, such that

$$\sqrt{\lambda}z_1=\sqrt{\lambda}z_2\cdot\sqrt{\lambda}z_3\cdot\sqrt{\lambda}z_4$$

holds. If we take the absolute value on both sides of the above equality, we obtain $\lambda^3=\lambda$ and since λ is positive it follows $\lambda=1$. □

Open problem 2. *Let $\mathcal{W}$ be a simple Hilbert triple system. When is $\mathcal{W}$ isomorphic to its negative?*

For a partial result see the next section.

5. Third structure theorem for associative Hilbert triple systems

In this section we describe the structure of simple associative Hilbert triple systems. A triple system $(\mathcal{W}, [\cdots])$ is called *associative*, if

$$[[abc]de]=[a[dcb]e]=[ab[cde]] \qquad (1)$$

holds for all $a, b, c, d, e \in \mathcal{W}$. If $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces, then the Hilbert triple system $\mathcal{W}=\mathbf{HS}(\mathcal{H}, \mathcal{K})$, described in the Introduction, is associative, as can easily be verified. A negative $\mathcal{W}_-$ of this triple system is also associative. Both of them are simple. This follows from the fact that operators with the finite rank are dense in $\mathcal{W}$. This section is devoted to prove the following result:

Every simple associative Hilbert triple system is isomorphic either to the triple $\mathbf{HS}(\mathcal{H}, \mathcal{K})$ *or to the triple* $\mathbf{HS}(\mathcal{H}, \mathcal{K})_-$ *for suitable complex Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$.

In the proof of this result we shall use the matrix version of Hilbert-Schmidt operators which goes as follows:

Let Λ and Γ be arbitrary sets. A mapping $M: \Lambda\times\Gamma\to\mathbf{C}$ is called a *Hilbert-Schmidt matrix,* if

$$\sum_{i,j}|M(i, j)|^2<\infty$$

holds. On the space of all such mappings we can define the inner product with

$$\langle M, N\rangle=\kappa\sum_{i,j}M(i, j)\overline{N(i, j)},$$

where $\kappa>0$ is some constant. The triple product can be defined with

$$[MNK](i, j)=\sum_{k\in\Gamma, l\in\Lambda}M(i, k)\overline{N(l, k)}K(l, j).$$

Long but straightforward computation shows that this space, with the above defined inner and triple product, forms an associative Hilbert triple system. Denote this triple system by $\mathcal{M}(\Lambda, \Gamma, \kappa)$.

If we define a matrix U_{ij} by

$$U_{ij}(i, j)=1,$$
$$U_{ij}(k, l)=0, \quad \text{if } k\neq i \text{ or } j\neq l,$$

then it is obvious that U_{ij} is a Hilbert-Schmidt matrix and that the family $\{U_{ij};\ i\in\Lambda,\ j\in\Gamma\}$ forms an orthogonal base for the triple system $\mathcal{M}(\Lambda, \Gamma, \kappa)$.

An easy computation also gives us

(i) $[U_{ij}U_{ij}U_{ij}]=U_{ij}$.

(ii) $[U_{ij}\mathcal{W}U_{kl}]=\mathbf{C}U_{il}$.

(iii) $[U_{ij}U_{kj}U_{kl}]=U_{il}$.

(vi) $[U_{ij}U_{kl}U_{rs}]=0$, if $j\neq l$ or $k\neq r$.

(v) Every Hilbert-Schmidt matrix M is of the form $M=\sum\lambda_{ij}U_{ij}$, where $\sum|\lambda_{ij}|^2<\infty$ and λ_{ij} belongs to $\boldsymbol{C}$.

In order to prove the announced result, we shall use the concept of a tripotent. A **nonzero** element $u\in\mathcal{W}$ is called a *tripotent*, if $[uuu]=u$ holds and an *antitripotent*, if $[uuu]=-u$ holds. Each Hilbert-Schmidt matrix U_{ij} for $i\in\Lambda$ and $j\in\Gamma$ is a tripotent as (i) tells us. Now we prove that a simple associative Hilbert triple system contains either tripotents or antitripotents.

Lemma 6. *Let T be a positive operator on the complex Hilbert space $\mathcal{H}$ and let $\|T\|=1$. Then a sequence $T^n(x)$ converges for all $x\in\mathcal{H}$.*

Proof. From the theory of operators acting on Hilbert spaces it is well-known that T is of the form $T=S^2$, with S being positive. Take $x\in\mathcal{H}$ and consider a sequence

$$\alpha_n=\langle T^n(x), x\rangle.$$

Obviously we have $\alpha_n\geqq 0$. We also have

$$\alpha_{n+1}=\langle T^{n+1}(x), x\rangle=\langle S^nTS^n(x), x\rangle=\langle TS^n(x), S^n(x)\rangle\leqq$$
$$\leqq\|T\|\|S^n(x)\|^2=\langle S^n(x), S^n(x)\rangle=\langle T^n(x), x\rangle=\alpha_n.$$

This implies that $\alpha=\inf\alpha_n=\lim_{n\to\infty}\alpha_n$ exists. Take any integers n and m. From

$$\|T^n(x)-T^m(x)\|^2=\langle T^{2n}(x), x\rangle-2\langle T^{n+m}(x), x\rangle+\langle T^{2m}(x), x\rangle,$$

it follows that when $n, m\to\infty$, we get

$$\|T^n(x)-T^m(x)\|^2\longrightarrow\alpha-2\alpha+\alpha=0.$$

This tells us that $T^n(x)$ forms a Cauchy sequence and since $\mathcal{H}$ is complete this sequence converges. □

Proposition 8. *Let $\mathcal{W}$ be an associative Hilbert triple system with zero annihilator. Suppose that a nonzero element x of $\mathcal{W}$ is contained in $\mathcal{L}\cap\mathcal{R}$ where $\mathcal{L}$ is a closed left ideal of $\mathcal{W}$ and $\mathcal{R}$ is a closed right ideal of $\mathcal{W}$. Then $\mathcal{L}\cap\mathcal{R}$ contains either tripotent or antitripotent.*

Remark. If we take $\mathcal{L}=\mathcal{R}=\mathcal{W}$, we obtain that every associative Hilbert triple system with zero annihilator contains either tripotent or antitripotent.

Proof. First we prove that $[xxx]\neq 0$. Suppose the contrary, i. e. $[xxx]=0$. Then we have first

$$\|[xxy]\|^2=\langle[xxy], [xxy]\rangle=\langle y, [xx[xxy]]\rangle=\langle y, [[xxx]xy]\rangle=0,$$

which tells us that $L(x, x)=0$. Next we have

$$\|[zyx]\|^2=\langle[zyx], [zyx]\rangle=\langle z, [[zyx]xy]\rangle=\langle z, [zy[xxy]]\rangle=0$$

and this finally yields $x\in\mathrm{Rann}(\mathcal{W})=(0)$. This contradiction tells us that $[xxx]\neq 0$ and also $L(x, x)\neq 0$.

This means that we can assume $\|L(x, x)\|=1$. Since the operator $S=L(x, x)$ is selfadjoint, the operator $T=S^2$ is positive and $\|T\|=\|S^2\|=\|S^*S\|=\|S\|^2=1$ holds.

Using the previous lemma, we can define

$$u=\lim_{n\to\infty} T^n(x), \qquad v=\lim_{n\to\infty} T^n([xxx]).$$

Since $\mathcal{L}$ and $\mathcal{R}$ are closed, it is obvious that u and v belongs to $\mathcal{L}\cap\mathcal{R}$. From the associativity of the triple system $\mathcal{W}$ it easily follows that $S([xyz])=[S(x)yz]$ holds for all $x, y, z\in\mathcal{W}$.

In our next step we prove that either $u\neq 0$ or $v\neq 0$. Assume the contrary, i.e. $u=v=0$. Since $[xxx]=L(x, x)(x)=S(x)$, this implies that $\lim_{n\to\infty} S^n(x)=0$. The well-known property of the operators on Hilbert spaces states that

$$\sup_{\|y\|=1} |\langle S^n(y), y\rangle|=\|S^n\|=1$$

holds. The continuity of the product in Hilbert triple systems implies that there exists some $M>0$ such that $\|[xyz]\|\leqq M\|x\|\|y\|\|z\|$ holds for all $x, y, z\in\mathcal{W}$. This gives us

$$1=\sup_{\|y\|=1}\|S^{n+1}(y)\|=\sup_{\|y\|=1}\|[S^n(x)xy]\|\leqq M\|S^n(x)\|\|x\|$$

and therefore

$$\frac{1}{M\|x\|}\leqq\|S^n(x)\|\longrightarrow 0,$$

which is a contradiction.

Using the associativity and continuity of the product, we can easily compute

$$[uuu]=[vvu]=[vuv]=[uvv]=v,$$

$$[vvv]=[uuv]=[uvu]=[vuu]=u.$$

Now consider the elements $w_1=(u+v)/2$ and $w_2=(u-v)/2$. A straightforward computation shows that the above identities imply that w_1 is either tripotent or zero and that w_2 is either antitripotent or zero. Since one of them must be nonzero, the proof is concluded. □

Proposition 9. *Let $\mathcal{W}$ be a simple associative Hilbert triple system.*

(i) *If $x, y\in\mathcal{W}$ are nonzero elements, then $[x\mathcal{W}y]\neq(0)$ holds.*

(ii) *$\mathcal{W}$ contains only tripotents or only antitripotents.*

(iii) *The triple systems $\mathcal{W}$ and $\mathcal{W}_-$ are not isomorphic.*

Proof. (i) Let us first define

$$\mathrm{Lann}(y)=\{a\in\mathcal{W};\ [a\mathcal{W}y]=(0)\}.$$

We shall prove that this set is a closed ideal of the triple system $\mathcal{W}$. Obviously this set is a closed linear subspace of $\mathcal{W}$. Take some $a\in\mathrm{Lann}(y)$. Then we have

$$[[\mathcal{W}\mathcal{W}a]\mathcal{W}y]=[\mathcal{W}\mathcal{W}[a\mathcal{W}y]]=(0),$$

$$[[a\mathcal{W}\mathcal{W}]\mathcal{W}y]=[a[\mathcal{W}\mathcal{W}\mathcal{W}]y]=(0).$$

Since $\mathrm{Lann}(y)$ is closed, it follows (see Proposition 2(iii)) that it is a closed ideal of $\mathcal{W}$. Since $\mathcal{W}$ is simple, we have two possibilities: $\mathrm{Lann}(y)=\mathcal{W}$ or $\mathrm{Lann}(y)=(0)$. In the first case we would have $[yyy]=0$. This is not possible as was shown in the proof of the previous proposition. Therefore $x\notin\mathrm{Lann}(y)$ and (i) is proved.

(ii) Suppose that $\mathcal{W}$ contains a tripotent u and an antitripotent v. Take also $x\in\mathcal{W}$. A simple computation

$$\begin{aligned}\|[u[vxu]v]\|^2&=\langle[u[vxu]v],\ [u[vxu]v]\rangle=\langle[[vxu]u[u[vxu]v]],\ v\rangle\\&=\langle[[[vxu]uu][vxu]v],\ v\rangle=\langle[[vxu][vxu]v],\ v\rangle\\&=\langle[vxu],\ [vv[vxu]]\rangle=-\langle[vxu],\ [vxu]\rangle=-\|[vxu]\|^2\end{aligned}$$

tells us that $[v\mathcal{W}u]=(0)$, which contradicts (i).

(iii) This follows directly from (ii). □

Proposition 10. *Let $\mathcal{W}$ be an associative Hilbert triple system and let $u\in\mathcal{W}$ be a tripotent. Denote $\mathcal{A}=[u\mathcal{W}u]$ and define an algebra product and an involution on $\mathcal{A}$ by*

$$[uxu]\circ[uyu]=[u[yux]u],$$

$$[uxu]^*=[u[uxu]u].$$

Then $(\mathcal{A},\ \circ,\ *)$ becomes an associative H^*-algebra with identity.

Proof. It is obvious that the mapping $*$ is well-defined. A simple calculation

$$\begin{aligned}[uxu]\circ[uyu]&=[u[yux]u]=[u[y[uuu]x]u] \qquad (2)\\&=[[ux[uuu]]yu]=[[[uxu]uu]yu]=[[uxu]u[uyu]]\end{aligned}$$

shows us that the algebra product on $\mathcal{A}$ is also well-defined. It is easy to see that this algebra is associative and that $u=[uuu]\in\mathcal{A}$ is the identity element. From the associativity of the triple $\mathcal{W}$ it also follows that $\mathcal{A}=\{x\in\mathcal{W};\ x=[uux]=[xuu]\}$. This implies that $\mathcal{A}$ is closed in $\mathcal{W}$ and therefore $\mathcal{A}$ itself is a Hilbert space.

Our next step is to show that the mapping $*$ is an algebra involution on the algebra $\mathcal{A}$. Using the associativity of $\mathcal{W}$ we get

$$[uxu]^{**}=[u[uxu]u]^{*}=[u[u[uxu]u]u]=[[uuu]x[uuu]]=[uxu],$$

$$(\lambda[uxu])^{*}=[u(\bar{\lambda}x)u]^{*}=[u[u(\bar{\lambda}x)u]u]=[u(\lambda[uxu])u]=\bar{\lambda}[u[uxu]u]=\bar{\lambda}[uxu]^{*}$$

and also

$$\begin{aligned}&[uxu]^{*}\circ[uyu]^{*}=[u[uxu]u]\circ[u[uyu]u]\\&=[u[[uyu]u[uxu]]u]=[u[u[x[uuu]y]u]u]\\&=[u[u[xuy]u]u]=[u[xuy]u]^{*}=([uyu]\circ[uxu])^{*}.\end{aligned}$$

Finally we prove that $(\mathcal{A}, \circ, *, \langle\,,\rangle)$ is an H^*-algebra. Using (1) and (2), we obtain

$$\begin{aligned}&\langle[uyu], [uxu]^{*}\circ[uzu]\rangle=\langle[uyu], [u[uxu]u]\circ[uzu]\rangle\\&=\langle[uyu], [u[zu[uxu]]u]\rangle=\langle[uyu], [[u[uxu]u]zu]\rangle\\&=\langle[uyu], [u[uxu][uzu]]\rangle=\langle[[uxu]u[uyu]], [uzu]\rangle\\&=\langle[uxu]\circ[uyu], [uzu]\rangle.\end{aligned}$$

In a similar way we can prove that

$$\langle[uxu], [uzu]\circ[uyu]^{*}\rangle=\langle[uxu]\circ[uyu], [uzu]\rangle$$

holds which concludes the proof. □

According to Proposition 9, we can restrict our attention only to those simple triple systems which contains tripotents. The crucial role in the proof of our theorem will be played by the so called minimal tripotents. A tripotent $u\in\mathcal{W}$ is called a *minimal tripotent*, if $[u\mathcal{W}u]=Cu$ holds. Now we combine the previous previous proposition together with the known results about projections in associative H^*-algebras in order to prove that $\mathcal{W}$ contains "many" minimal tripotents.

Proposition 11. *Let $\mathcal{W}$ be a simple associative Hilbert triple system which contains tripotents. Let $\mathcal{R}$ be a right ideal of $\mathcal{W}$ and $\mathcal{L}$ a left ideal of $\mathcal{W}$. If $\mathcal{L}\cap\mathcal{R}=(0)$, then $\mathcal{L}=(0)$ or $\mathcal{R}=(0)$. If $\mathcal{R}$ and $\mathcal{L}$ are closed and nonzero, then $\mathcal{L}\cap\mathcal{R}$ contains a minimal tripotent.*

Proof. Suppose that $\mathcal{L}\cap\mathcal{R}=(0)$. This implies $[\mathcal{R}\mathcal{W}\mathcal{L}]\subset\mathcal{R}\cap\mathcal{L}=(0)$ and by Proposition 9 we have $\mathcal{L}=(0)$ or $\mathcal{R}=(0)$.

If $\mathcal{L}$ and $\mathcal{R}$ are closed and nonzero, then $\mathcal{L}\cap\mathcal{R}\neq(0)$. By Proposition 8 it follows that $\mathcal{L}\cap\mathcal{R}$ contains a tripotent u. If we form an H^*-algebra $\mathcal{A}=[u\mathcal{W}u]$ defined in the previous proposition, then u is a projection of $\mathcal{A}$. By the well-known results from the theory of associative H^*-algebras (see [1, Theorem 3.2]), u can be expressed as a finite sum of projections $u=p_1+p_2+\cdots+p_n$ with $p_i\circ\mathcal{A}\circ p_i=\boldsymbol{C}p_i$ and $p_i\circ p_j=0$ if $i\neq j$. This gives us (see Proposition 10)

$$[p_1p_1p_1]=[p_1p_1^*p_1]=[p_1[up_1u]p_1]=[[p_1up_1]up_1]$$
$$=[(p_1\circ p_1)up_1]=[p_1up_1]=p_1.$$

Therefore p_1 is a tripotent. From $p_1=[p_1uu]=[uup_1]$ we see that p_1 belongs to $\mathcal{L}\cap\mathcal{R}$. Finally we have

$$[p_1\mathcal{W}p_1]=[[p_1uu]\mathcal{W}[uup_1]]=[[p_1u[u\mathcal{W}u]]up_1]=p_1\circ\mathcal{A}\circ p_1=\boldsymbol{C}p_1$$

which completes the proof. □

Now we define two concepts which we need in the sequel. For minimal tripotents u and v we shall say that they are *horizontally connected*, if $\langle u, v\rangle=0$, $[u\mathcal{W}v]=\boldsymbol{C}u$ and $[v\mathcal{W}u]=\boldsymbol{C}v$ holds.

For minimal tripotents u and v we shall say that they are *vertically connected*, if $\langle u, v\rangle=0$, $[u\mathcal{W}v]=\boldsymbol{C}v$ and $[v\mathcal{W}u]=\boldsymbol{C}u$ holds. According to this definition, u is not vertically connected to itself. This is done to simplify further statements.

Proposition 12. *Let $\mathcal{W}$ be a simple associative Hilbert triple system and $u, v\in\mathcal{W}$ horizontally connected minimal tripotents. Then the following holds:*

(i) $[uvu]=[vuv]=0$.

(ii) $[uuv]=[vvu]=0$.

(iii) $[uvv]=u$, $[vuu]=v$.

(iv) $\|u\|=\|v\|$.

Proof. (i) Since u is minimal, there exists some constant $\alpha\in\boldsymbol{C}$ such that $[uvu]=\alpha u$ holds. From

$$\alpha\|u\|^2=\langle\alpha u, u\rangle=\langle[uvu], u\rangle=\langle u, [vuu]\rangle=\langle[uuu], v\rangle=\langle u, v\rangle=0,$$

it follows $\alpha=0$. In a similar way we can prove that $[vuv]=0$.

(ii) Using (i) and the horizontal connectedness we can compute

$$\|[uuv]\|^2=\langle[uuv], [uuv]\rangle=\langle u, [[uuv]vu]\rangle=$$
$$=\langle u, [u[vvu]u\rangle\subset\langle u, [u(\boldsymbol{C}v)u]\rangle=(0).$$

In a similar way we get $[vvu]=0$.

(iii) By the horizontal connectedness, there exists some $\alpha\in\boldsymbol{C}$ such that $[vuu]=\alpha v$ holds. From the equality

$$|\alpha|^2\|v\|^2=\langle\alpha v, \alpha v\rangle=\langle[vuu], [vuu]\rangle=\langle v, [[vuu]uu]\rangle$$
$$=\langle v, [vu[uuu]]\rangle=\langle v, [vuu]\rangle=\bar{\alpha}\|v\|^2$$

we get that $\alpha=0$ or $\alpha=1$ holds. Since $[u\mathcal{W}v]=\mathbf{C}u$ holds, there exists some $x\in\mathcal{W}$ such, that $[uxv]=u$ holds. This gives us

$$\|u\|^2=\langle[uxv], u\rangle=\langle v, [xuu]\rangle=\langle[vuu], x\rangle=\alpha\langle v, x\rangle,$$

which means that $\alpha\neq 0$. In a similar way we can prove that $[uvv]=u$.

(iv) Using (iii) we immediately get

$$\|u\|^2=\langle u, u\rangle=\langle[uvv], u\rangle=\langle v, [vuu]\rangle=\langle v, v\rangle=\|v\|^2. \quad \square$$

Proposition 13. *Let $\mathcal{W}$ be a simple associative Hilbert triple system and $u, v\in\mathcal{W}$ vertically connected minimal tripotents. Then we have*

(i) $[uvu]=[vuv]=0.$

(ii) $[uuv]=v, [vvu]=u.$

(iii) $[uvv]=[vvu]=0.$

(iv) $\|u\|=\|v\|.$

This can be proved in a similar way as Proposition 12. Now we have everything prepared to prove the main theorem.

Theorem 8. *Let $\mathcal{W}$ be a simple associative Hilbert triple system. Then $\mathcal{W}$ is isomorphic either to the triple system* $\mathbf{HS}(\mathcal{H}, \mathcal{K})$ *or to the triple system* $\mathbf{HS}(\mathcal{K}, \mathcal{K})_-$ *for suitable $\mathcal{H}$ and $\mathcal{K}$.*

Proof. We shall assume that $\mathcal{W}$ contains only tripotents (see Proposition 9) which is equivalent to the fact that $\mathcal{W}_-$ contains only antitripotents.

Choose a minimal tripotent in $\mathcal{W}$ and denote it by u_{00}. Let $\{u_{00}\}\cup\{u_{i0}; i\in\Lambda\}$ be a maximal family of pairwise horizontally connected minimal tripotents and $\{u_{00}\}\cup\{u_{0j}; j\in\Gamma\}$ a maximal family of pairwise vertically connected minimal tripotents. The existence of such families follows from the Zorn lemma. Without loss of generality we may assume that $0\notin\Lambda$ and $0\notin\Gamma$.

Now take any $i\in\Lambda$ and $j\in\Gamma$. We define an element u_{ij} by $u_{ij}=[u_{i0}u_{00}u_{0j}]$. We shall prove that this element is a minimal tripotent, which has the same norm as u_{00}. We shall also prove that this element is horizontally connected with the element u_{0j} and vertically connected with the element u_{i0}. Besides it is orthogonal to the elements u_{00}, u_{i0} and u_{0j}.

First we have

$$[u_{ij}u_{ij}u_{ij}]=[[u_{i0}u_{00}u_{0j}][u_{i0}u_{00}u_{0j}][u_{i0}u_{00}u_{0j}]]$$
$$=[u_{i0}u_{00}[u_{0j}[u_{i0}u_{00}u_{0j}][u_{i0}u_{00}u_{0j}]]]$$

$$=[u_{i0}u_{00}[[u_{0j}u_{0j}u_{00}]u_{i0}[u_{i0}u_{00}u_{0j}]]]$$
$$=[u_{i0}u_{00}[u_{00}u_{i0}[u_{i0}u_{00}u_{0j}]]]=[u_{i0}u_{00}[u_{00}[u_{00}u_{i0}u_{i0}]u_{0j}]]$$
$$=[u_{i0}u_{00}[u_{00}u_{00}u_{0j}]]=[u_{i0}u_{00}u_{0j}]=u_{ij},$$

which means that u_{ij} is a tripotent. In this computation we used Proposition 12 and Proposition 13. Next we have

$$[u_{ij}\mathcal{W}u_{ij}]=[[u_{i0}u_{00}u_{0j}]\mathcal{W}[u_{i0}u_{00}u_{0j}]]$$
$$=[u_{i0}[\mathcal{W}u_{0j}u_{00}][u_{i0}u_{00}u_{1j}]]=[u_{i0}[u_{00}u_{i0}[\mathcal{W}u_{0j}u_{00}]]u_{0j}]$$
$$=[u_{i0}[u_{00}[u_{0j}\mathcal{W}u_{i0}]u_{00}]u_{0j}]\subset \boldsymbol{C}[u_{i0}u_{00}u_{0j}]=\boldsymbol{C}u_{ij}.$$

Using Proposition 9 we obtain $[u_{ij}\mathcal{W}u_{ij}]=\boldsymbol{C}u_{ij}$, which means that u_{ij} is minimal. In a similar way we can prove that $[u_{ij}\mathcal{W}u_{0j}]=\boldsymbol{C}u_{ij}$, $[u_{ij}\mathcal{W}u_{i0}]=\boldsymbol{C}u_{i0}$, $[u_{0j}\mathcal{W}u_{ij}]=\boldsymbol{C}u_{0j}$ and $[u_{i0}\mathcal{W}u_{ij}]=\boldsymbol{C}u_{ij}$ holds, which tells us that u_{i0}, u_{ij} are vertically connected and u_{0j}, u_{ij} are horizontally connected, since it is also easy to verify $\langle u_{ij}, u_{00}\rangle=\langle u_{ij}, u_{i0}\rangle=\langle u_{ij}, u_{0j}\rangle=0$. From Proposition 12 and Proposition 13 we finally get $\|u_{ij}\|=\|u_{i0}\|=\|u_{0j}\|=\|u_{00}\|$.

Thus we obtained a system of elements $\{u_{ij}; i\in\{0\}\cup\Lambda, j\in\{0\}\cup\Gamma\}$. Now we must prove that they form an orthogonal system in $\mathcal{W}$. We already know this for the subsystem $\{u_{00}\}\cup\{u_{i0}; i\in\Lambda\}\cup\{u_{0j}; j\in\Gamma\}$ because they were chosen in such a way. Take two pairs $(i, j)\neq(k, l)\in\Lambda\times\Gamma$. Suppose that $i\neq k$. Then we know that the element u_{i0} is orthogonal to the element u_{k0}. Using the definition of horizontal connectedness, we obtain

$$\langle u_{ij}, u_{kl}\rangle=\langle[u_{i0}u_{00}u_{0j}\rangle, [u_{k0}u_{00}u_{0l}]\rangle$$
$$=\langle u_{i0}, [[u_{k0}u_{00}u_{0l}]u_{0j}u_{00}]\rangle=\langle u_{i0}, [u_{k0}[u_{0j}u_{0l}u_{00}]u_{00}]\rangle$$
$$\subset\langle u_{j0}, \boldsymbol{C}u_{k0}\rangle=(0).$$

In a similar way we can treat the case $j\neq l$ and also prove that $\langle u_{ij}, u_{0l}\rangle=\langle u_{ij}, u_{k0}\rangle=0$ for all $k\in\Lambda$ and $l\in\Gamma$. This means that

$$\mathcal{S}=\{u_{ij}; i\in\{0\}\cup\Lambda, j\in\{0\}\cup\Gamma\}$$

is an orthogonal system in $\mathcal{W}$ where all elements have that same norm.

Now we define $\mathcal{L}=\bigoplus_{j\in\{0\}\cup\Gamma}[\mathcal{W}\mathcal{W}u_{0j}]$. We must prove that this sum is in fact orthogonal. Assume that $j\neq l$. Take any $x\in\mathcal{W}$ and compute

$$\|[u_{0j}u_{0l}x]\|^2=\langle[u_{0j}u_{0l}x], [u_{0j}u_{0l}x]\rangle=\langle u_{0j}, [[u_{0j}u_{0l}x]xu_{0l}]\rangle$$
$$=\langle u_{0j}, [u_{0j}[xxu_{0l}]u_{0l}]\rangle\subset\langle u_{0j}, \boldsymbol{C}u_{0l}\rangle=(0).$$

This further implies

$$\langle[\mathcal{W}\mathcal{W}u_{0j}], [\mathcal{W}\mathcal{W}u_{0l}]\rangle=\langle\mathcal{W}, [[\mathcal{W}\mathcal{W}u_{0l}]u_{0j}\mathcal{W}]\rangle$$
$$=\langle\mathcal{W}, [\mathcal{W}[u_{0j}u_{0l}\mathcal{W}]\mathcal{W}]\rangle=(0).$$

Now it is easy to see that $\mathcal{L}$ is a left ideal and so $\mathcal{L}^{\perp}$ is also a (closed) left ideal. Take also a right ideal $\mathcal{R}$, defined by $\mathcal{R}=[u_{00}\mathcal{W}\mathcal{W}]$. From the associativity of the triple system $\mathcal{W}$ it is obvious that $\mathcal{R}=\{a\in\mathcal{W};\ a=[u_{00}u_{00}a]\}$ and therefore $\mathcal{R}$ is closed.

Suppose that $\mathcal{L}^{\perp}\cap\mathcal{R}$ contains a minimal tripotent v. From Proposition 12 and Proposition 13 we see that $\mathcal{S}\subset\mathcal{L}$ holds and therefore $v\perp\mathcal{S}$. Furthermore v is of the form $v=[u_{00}ab]$ for some $a, b\in\mathcal{W}$. This gives us

$$[v\mathcal{W}u_{0j}]=[[u_{00}ab]\mathcal{W}u_{0j}]=[u_{00}[\mathcal{W}ba]u_{0j}]\subset\boldsymbol{C}u_{0j},$$

$$[u_{0j}\mathcal{W}v]=[u_{0j}\mathcal{W}[u_{00}ab]]=[[u_{0j}\mathcal{W}u_{00}]ab]=\boldsymbol{C}[u_{00}ab]=\boldsymbol{C}v.$$

Using Proposition 9 we obtain that v is vertically connected with all u_{0j} where $j\in\{0\}\cup\Gamma$. Because of the maximality of the family $\{u_{0j}\}$ this is impossible. This means that $\mathcal{L}^{\perp}\cap\mathcal{R}$ does not contain minimal tripotents. By Proposition 11 and Proposition 9 we have either $\mathcal{L}^{\perp}=(0)$ or $\mathcal{R}=(0)$. Since $u_{00}\in\mathcal{R}$, we get $\mathcal{L}^{\perp}=(0)$ and hence $\mathcal{W}=\bigoplus_{j\in\{0\}\cup\Gamma}[\mathcal{W}\mathcal{W}u_{0j}]$. In a similar way we can prove that $\mathcal{W}=\bigoplus_{i\in\{0\}\cup\Lambda}[u_{i0}\mathcal{W}\mathcal{W}]$ holds.

Let $\mathcal{V}$ be a closure of the linear span of $\mathcal{S}$. Actually we have

$$\mathcal{V}=\{\sum_{i,j}\lambda_{ij}u_{ij};\ \sum_{i,j}|\lambda_{ij}|^2<\infty\}.$$

Take some $[u_{i0}xy]\in[u_{i0}\mathcal{W}\mathcal{W}]$. The element y can be expressed as a sum $y=\sum_j[a_jb_ju_{0j}]$ with $a_j, b_j\in\mathcal{W}$. This gives us

$$\begin{aligned}[u_{i0}xy]&=[u_{i0}x(\sum_j[a_jb_ju_{0j}])]\\&=\sum_j[u_{i0}[b_ja_jx]u_{0j}]=\sum_j\alpha_{ij}u_{ij}\in\mathcal{V}.\end{aligned}$$

Finally we get

$$\mathcal{W}=\bigoplus[u_{i0}\mathcal{W}\mathcal{W}]\subset\mathcal{V},$$

which means that $\mathcal{W}=\mathcal{V}$. Now we can define a mapping $\Phi:\mathcal{W}\to\mathcal{M}(\{0\}\cup\Lambda, \{0\}\cup\Gamma, \sqrt{\|u_{00}\|})$ with $\sum\lambda_{ij}u_{ij}\mapsto\sum\lambda_{ij}U_{ij}$. It can easily be verified that Φ is an isometrical isomorphism. □

We conclude our paper by a remark concerning Hilbert modules introduced in [41] by Saworotnow. Let $\mathcal{A}$ be an associative H^*-algebra. A faithful left $\mathcal{A}$-module $\mathcal{W}$ is called a *Hilbert module* if there exists a mapping $[\,|\,]:\mathcal{W}\times\mathcal{W}\to\mathcal{A}$ such that the following holds:

(i) $[\lambda x|y]=\lambda[x|y]$.

(ii) $[x_1+x_2|y]=[x_1|y]+[x_2|y]$.

(iii) $[y|x]=[x|y]^*$.

(iv) $[e\circ x|y]=e[x|y]$.

(v) If x is nonzero, then $[x|x]=e^*e$ for some nonzero $e\in\mathcal{A}$.

(vi) $\mathcal{W}$ is a Hilbert space with the inner product $\langle x, y\rangle=\mathrm{trace}([x|y])$.

The structure of Hilbert modules was described in [43]. Let $\mathcal{W}$ be a Hilbert module. Define a triple product on $\mathcal{W}$ with $[xyz]=[x\,|\,y]\circ z$. Then $\mathcal{W}$ becomes an associative Hilbert triple system. Thus the Smith's results from [43] can be derived from Theorem 8.

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