YOKOHAMA MATHEMATICAL
JOURNAL VOL. 41, 1993

ON p-QUASIHYPONORMAL OPERATORS FOR $0<p<1$

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(Received September 18, 1992; Revised May 1, 1993)

Abstract. For $0<p<1$ the notion of p-quasihyponormal operators on a Hilbert space is introduced and studied. It is proved that if T is a p-quasihyponormal operator with polar decomposition $T=U|T|$ then the operator $|T|^{1/2}U|T|^{1/2}$ is quasihyponormal for $1/2\leqq p<1$ and it is $(p+(1/2))$-quasihyponormal for $0<p<1/2$.

A bounded linear operator T on a Hilbert space H is said to be hyponormal if

$$\|T^{*}x\|\leqq\|Tx\| \quad \text{for all } x\in H$$

or equivalently if

$$T^{*}T-TT^{*}\geqq 0$$

and is said to be quasihyponormal if

$$\|T^{*}Tx\|\leqq\|TTx\| \quad \text{for all } x\in H$$

or equivalently

$$T^{*}(T^{*}T-TT^{*})T=T^{*2}T^{2}-(T^{*}T)^{2}\geqq 0$$

(see [5]). For $0<p<1$ T is said to be p-hyponormal if

$$(T^{*}T)^{p}-(TT^{*})^{p}\geqq 0.$$

Here H denotes a separable complex infinite dimensional Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Throughout the paper we consider those operators T for which $R(T)$, the range space of T, is a closed linear subspace of H. We begin with the following definition.

Definition. An operator T on the space H is said to be p-quasihyponormal if

$$T^{*}((T^{*}T)^{p}-(TT^{*})^{p})T\geqq 0.$$

If $p=1$ then T is quasihyponormal ([5], [6]) and if $p=1/2$ then T is called

1991 Mathematics Subject Classification: Primary 47B20, 47B65; Secondary 47A05.
Key Words and phrases: Hilbert space, hyponormal, p-quasihyponormal, polar decomposition, semi-quasihyponormal, unitary operator, positive operator, Furuta Inequality.

semi-quasihyponormal and for $p=1/4$ T is called quarter-quasihyponormal. Also for $q \leqq p$ any p-quasihyponormal is q-quasihyponormal. It is immediate that every p-hyponormal operator is p-quasihyponormal but not necessarily conversly. If T is semi-quasihyponormal but not quasihyponormal and if $T=U|T|$ is the polar decomposition of T, where $|T|=(T^*T)^{1/2}$, then the operator $T_0=U|T|^2$ is quarter-quasihyponormal but not semi-quasihyponormal.

Let T be a p-quasihyponormal operator. Let $T=U|T|$ be the polar decomposition of T and U be unitary and also let

$$\tilde{T}=|T|^{1/2}U|T|^{1/2}.$$

Then

(i) $\tilde{T}$ is quasihyponormal for $1/2 \leqq p<1$,

(ii) $\tilde{T}$ is $(p+(1/2))$-quasihyponormal for $0<p<1/2$.

We begin with the following lemma.

Lemma 1. *For* $T=U|T|$, $R(\tilde{T}) \subset R(|T|)$.

Proof. As $R(T)$ is assumed to be closed, $R(T^*)$ is closed ([4]). By [2, Theorem 2.2]

$$R(T^*)+R(T^*)=R(\sqrt{T^*T+T^*T}).$$

This implies that

$$R(T^*)=R(\sqrt{2T^*T})=R(\sqrt{2}|T|) \subset R(|T|).$$

Thus $R(|T|)$ is closed. Also by [2, Corollary 1] it follows that $R(|T|^{1/2})=R(|T|)$, since $|T|$ is a positive operator and $R(|T|)$ is closed. Therefore

$$R(\tilde{T}) \subset R(|T|^{1/2})=R(|T|).$$

Aluthge [1] proved that if T is p-hyponormal for $1/2 \leqq p<1$ and U is unitary, then the operator $\tilde{T}$ is hyponormal. We prove the following for p-quasihyponormal operators.

Theorem 2. *Let* $T=U|T|$ *be* p*-quasihyponormal;* $1/2 \leqq p<1$, *and* U *be unitary, then* $\tilde{T}=|T|^{1/2}U|T|^{1/2}$ *is quasihyponormal.*

Proof. As any p-quasihyponormal operator for $1/2 \leqq p<1$ is semi-quasihyponormal, we have

$$T^*((T^*T)^{1/2}-(TT^*)^{1/2})T \geqq 0.$$

This implies that

$$|T|U^*(|T|-U|T|U^*)U|T| \geqq 0.$$

This is equivalent to

$$|T|(U^*|T|U-|T|)|T| \geqq 0.$$

Thus $U^*|T|U-|T|\geqq 0$ on $R(|T|)$. Therefore by Lemma 1 it follows that on $R(\tilde{T})$

$$U^*|T|U\geqq|T|$$

or equivalently

$$U|T|U^*\leqq|T|.$$

Hence on $R(\tilde{T})$ we have

$$U^*|T|U\geqq|T|\geqq U|T|U^*.$$

Therefore on $R(\tilde{T})$ we get

$$\tilde{T}^*\tilde{T}-\tilde{T}\tilde{T}^*=|T|^{1/2}(U^*|T|U-U|T|U^*)|T|^{1/2}\geqq 0.$$

Hence $\tilde{T}$ is quasihyponormal.

Aluthge [1] has proved that if T is p-hyponormal for $0<p<1/2$ and U is unitary then $\tilde{T}$ is $(p+(1/2))$-hyponormal. To see through such a result for p-quasihyponormal operators we need the following famous and useful Furuta Inequality [3].

Theorem A. *If A and B are bounded self-adjoint operators such that $A\geqq B\geqq 0$. Then*

$$(B^r A^p B^r)^{1/2}\geqq B^{(p+2r)/q}$$

and

$$A^{(p+2r)/q}\geqq(A^r B^p A^r)^{1/q}$$

hold for each $r\geqq 0$, $p\geqq 0$, $q\geqq 1$ such that $(1+2r)q\geqq p+2r$.

Theorem 3. *Let $T=U|T|$ be p-quasihyponormal, $0<p<1/2$ and U be unitary. Then $\tilde{T}=|T|^{1/2}U|T|^{1/2}$ is $(p+(1/2))$-quasihyponormal.*

Proof. We have only to employ the ingenious proof of Theorem 2 in [1] based on Theorem A. Since T is p-quasihyponormal, therefore

$$T^*((T^*T)^p-(TT^*)^p)T\geqq 0.$$

This implies that

$$|T|U^*(|T|^{2p}-U|T|^{2p}U^*)U|T|\geqq 0.$$

This is equivalent to

$$|T|(U^*|T|^{2p}U-|T|^{2p})|T|\geqq 0.$$

Thus on $R(|T|)$

$$U^*|T|^{2p}U\geqq|T|^{2p}.$$

By Lemma 1 it follows that on $R(\tilde{T})$

$$U^*|T|^{2p}U\geqq|T|^{2p}$$

or equivalently

$$U|T|^{2p}U^*\leqq|T|^{2p}.$$

Hence on $R(\tilde{T})$, we have

$$U^*|T|^{2p}U \geqq |T|^{2p} \geqq U|T|^{2p}U^*.$$

Let $A=U^*|T|^{2p}U$, $B=|T|^{2p}$ and $C=U|T|^{2p}U^*$. Then using Theorem A, we get that on $R(\hat{T})$, we have

$$\begin{aligned}(\hat{T}^*\hat{T})^{p+1/2}&=(|T|^{1/2}U^*|T|U|T|^{1/2})^{p+1/2}\\&=(B^{1/4p}A^{1/2p}B^{1/4p})^{p+1/2}\\&\geqq B^{(1/2p+2/4p)(p+1/2)}=B^{1+1/2p}\end{aligned}$$

and

$$\begin{aligned}(\hat{T}\hat{T}^*)^{p+1/2}&=(|T|^{1/2}U|T|U^*|T|^{1/2})^{p+1/2}\\&=(B^{1/4p}C^{1/2p}B^{1/4p})^{p+1/2}\\&\leqq B^{(1/2p+2/4p)(p+1/2)}=B^{1+1/2p}.\end{aligned}$$

Hence on $R(\hat{T})$

$$(\hat{T}^*\hat{T})^{p+1/2} \geqq (\hat{T}\hat{T}^*)^{p+1/2}.$$

This implies that

$$\hat{T}^*((\hat{T}^*\hat{T})^{p+1/2}-(\hat{T}\hat{T}^*)^{p+1/2})\hat{T} \geqq 0.$$

Hence $\hat{T}$ is $(p+(1/2))$-quasihyponormal.

As a consequence of Theorems 2 and 3, we obtain

Corollary 4. *If T is a p-quasihyponormal operator for $0<p<1/2$, then the operator $|\hat{T}|^{1/2}\hat{U}|\hat{T}|^{1/2}$ is quasihyponormal, where $\hat{T}=|T|^{1/2}U|T|^{1/2}$ and $\hat{T}=\hat{U}|\hat{T}|$ is the polar decomposition of $\hat{T}$.*

Finally we give an example to show that the class of p-hyponormal operators is properly contained in the class of p-quasihyponormal operators.

Example 5. Let K be the direct sum of a denumerable number of copies of H. For given positive operators A and B defined on H, define the operator $T_{A,B}$ on K as follows

$$T_{A,B}(x_1, x_2, \cdots) = (0, Ax_1, Ax_2, \cdots, Ax_n, Bx_{n+1}, \cdots).$$

The operator $T_{A,B}$ is p-hyponormal if and only if $B^{2p} \geqq A^{2p}$ and is p-quasihyponormal if and only if $AB^{2p}A \geqq A^{2(p+1)}$.

Let H be a two-dimensional Hibert space with

$$A=\begin{bmatrix}4 & 0\\ 0 & 0\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}29 & 12\\ 12 & 5\end{bmatrix}$$

and let $p=1/2$. Then

$$B^{2p}-A^{2p}=B-A=\begin{bmatrix}25 & 12\\ 12 & 5\end{bmatrix}$$

which is not positive. Therefore $T_{A,B}$ is not semihyponormal. But

$$AB^{2p}A-A^{2(p+1)}=ABA-A^3$$

$$=\begin{bmatrix}464 & 0\\ 0 & 0\end{bmatrix}-\begin{bmatrix}64 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}400 & 0\\ 0 & 0\end{bmatrix}$$

which is positive. Therefore $T_{A,B}$ is semi-quasihyponormal.

The authors are thankful to Professor B. S. Yadav for his kind inspiration. The authors are also extremely grateful to the referee for his useful suggestions over the improvement of the paper.

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