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ON SOLUTIONS OF SOME NONLINEAR STOCHASTIC INTEGRAL EQUATIONS

By

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Summary. We study stochastic integral equations of [10] under less restrictive assumptions. There are given sufficient conditions for the existence, uniqueness and stability of solutions to these stochastic integral equations. We use the Banach space of tempered functions which contains the space $D([0, +\infty))$ and Banach fixed point principle.

1. Introduction

Theoretical treatments of problems concerning stochastic differential and integral equations have nowadays a large literature, cf. [2], [3], [4], [6]-[10]. Most of them concern the existence and uniqueness of solutions to the examined equations.

The aim of this paper is to give a new existence and uniqueness theorem to stochastic integral equations and to investigate the asymptotic behaviour of their solutions. Our approach bases on a construction of the real Banach space of tempered functions which contains the space $D([0, +\infty))$ of real, right continuous functions having left hand limits. The results of this paper generalize one of results given in [11]. Our proof does not use the concepts of contractor and we omit the assumptions on L-continuity of stochastic processes.

2. Preliminaries

Let $(\boldsymbol{R}, B, \nu)$ be a measurable space with the Lebesgue meaure ν on $(\boldsymbol{R}, B)$, where B denotes the Borel σ-field of subsets of $\boldsymbol{R}$. By $L_p(\boldsymbol{R}, B, \nu)$, $1\leqq p<\infty$, we denote the set of all ν-measurable functions $x:\boldsymbol{R}\rightarrow\boldsymbol{R}$, such that the functions $|x(\cdot)|^p$ is ν-integrable. The norm of $x\in L_p(\boldsymbol{R}, B, \nu)$ is defined by

$$\|x\|_{L_p}=\left(\int_{\boldsymbol{R}}|x(t)|^p d\nu(t)\right)^{1/p}.$$

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Let $L([0, +\infty)) := L_\infty([0, +\infty), B, \nu)$ be the space of ν-essentially bounded function on $[0, +\infty)$. Assume that $p(\cdot) \in L([0, +\infty))$ is a fixed positive function. A triple (Ω, A, P) denotes a complete probability space. By $\mathcal{L}_1^p(\boldsymbol{R}_+, L_2(\Omega, A, P), p)$ (or shortly $\mathcal{L}_1^p$) we mean a space of all functions $x := x(t, \cdot)$ in $\boldsymbol{R}_+$ which are integrable with respect to Lebesgue measure ν, with values $X(t)$ being random variables in $L_2(\Omega, A, P)$ and the topology generated by

$$\|x\|_p = \int_0^\infty p(t)\, \nu\text{-ess}\sup_{s \in [0, t]} \|x(s)\|_{L_2} d\nu(t)\,, \tag{2.1}$$

where $\nu\text{-ess}\sup_{s\in[0,t]})\|x(s)\|_{L_2}$ is taken with respect to the Lebesgue measure ν.

We prove that $\mathcal{L}_1^p$ is a Banach space. The following lemma will be used.

Lemma 2.1 ([1], p. 104). *Let $\{\varepsilon_n, n \geqq 1\}$ be a subsequence of positive numbers such that $\sum_{n=1}^\infty \varepsilon_n < \infty$. Assume that $\mathcal{X}$ is a linear normed space. If every series $\sum_{n=1}^\infty x_n$ with $x_n \in \mathcal{X}$ such that $\|x_n\| \leqq \varepsilon_n$ converges in $\mathcal{X}$, then $\mathcal{X}$ is a complete metric space.*

Lemma 2.2. *$\mathcal{L}_1^p$ is a Banach space.*

Proof. Let $x_n \in \mathcal{L}_1^p$ be such that $\|x_n\| \leqq 2^{-n}$, $n \geqq 1$. It is obvious that

$$\|x_n(s)\|_{L_2} \leqq \nu\text{-ess}\sup \|x_n(s)\|_{L_2}, \qquad n \geqq 1,$$

except of a ν-Lebesgue measure zero set Z_n. Hence

$$\|x_q(s) + \cdots + x_r(s)\|_{L_2} \leqq \nu\text{-ess}\sup\|x_q(s)\|_{L_2} + \cdots + \nu\text{-ess}\sup\|x_r(s)\|_{L_2} \tag{2.2}$$

except of a set having zero Lebesgue measure. By (2.1), (2.2) and the assumption that $\|x_n\|_p \leqq 2^{-n}$, $n \geqq 1$, we have

$$\|x_q + \cdots + x_r\|_p \leqq 2^{-q+1}.$$

Therefore

$$\sum_{j=q}^r \|x_j\|_p \longrightarrow 0, \qquad \text{as} \quad q, r \to \infty,$$

which implies that the series $\sum_{n=1}^\infty x_n$ converges. Hence, by Lemma 2.1, we conclude that $\mathcal{L}_1^p$ is a Banach space.

3. Main results

We deal with the existence, uniqueness and stability of random solutions to the following stochastic integral equation

$$X(t;\omega)=h(t, X(t;\omega))+\int_0^t f(t, s, X(s;\omega);\omega)ds$$
$$+\int_0^t g(t, s, X(s;\omega);\omega)d\beta(s;\omega), \qquad t\geq 0, \tag{3.1}$$

where

(i) $\omega\in\Omega$, and Ω is the supporting set of a complete probability measure space (Ω, A, P) with A being the σ-algebra and P the probability measure,

(ii) $X(t;\omega)$ is an unknown random process,

(iii) $h(t, x)$ is a map from $\boldsymbol{R}_+\times\boldsymbol{R}$ into $\boldsymbol{R}$,

(iv) $f(t, s, X;\omega)$ and $g(t, s, X;\omega)$ are maps from $\boldsymbol{R}_+\times\boldsymbol{R}_+\times\boldsymbol{R}\times\Omega$ into $\boldsymbol{R}$,

(v) $\beta(t;\omega)$, where $t\in\boldsymbol{R}_+$, is a martingale process.

The first part of the stochastic system (3.1) is to be understood as an ordinary Lebesgue integral with probabilistic characterization while the second part is the Ito-Doob stochastic integral.

With respect to the random process $\beta(t;\omega)$ we shall assume that for each $t\in\boldsymbol{R}_+$, a minimal σ-algebra $A_t\subset A$, is defined such that $\beta(t;\omega)$ is measurable with respect to A_t. In addition, we shall assume that $\{A_t, t\in\boldsymbol{R}_+\}$ is an increasing family such that

(i) the random process $\{\beta(t;\omega), A_t: t\in\boldsymbol{R}_+\}$ is a real martingale,

(ii) there is a real continuous non-decreasing function $F(t)$ such that for $s<t$ we have

$$E\{|\beta(t;\omega)-\beta(s;\omega)|^2\}=E\{|\beta(t;\omega)-\beta(t;s)|^2|A_s\}=F(t)-F(s),$$

P-a. e. (cf. [4]).

Definition 3.1. A process $X(t;\omega)$ such that

$$\|X(t)\|_{L_2}\in L_1([0, +\infty))$$

and satisfying (3.1) a. s. is said to be a random solution to that equation.

Definition 3.2. A random solution $X(t;\omega)$ is said to be asymptotically stable in mean square sense if

$$\lim_{T\to\infty}\int_T^\infty \|X(t)\|_{L_2}d\nu(t)=0.$$

Theorem 3.1. *Suppose that the functions h, f and g in* (3.1) *satisfy the following Lipschitz conditions: For* $X(t;\omega)\in\mathcal{L}_1^p$ *and* $Y(t;\omega)\in\mathcal{L}_1^p$

(i) $|h(t, X(t;\omega))-h(t, Y(t;\omega))|\leq K|X(t;\omega)-Y(t;\omega)|$ *P-a. s.*,

where $K\in[0, 1)$,

(ii) $|f(t, s, X(s;\omega);\omega)-f(t, s, Y(s;\omega);\omega)|$
$\leq\alpha(t, s;\omega)|X(t;\omega)-Y(t;\omega)|$ *P-a. s.*,

for some nonnegative function $\alpha(t, s; \omega)$ *belonging to* $L_\infty(\Omega, A, P)$ *with* $\|\alpha(t, s)\| = P\text{-ess}\sup_{\omega\in\Omega}|\alpha(t, s; \omega)|$, *and* $\alpha(t, s; \omega)$ *is continuous for* $t\in\boldsymbol{R}_+$,

(iii) $$|g(t, s; X(s; \omega)-g(t, s; Y(s; \omega); \omega)| \leqq \alpha_1(t, s; \omega)|X(s; \omega)-Y(s; \omega)| \quad P\text{-}a.s.,$$

for some nonnegative function $\alpha_1(t, s; \omega)$ *belonging to* $L_\infty(\Omega, A, P)$, *and* $\alpha_1(t, s, \omega)$ *is continuous for* $t\in\boldsymbol{R}_+$.

Set

$$M := K + \sup_{t\in[0,+\infty)}\left(\int_0^t \|\alpha_1(t, s)\|^2 dF(s)\right)^{1/2} + \sup_{t\in[0,+\infty)}\int_0^t \|\alpha(t, s)\| ds,$$

and suppose that

(iv) $0 < M < 1$.

Then there exists one and only one solution $X\in\mathcal{L}_1^p$ *to equation* (3.1).

Proof. For processes $X, Y\in\mathcal{L}_1^p$, define the process $GX-GY$ by

$$\begin{aligned}(GX)(t; \omega)&-(GY)(t; \omega)\\ &= h(t, X(t; \omega)) - h(t, Y(t; \omega))\\ &\quad + \int_0^t (f(t, s, X(s; \omega); \omega) - f(t, s, Y(s; \omega); \omega))ds\\ &\quad + \int_0^t (g(t, s, X(s; \omega); \omega) - g(t, s, Y(s; \omega); \omega))d\beta(s; \omega).\end{aligned}$$

The assumptions concerning $\beta(t; \omega)$ and $X\in\mathcal{L}_1^p$ allow us to give the following estimate

$$\left\|\int_0^t \alpha_1(t, s)X(s)\beta(s)\right\|_{L_2} \leqq \left(\int_0^t \|\alpha_1(t, s)\|\,\|X(s)\|_{L_2} dF(s)\right)^{1/2} \tag{3.2}$$

Write

$$K(t) := \int_0^t (g(t, s)X(s; \omega); \omega) - g(t, s, Y(s; \omega); \omega))d\beta(s; \omega).$$

Now, by (iii) and (3.2) we obtain

$$\int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|K(s)\|_{L_2} d\nu(t)$$

$$= \int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\left\|\int_0^s |g(s, s_1; X(s_1; \omega); \omega) - g(s, s_1; Y(s_1; \omega); \omega)|\, d\beta(s_1; \omega)\right\|_{L_2}$$

$$\leqq \int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\left\|\int_0^s \alpha_1(s, s_1; \omega)|X(s_1; \omega) - Y(s_1; \omega)|\, d\beta(s_1; \omega)\right\|_{L_2}$$

$$\leqq \int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\left(\int_0^s \|\alpha_1(s, s_1)\|^2 \|X(s) - Y(s)\|_{L_2}^2 dF(s)\right)^{1/2}$$

$$\leqq\int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\nu\text{-ess}\sup_{s_1\in[0,s]}\|X(s_1)-Y(s_1)\|_{L_2}\Big(\int_0^s\|\alpha_1(s,s_1)\|^2dF(s_1)\Big)^{1/2}d\nu(t)$$

$$\leqq\int_0^\infty\sup_{t\in[0,+\infty)}\Big(\int_0^t\|\alpha_1(t,s)\|^2dF(s)\Big)^{1/2}p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|X(s)-Y(s)\|_{L_2}d\nu(t)$$

$$\leqq\sup_{t\in[0,+\infty)}\Big(\int_0^t\|\alpha_1(t,s_1)\|^2dF(s_1)\Big)^{1/2}\int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|X(s)-Y(s)\|_{L_2}d\nu(t) \tag{3.3}$$

Put

$$L(t):=\int_0^t(f(t,s;X(s;\omega);\omega)-f(t,s;Y(s;\omega);\omega))ds.$$

By the assumption (ii) we get

$$\int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|L(s)\|_{L_2}d\nu(t)$$

$$\leqq\int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\int_0^s\|\alpha(s,s_1)\|\,\|X(s_1)-Y(s_1)\|_{L_2}ds_1d\nu(t)$$

$$\leqq\sup_{t\in[0,+\infty)}\int_0^t\|\alpha(t,s)\|ds\int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|X(s)-Y(s)\|_{L_2}d\nu(t) \tag{3.4}$$

Now, combining (3.2), (3.3), (3.4) and (i) we conclude that

$$\begin{aligned}\|GX-GY\|_p&=\int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|(GX)(s)-(GY)(s)\|_{L_2}d\nu(t)\\&\leqq K\int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|X(s)-Y(s)\|_{L_2}d\nu(t)\\&\quad+\int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|L(s)\|_{L_2}d\nu(t)\\&\quad+\int_0^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|K(s)\|_{L_2}d\nu(t)\\&\leqq M\|X-Y\|_p,\end{aligned}$$

which proves that G is a continuous function. This fact, by the Banach fixed point principle, completes the proof of Theorem 3.1.

Remark 3.1. Let $h(t, X(t;\omega))\in D([0,+\infty))$. By Theorem 3.1 the solution $X(t;\omega)$ to equation (3.1) belongs to $D([0,+\infty))$, satisfying

$$\lim_{T\to\infty}\int_T^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|X(s)\|_{L_2}d\nu(t)=0.$$

Remark 3.2. If $p(t)\equiv1$ for $t\in\boldsymbol{R}_+$ then the random solution to equation (3.1) is asymptotically stable in the sense of Definition 3.2.

Now we are going to the stochastic integral equation of the form

$$X(t;\omega)=h(t, X(t;\omega))+\int_0^t g(t-s, X(t-s;\omega);\omega)\iota(s;\omega)ds \tag{3.5}$$

for $t\geqq 0$, which is equivalent to

$$X(t;\omega)=h(t, X(t;\omega))+\int_0^t g(u, X(u,\omega);\omega)\iota(t-u;\omega)du$$

where $\iota(t-u;\omega)\in L_\infty(\Omega, A, P)$.

Theorem 3.2. *Suppose that for* $X(t;\omega)\in\mathcal{L}_1^p$ *and* $Y(t;\omega)\in\mathcal{L}_1^p$

(i) $|g(u, X(u;\omega);\omega)-g(u, Y(u;\omega);\omega)|$

$$\leqq\alpha(u;\omega)|X(u;\omega)-Y(u;\omega)| \quad P\text{-a.s.},$$

where $\alpha(u;\omega)\in L_\infty(\Omega, A, P)$,

(ii) $|h(t, X(t;\omega))-h(t, Y(t;\omega))|$

$$\leqq K|X(t;\omega)-Y(t;\omega)| \quad P\text{-a.s.}.$$

Set

$$M:=K+\sup_{t\in[0,+\infty)}\int_0^\infty \|\alpha(t)\|\,\|\iota(t-u)\|du,$$

and suppose that

(iiii) $0<M<1$.

Then there exists one and only one solution $X\in\mathcal{L}_1^p$ *to equation* (3.5) *such that*

$$\lim_{T\to\infty}\int_T^\infty p(t)\,\nu\text{-ess}\sup_{s\in[0,t]}\|X(s)\|_{L_2}d\nu(t)=0.$$

Remark 3.3. If $p(t)\equiv 1$ for $t\in \boldsymbol{R}_+$ then the random solution to Equation (3.5) is asymptotically stable in the sense that

$$\limsup_{t\to\infty}\frac{\|x(t)\|_{L_2}}{u(t)}\leqq K,$$

where $K>0$, whenever $\int_0^\infty u(t)d\nu(t)<\infty$. Hence, we conclude that exponential stability in the sense of ([5], [7]) is a particular case of our results.

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