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UNCOUNTABLY MANY LOOP SPACES OF THE SAME N-TYPE FOR ALL N, I

(Dedicated to Professor Seiya Sasao on his 60th bithday)

By

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Abstract. Let $C(f)$ be the mapping cone of a continuous map f from an infinite dimensional complex projective space CP^{∞} to a three dimensional sphere S^3. For any non-negative integer k, we classify the homotopy types of k-time loop spaces $\{\Omega^k\Sigma^k C(f)\,|\,f\}$ which have the same n-type for all n.

Introduction

Two spaces X and Y are called to have the same n-type if their Postnikov n-stages X_n and Y_n are homotopy equivalent. J. H. C. Whitehead posed a question: "If two CW-complexes are of the same n-type for all n, are they necessarily of the same homotopy type?"

J. F. Adams [1] answered it negatively by constructing two spaces satisfying the following conditions:

(1) They have the same n-type for all n.

(2) They are not homotopy equivalent.

Since spaces in his example were not of finite type, he posed a problem: "Do there exist CW-complexes which satisfy the conditions (1), (2) and are H-spaces of finite type?"

B. Gray [3] gave two CW-complexes satisfying the above conditions (1), (2) and the next conditions:

(3) They have finite cells at each dimension.

(4) They are loop spaces.

Spaces in his example were given by the reduced products of the mapping cones of a constant map and a non-trivial phantom map from CP^{∞} to S^3. Since one has a retraction and the other has not any retraction, they are distinguished.

In this paper, we give uncountably many k-time loop spaces satisfying the above conditions (1), (2), (3), and classify completely the homotopy types of these spaces. This is the first result to classify concretely the homotopy types of loop spaces of the same n-types for all n. Now we state our main theorems.

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Let $\Sigma^k CP^\infty$ and S^n be the k-time suspension of CP^∞ and an n-dimensional sphere respectively.

Theorem 2.1. *For $k\geqq 0$, let $f, g: \Sigma^k CP^\infty \to S^{k+3}$ be continuous maps, and $C(f)$, $C(g)$ mapping cones of f and g respectively. Then $C(f)$ and $C(g)$ are homotopy equivalent if and only if f and $\pm g$ are homotopic. The spaces $\{C(f) | f: \Sigma^k CP^\infty \to S^{k+3}\}$ are of finite type and have the same n-type for all n.*

Theorem 3.2. *For $k\geqq 0$, let $f, g: \Sigma^k CP^\infty \to S^{k+3}$ be continuous maps, and $C(f)$, $C(g)$ mapping cones of f and g respectively. For $0\leqq m\leqq k$, $\Omega^m C(f)$ and $\Omega^m C(g)$ are homotopy equivalent if and only if f and $\pm g$ are homotopic. The spaces $\{\Omega^m \Sigma^m C(f) | f: CP^\infty \to S^3\}$ are of finite type and have the same n-type for all n.*

From this theorem and the fact that the homotopy set $[\Sigma^k CP^\infty, S^{k+3}]=\hat{Z}/Z$ by Theorem D of [6], we can easily obtain the next theorem.

Theorem 3.3. *For each $k\geqq 0$, there exist, up to homotopy type, uncountably many k-time loop spaces which satisfy the conditions* (1), (2) *and* (3).

Let $SNT(X)$ be the collection of spaces which have the same n-type for all n as X. C. A. McGibbon and J. M. Møller [4] proved that $SNT(\Sigma^k(K(Z, n)\vee S^n))$ are uncountable for $k>0$, $n>1$, and got some theorems about the cardinality of $SNT(X)$. Our theorem can be restate that $SNT(\Omega^k\Sigma^k(\Sigma CP^\infty \vee S^3))$ is uncountable for any $k\geqq 0$.

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1. Preliminaries

In this paper, we work in the category of CW-complexes with base points and base point preserving continuous maps. We use the terminologies and the notations used in [5], [6].

Let $\hat{Z}$ be the completion of the ring Z of integers and $\hat{Z}/Z$ its quotient group. Let a_n be an integer and $a_n: Z\to Z$ the homomorphism defined by $a_n(k)=a_n k$. Hereafter we identify the integer a_n and the homomorphism a_n. Let $\{Z, a_n\}$ be an inverse system. Now we prepare an algebraic lemma.

Lemma 1.1. *Let $\{j_n\}: \{Z, a_n\}\to\{Z, b_n\}$ be a monomorphism between two inverse systems where $\varprojlim^1\{Z, a_n\}=\hat{Z}/Z$. Then, the image group $j(\hat{Z}/Z)$ is the 0-group or an uncountable group. Moreover, if $j_n: Z\to Z$ is not a 0-map for every $n>0$ and $\varprojlim^1\{Z, b_n\}=\hat{Z}/Z$, then $j=\varprojlim^1 j_n$ is an isomorphism.*

Proof. If $j_n=0$ for infinitely many integers $n>0$, then we have that $j(\hat{Z}/Z)=0$. If $j_n=0$ for finitely many integers $n>0$, then we may assume $j_n\neq 0$ for every $n>0$. We prove this lemma for a map $\{j_n\}:\{Z, a_n\}\to\{Z, b_n\}$ such that $a_n\neq 0$, $b_n\neq 0$. There are the canonical maps $\{f_n\}:\{Z, a_n\}\to\{Z, Id_n\}$, $\{g_n\}:\{Z, b_n\}\to\{Z, Id_n\}$ where $f_n=a_{n-1}\cdots a_1$, $g_n=b_{n-1}\cdots b_1$ for $n>1$ and $f_1=Id$, $g_1=Id$. Hence we obtain the next commutative diagram where the horizontal sequences are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \{Z, a_n\} & \longrightarrow & \{Z, Id_n\} & \longrightarrow & \{Z/f_{n-1}Z, Proj_n\} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \{Z, b_n\} & \longrightarrow & \{Z, Id_n\} & \longrightarrow & \{Z/g_{n-1}Z, Proj_n\} & \longrightarrow & 0 \end{array}$$

From this diagram, we obtain the next diagram by Proposition 2.3 of Chapter 9 in [2].

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & \varprojlim\{Z/f_{n-1}Z\} & \longrightarrow & \varprojlim^1\{Z, a_n\} & \longrightarrow & 0 \\ & & \downarrow Id & & \downarrow \phi & & \downarrow \psi & & \\ 0 & \longrightarrow & Z & \longrightarrow & \varprojlim\{Z/g_{n-1}Z\} & \longrightarrow & \varprojlim^1\{Z, b_n\} & \longrightarrow & 0 \end{array}$$

Since the above $\varprojlim$-groups have compact topology and ϕ is a dense map, ϕ is an onto-map. Hence ψ is also an onto-map. Since $\varprojlim^1\{Z, a_n\}$ is an uncountable group, we obtain the former part of the lemma. By $\varprojlim^1\{Z, b_n\}=\hat{Z}/Z$, we have $\varprojlim\{Z/g_{n-1}Z\}=\hat{Z}$. Hence ϕ is monic by considering on the p-adic integer for each prime number p. Finally, we obtain $\mathrm{Ker}\,\psi=0$.

Let $j: S^{k+2}\to\Sigma^k CP^\infty$ be the canonical inclusion. This induces the map of inverse systems $(j_n)_*:[\Sigma^k CP^n, S^{k+2}]\to[\Sigma^k CP^n, \Sigma^k CP^\infty]$ and $j_*:[\Sigma^{k-1}CP^\infty, S^{k+2}]\to[\Sigma^{k-1}CP^\infty, \Sigma^k CP^\infty]$. The group $[\Sigma^{k-1}CP^\infty, S^{k+2}]$ is isomorphic to $\hat{Z}/Z$ and $[\Sigma^k CP^n, \Sigma^k CP^\infty]$ is a finitely generated nilpotent group for $k>0$.

Proposition 1.2. *The canonical inclusion* $j: S^{2+k}\to\Sigma^k CP^\infty$ *induces a monomorphism* $j_*:[\Sigma^{k-1}CP^\infty, S^{k+2}]\to[\Sigma^{k-1}CP^\infty, \Sigma^k CP^\infty]$ *for* $k\geqq 1$.

Proof. It is sufficient to prove that the map

$$\{(j_n)_*\}:\{[\Sigma^k CP^n, S^{k+2}]\}\longrightarrow\{[\Sigma^k CP^n, \Sigma^k CP^\infty]\}$$

induces an into-isomorphism of $\varprojlim^1$-groups for $k\geqq 1$. Since $[\Sigma^k CP^n, \Sigma^k CP^\infty]\otimes Q$ is equal to $[\Sigma^k CP^n_Q, \Sigma^k CP^\infty_Q]=[\vee_{j=1}^n(S^{k+2j})_Q, \vee_{j=1}^\infty(S^{k+2j})_Q]$, there exists a free generator $g:\Sigma^k CP^n\to\Sigma^k CP^\infty$ where g_Q represents

$$(\text{inclusion})\cdot(\text{projection}): \vee_{j=1}^n(S^{k+2j})_Q\longrightarrow(S^{k+2})_Q\longrightarrow\vee_{j=1}^\infty(S^{k+2j})_Q$$

up to a finite degree. The free part of $[\Sigma^k CP^n, S^{k+2}]$ is Z and is mapped into the free part G_n of $[\Sigma^k CP^n, \Sigma^k CP^\infty]$ which is generated by the map g. It is sufficient to prove that the homological degree of the image of $\kappa_n : G_n \to [\Sigma^k S^2, \Sigma^k CP^\infty]$ is increasing as n is increasing for $k \geqq 1$. Since this value is evaluated by the ordinary cohomology $H^{2+k}(\kappa_n)$ or the K-theory $K(\kappa_n)$, we obtain the result for $k>1$ from the next Lemma 1.3. For $k=1$, we project $[\Sigma CP^n, \Sigma CP^\infty]$ to its abelianized group. Then, we can apply Lemma 1.1 to this case, because the map g is a generator.

Lemma 1.3. *If a map $h: \Sigma^k CP^n \to \Sigma^k CP^\infty$ satisfies $H^m(h)=0$ for $m>k+2$, then the degree $d(n, k)$ of the map $H^{k+2}(h): H^{k+2}(\Sigma^k CP^\infty) \to H^{k+2}(\Sigma^k CP^n)$ satisfies*

$$\nu_p(d(n, k)) \geqq \mathrm{Max}\{\nu_p(j): j=1, 2, \cdots, n\}$$

for all prime number p, where $\nu_p(j)$ is the exponent of p in the decomposition of j to prime factors.

Proof. We may prove this lemma for even k by considering the suspension. We shall prove only in the case of $k=2$, because in the other case the proof is similar. Since the Chern character map is monomorphic, we use K-theory. We set

$$h'(B\mu)=\sum_{j=1}^n a_j B\mu^j$$

where B is the Bott isomorphism and η is the canonical line bundle over the complex projective space and $\mu=\eta-1$. Clearly a_1 is equal to the degree of $H^4(h)$. Since the Chern character map is monomorphic, it holds $h'(B\mu^j)=0$ for $j>1$ by the assumption. We calculate the next formulas.

$$\begin{aligned} \psi^2 h'(B\mu) &= \psi^2(\textstyle\sum_{j=1}^n a_j B\mu^j)=2B\psi^2(\sum_{j=1}^n a_j \mu^j) \\ &= 2\textstyle\sum_{j=1}^n a_j B(2\mu+\mu^2)^j \\ h'\psi^2(B\mu) &= 2h'B\psi^2(\mu)=2h'(B\mu^2+2B\mu)) \\ &= 4(\textstyle\sum_{j=1}^n a_j B\mu^j) \end{aligned}$$

By $\psi^2 h'=h'\psi^2$, we obtain

$$T(\mu^2+2\mu)=2T(\mu)$$

where $T(\mu)=\sum_{j=1}^n a_j\mu^j$. Hence $T(\mu)$ must be $a_1 \log(1+\mu)=a_1 \sum_{j=1}^\infty \{(-1)^{j+1}/j\}\mu^j$ mod μ^{n+1} and $a_j=a_1(-1)^{j+1}/j$. Since $a_j=a_1(-1)^{j+1}/j$ is an integer for $j=1, 2, \cdots, n$. We obtain the result.

2. *CW*-complexes of the same *n*-type for all *n*

A map $f: X \to Y$ is called a phantom map if the restriction map $f|X^n$ on the skeleton X^n is homotopic to a constant map for all $n \geqq 0$. We remark that the canonical inclusion $Y\to\Omega\Sigma Y$ induces the suspension map $[X, Y]\to[X, \Omega\Sigma Y]=[\Sigma X, \Sigma Y]$ and $[\Sigma^k CP^\infty, S^{k+3}]=[\Sigma^{k+1}CP^\infty, S^{k+4}]$ by Theorem D of [6] or Theorem 2.2 of [5].

Theorem 2.1. *For $k\geqq 0$, let $f, g: \Sigma^k CP^\infty\to S^{k+3}$ be continuous maps, and $C(f)$, $C(g)$ mapping cones of f and g respectively. Then $C(f)$ and $C(g)$ are homotopy equivalent if and only if f and $\pm g$ are homotopic. The spaces $\{C(f)| f: \Sigma^k CP^\infty\to S^{k+3}\}$ are of finite type and have the same n-type for all n.*

Proof. By Theorem D of A. Zabrodsky [6], the homotopy set $[\Sigma^k CP^\infty, S^{k+3}]$ is equal to $\mathrm{Ext}(H_{k+2}(\Sigma^k CP^\infty; Q), \pi_{k+3}(S^{k+3}))=\mathrm{Ext}(Q, Z)=\hat{Z}/Z$ and all maps $f: \Sigma^k CP^\infty\to S^{k+3}$ are phantom maps. Pairs of homotopy equivalences $\Sigma^k CP^\infty\to\Sigma^k CP^\infty$ and $S^{k+3}\to S^{k+3}$ induce an action on $\mathrm{Ext}(H_{k+2}(\Sigma^k CP^\infty; Q), \pi_{k+3}(S^{k+3}))=\mathrm{Ext}(Q, Z)=\hat{Z}/Z$. In general, algebraically, a finite group $\mathrm{Aut}(H_{k+2}(\Sigma^k CP^\infty; Z))\times\mathrm{Aut}(\pi_{k+3}(S^{k+3}))=\{\pm 1\}\times\{\pm 1\}$ acts on uncountable group $\mathrm{Ext}(H_{k+2}(\Sigma^k CP^\infty; Q), \pi_{k+3}(S^{k+3}))=\mathrm{Ext}(Q, Z)=\hat{Z}/Z$. Hence the cardinality of its equivalence classes under the action is uncountable.

If f and $\pm g$ are homotopic, we denote $f\sim\pm g$, $C(f)$ and $C(g)$ are clearly homotopy equivalent. If there is a homotopy equivalence $\beta: C(f)\to C(g)$, we set $\beta^*(V)=aV+b\Sigma^{k+1}U$ and $\beta^*(\Sigma^{k+1}U)=cV+d\Sigma^{k+1}U$, $ad-bc=\pm 1$ where V is the generator of $H^{k+3}(S^{k+3}; Z)$ and $\Sigma^{k+1}U$ is the generator of $H^{k+3}(\Sigma^{k+1}CP^\infty; Z)$. By using the reduced power operation, we have $b=0$ and $a=\pm 1$ by $ad-bc=\pm 1$. Hence we have $d=\pm 1$. Moreover, β induces a map $\alpha: S^{k+3}\to S^{k+3}$ of degree ± 1, and $c=0$ follows by the following consideration. There is the following diagram.

$$\begin{array}{ccccccccccc}
\Sigma^k CP^\infty & \xrightarrow{f} & S^{k+3} & \xrightarrow{j} & C(f) & \xrightarrow{p} & \Sigma^{k+1}CP^\infty & \xrightarrow{-\Sigma f} & S^{k+4} & \longrightarrow & \Sigma C(f) \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \\
\Sigma^k CP^\infty & \xrightarrow[g]{} & S^{k+3} & \xrightarrow[k]{} & C(g) & \xrightarrow[q]{} & \Sigma^{k+1}CP^\infty & \xrightarrow[-\Sigma g]{} & S^{k+4} & \longrightarrow & \Sigma C(g)
\end{array}$$

The existence of α is proved by the following consideration. When $f\sim g\sim 0$, the statement is clear. We may assume that f is not homotopic to a constant map. $q\beta jf: \Sigma^k CP^\infty\to\Sigma^{k+1}CP^\infty$ is homotopic to the constant map by $jf\sim 0$. Since $q\beta jf$ is homotopic to $sf: \Sigma^k CP^\infty\to S^{k+3}\to\Sigma^{k+1}CP^\infty$ where $s: S^{k+3}\to\Sigma^{k+1}CP^\infty$ is the map of degree s, the map s is homotopic to the constant map by Proposition 1.2. Hence β induces the above commutative diagram where α, β, γ and δ are maps of homotopy equivalences.

By the above argument, we have that Σf and Σg are equivalent under the action of homotopy equivalences on $\mathrm{Ext}(H_{k+3}(\Sigma^{k+1}CP^\infty; Q), \pi_{k4}(S^{k+4}))=\mathrm{Ext}(Q, Z)=\hat{Z}/Z$. From this we have $\Sigma f\sim\pm\Sigma g$ and hence $f\sim\pm g$ by the suspension isomorphism.

Remark. We shall discuss the similar statements for Proposition 1.2 and Theorem 2.1 in the case of the non-trivial connected Lie group G in a forthcoming paper.

3. Main results

In this section, we prove our main theorem. The next lemma is elementary.

Lemma 3.1. *If two simply connected spaces X and Y have the same n-type for all n, their loop spaces ΩX and ΩY have the same n-type for all n.*

Theorem 3.2. *For $k\geqq 0$, let $f, g: \Sigma^k CP^\infty \to S^{k+3}$ be continuous maps, and $C(f)$, $C(g)$ mapping cones of f and g respectively. For $0\leqq m\leqq k$, $\Omega^m C(f)$ and $\Omega^m C(g)$ are homotopy equivalent if and only if f and $\pm g$ are homotopic. The spaces $\{\Omega^k\Sigma^k C(f) \mid f: CP^\infty \to S^3\}$ are of finite type and have the same n-type for all n.*

Proof. It is sufficient to prove the statement for $m=k$. If $f\sim\pm g$, $\Omega^k C(f)$ and $\Omega^k C(g)$ are clearly homotopy equivalent. Conversely, if $\Omega^k C(f)$ and $\Omega^k C(g)$ are homotopy equivalent, there is a homotopy equivalence $\rho: \Omega^k C(f)\to\Omega^k C(g)$. By the suspension isomorphism, we have $C(f)=\Sigma^k C(f')$ and $C(g)=\Sigma^k C(g')$ where $f', g': CP^\infty\to S^3$. Then, there is a map:

$$\phi: \Sigma^k C(f') \xrightarrow{\Sigma^k\eta} \Sigma^k\Omega^k\Sigma^k C(f') \xrightarrow{\Sigma^k\rho} \Sigma^k\Omega^k\Sigma^k C(g') \xrightarrow{\varepsilon} \Sigma^k C(g')$$

where $\Sigma^k\eta$ is the canonical inclusion and ε is the evaluation map. Since η induces an isomorphism $H_3(C(f'); Z)\to H_3(\Omega^k\Sigma^k C(f'); Z)$ by Hurewicz theorem. Hence ϕ induces the isomorphism of ϕ^* at $k+3$ dimension. As same as proof of Theorem 2.1, we have $\phi^*(V)=\pm V$ and $\phi^*(\Sigma^{k+1}U)=\pm\Sigma^{k+1}U$, where V is the generator of $H^{k+3}(S^{k+3}; Z)$ and $\Sigma^{k+1}U$ is the generator of $H^{k+3}(\Sigma^{k+1}CP^\infty; Z)$. Moreover, ϕ induces a map $S^{k+3}\to S^{k+3}$ of degree ± 1. Hence we obtain the next commutative diagram.

$$\begin{array}{ccccccccccc}
\Sigma^k CP^\infty & \xrightarrow{f} & S^{k+3} & \xrightarrow{j} & C(f) & \xrightarrow{p} & \Sigma^{k+1}CP^\infty & \xrightarrow{-\Sigma f} & S^{k+4} & \longrightarrow \\
 & & \downarrow \pm 1 & & \downarrow \phi & & \downarrow \gamma & & \downarrow \pm 1 & \\
\Sigma^k CP^\infty & \xrightarrow[g]{} & S^{k+3} & \xrightarrow[k]{} & C(g) & \xrightarrow[q]{} & \Sigma^{k+1}CP^\infty & \xrightarrow[-\Sigma g]{} & S^{k+4} & \longrightarrow
\end{array}$$

Here ϕ and γ are rational equivalences by Theorem 3.6. Moreover, it holds $\gamma^*(\Sigma^{k+1}U)=\pm\Sigma^{k+1}U$. Hence γ induces the isomorphism $[\Sigma^{k+1}CP^\infty, S^{k+4}]=[\Sigma^{k+1}CP^\infty, S^{k+4}]$ by Theorem D of [6]. We have also $\Sigma f\sim\pm\Sigma g$ by $\Sigma f=\Sigma g\gamma$. Finally we have $f\sim\pm g$ if $\Omega^k C(f)$ and $\Omega^k C(g)$ are homotopy equivalent.

Theorem 3.3. *For any $k \geqq 0$, there exist uncountably many CW-complexes which satisfy the next conditions*:

(1) *They have finite cells for each dimensions.*

(2) *They are k-time iterated loop spaces.*

(3) *They have the same n-type for all n.*

(4) *They are not homotopy equivalent each other.*

Proof. Let $C(f)$ be the mapping cone of $f: \Sigma^k CP^\infty \to S^{k+3}$ and $X(f) = \Omega^k C(f)$ the k-time iterated loop space of $C(f)$. We choose a representative map f for each equivalence class of $[\Sigma^k CP^\infty, S^{k+3}] = \hat{Z}/Z$ under the action $\{\pm 1\}$. Then $\{X(f) | f\}$ satisfy the conditions of this theorem by Theorem 3.2.

We show that ϕ in the proof of Theeorem 3.2 is a rational equivalence.

Lemma 3.4. *Let $h_1, h_2, \cdots, h_k$ and $i_1, i_2, \cdots, i_k$ be natural numbers. Set*

$$H = (h_1, h_2, \cdots, h_k), \qquad I = (i_1, i_2, \cdots, i_k),$$

$$h = h(H) = h_1 + h_2 + \cdots + h_k, \qquad j = j(H, I) = h_1 i_1 + h_2 i_2 + \cdots + h_k i_k.$$

Then,

$$\alpha(H, I) = \{h!/(h_1!)(h_2!) \cdots (h_k!)\} \times \{j!/(i_1!)^{h_1}(i_2!)^{h_2} \cdots (i_k!)^{h_k}\}$$

is an even numbers for $h(H) > 1$.

Proof. At first, we remark that $(2i)!/(i!)(i!)$ is an even number for $i > 0$. If a number h_s is greater than 1, for example $h_1 > 1$, we decompose $j!/(i_1!)^{h_1}(i_2!)^{h_2} \cdots (i_k!)^{h_k}$ such as

$$\{j!/(2i_1)!(i_1!)^{h_1-2}(i_2!)^{h_2} \cdots (i_k!)^{h_k}\} \times \{(2i_1)!/(i_1!)(i_1!)\}.$$

Here the first part is a natural number and the second part is an even number. Hence the result follows. If $h_1 = h_2 = \cdots = h_k = 1$, $h!/(h_1!)(h_2!) \cdots (h_k!)$ is an even number for $h > 1$. Hence, $\alpha(H, I)$ is an even number for $h > 1$.

Proposition 3.5. *Let $f: \Sigma^k CP^\infty \to \Sigma^k CP^\infty$ be a map such that $f^*(\Sigma^k U) = \pm \Sigma^k U$. Then, β_j is an odd number for $j > 0$, where $f^*(\Sigma^k U^j) = \beta_j \Sigma^k U^j$. Hence the map f is a rational equivalence.*

Proof. We shall prove only for $k=2$. In the other cases, the proofs are similar. We put

$$f'(B\mu) = \sum_{j=1}^\infty b_j B\mu^j$$

$$f^*(\Sigma^2 U^j) = \beta_j \Sigma^2 U^j$$

for K-theory and the ordinary cohomology theory respectively, where b_j and β_j

are integers. We have the next formulas,

$$\begin{aligned}\mathrm{Ch}\, f'(B\mu) &= \mathrm{Ch}\,(\textstyle\sum_{j=1}^{\infty} b_j B\mu^j) \\ &= \textstyle\sum_{j=1}^{\infty} b_j \{\Sigma^2(e^U-1)^j\}\,, \\ f^*\mathrm{Ch}\,(B\mu) &= f^*(\Sigma^2(e^U-1)) \\ &= f^*(\Sigma^2(U+U^2/2!+U^3/3!+\cdots+)) \\ &= \textstyle\sum_{j=1}^{\infty} \beta_j \{\Sigma^2 U^j/j!\}\,.\end{aligned}$$

By $\mathrm{Ch}\, f'(B\mu)=f^*\mathrm{Ch}\,(\beta\mu)$, we have

$$\textstyle\sum_{j=1}^{\infty} \beta_j U^j/j! = \sum_{j=1}^{\infty} b_j (e^U-1)^j\,.$$

By comparing the both sides, we have $\beta_j=\Sigma b_h \alpha(H, I)$ where the summation takes for every h, H and I such that $h=h(H)$, $j=j(H, I)$. Specially, we have $\beta_1=b_1=\pm 1$. Since $\alpha(H, I)$ is an even number for $h>1$, β_j is odd number for $j>0$. Hence f is a rational equivalence.

By using this result, we have the next theorem.

Theorem 3.6. *Let $\phi: C(f)\to C(g)$ be a map satisfying $\phi^*(\Sigma^{k+1}U)=\pm\Sigma^{k+1}U$ and $\phi^*(V)=\pm V$, where V and $\Sigma^{k+1}U$ are the generators of $H^{k+3}(S^{k+3}; Z)$ and $H^{k+3}(\Sigma^{k+1}CP^{\infty}; Z)$ respectively. Then ϕ is a rational equivalence.*

Remark. We shall discuss a generalization of Theorem 3.2 in the case of Lie groups G, and the case $k=\infty$ in forthcoming papers.

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