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MONOID STRUCTURE OF ENDOMORPHISMS OF $HP^{\infty}\times S^{n}$

By

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Summary. We calculate the homotopy set $[HP^{\infty}\times S^{n}, HP^{\infty}\times S^{n}]$ and determine its monoid structure given by the composition of maps for all n. From the results, we get the group $\mathrm{Aut}(HP^{\infty}\times S^{n})$ of self-homotopy equivalences and the group $\mathrm{WI}(HP^{\infty}\times S^{n})$ of weak identities for all n.

Introduction

Two continuous maps $f, g: X\rightarrow Y$ are called a phantom pair, if the restriction maps $f|X^{n}$, $g|X^{n}$ on the n-skeleton X^{n} are homotopic for all n, or equivalently $q_{n}f$ and $q_{n}g$ are homotopic for all n, where $q_{n}: Y\rightarrow Y_{n}$ is the Postnikov n-stage of Y. When g is the constant map, f is called a phantom map. When g is an identity map, f is called a weak identity map. J. Roitberg [6, 7] studied the relation of phantom maps and weak identities and calculated the group $\mathrm{Aut}(K(Z, 2)\times S^{3})$ of the self-homotopy equivalences of $K(Z, 2)\times S^{3}$ as the group extension of the groups $\mathrm{WI}(K(Z, 2)\times S^{3})$ of weak identities of $K(Z, 2)\times S^{3}$ and $Z/2Z\times Z/2Z$. C. A. McGibbon and J. M. Møller [3] determined also $\mathrm{Aut}(K(Z, m)\times S^{m+1})$. The author [8] calculated the homotopy set $[K(Z, m)\times S^{n}, K(Z, m)\times S^{n}]$ for all $m, n\geqq 1$, which is denoted by $\mathrm{End}(K(Z, m)\times S^{n})$ and also determined the monoid structure of $\mathrm{End}(K(Z, m)\times S^{n})$ given by the composition of maps. From the results, we got the groups $\mathrm{Aut}(K(Z, m)\times S^{n})$ and $\mathrm{WI}(K(Z, m)\times S^{n})$ for all $m, n\geqq 1$.

In this paper, we calculate the homotopy set

$$\mathrm{End}(HP^{\infty}\times S^{n})=[HP^{\infty}\times S^{n}, HP^{\infty}\times S^{n}]$$

for all n and also determine the monoid structure of $\mathrm{End}(HP^{\infty}\times S^{n})$. Our main theorems are the followings.

Theorem 0.1. *The elements of* $\mathrm{End}(HP^{\infty}\times S^{n})$ *are in one-to-one correspondence with* 2×2 *matrices of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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where a, b, c and d are defined as follows:

(1) *For* $n\neq 1, 3, 4i+2$ *or* $n=2$, $a\in[HP^\infty, HP^\infty]$, $b\in[S^n, HP^\infty]$, $c\in[HP^\infty, S^n]$ *and* $d\in[S^n, S^n]$ *are defined by the compositions of injections, projections and the map of endomorphism.*

(2) *For* $n=4i+2>2$, a, b, d *are defined as* (1) *and* $c: HP^\infty\times S^n\to S^n$ *is the class of a phantom map with* $cd=0$.

(3) *For* $n=1, 3$, a, d *are defined as* (1) *and* $b\in[S^n, \mathrm{Map}(HP^\infty, HP^\infty; a)]$, $c=0$.

We determine also the multiplications of matrices corresponding to the composition of maps (Theorems 3.2, 3.3 and 3.5). From these results, we can easily get the groups $\mathrm{Aut}(HP^\infty\times S^n)$ and $\mathrm{WI}(HP^\infty\times S^n)$.

Theorem 0.2. *The multiplications of* $\mathrm{End}(HP^\infty\times S^n)$ *are given as follows:*

(1) *For* $n\neq 1, 3, 4i+2$ *or* $n=2$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae & af+bh \\ ce^i+dg & dh \end{pmatrix},$$

where i *is given for* $n=4i+1$ *and* $c=g=0$ *for the other cases.*

(2) *For* $n=4i+2>2$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae & af+bh \\ ce^ih+d^2g & dh \end{pmatrix}.$$

(3) *For* $n=1, 3$,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} e & f \\ 0 & h \end{pmatrix}=\begin{pmatrix} ae & af+beh \\ 0 & dh \end{pmatrix}.$$

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1. Preliminaries

We shall work in the category of nilpotent CW-complexes with base point and base point preserving continuous maps except for some special cases. We use the notations and terminologies of [8, 9] and the notations for mapping spaces as follows:

$\mathrm{Map}(X, Y)$; the space of continuous maps from X to Y,
$\mathrm{Map}_*(X, Y)$; the space of based continuous maps from X to Y,
$\mathrm{Map}(X, Y; f)$; the connected component of $\mathrm{Map}(X, Y)$ which contains f,
$\mathrm{Map}_*(X, Y; f)$; the connected component of $\mathrm{Map}_*(X, Y)$ which contains f.

The next lemma is proved by using the fibration $\mathrm{Map}_*(S^n, S^n; k)\to$

$\mathrm{Map}(S^n, S^n; k) \to S^n$ of the evaluation map and the non-triviality of the Whitehead products $[k, h]$ for all $k \neq 0$, $h \neq 0$ and even n (cf. [8]).

Lemma 1.1. *Let $k: S^n \to S^n$ be the map of degree k. The free parts of homotopy groups of* $\mathrm{Map}(S^n, S^n; k)$ *are given as follows:*

$$\pi_j(\mathrm{Map}(S^n, S^n; k)) \otimes Q = Q \quad \begin{cases} j=n \text{ for odd } n, \\ j=2n-1 \text{ for even } n \text{ and } k \neq 0, \\ j=n-1,\ n,\ 2n-1 \text{ for even } n \text{ and } k=0, \end{cases}$$
$$=0 \quad \text{otherwise.}$$

It suffices to determine $[HP^\infty \times S^n, HP^\infty]$ and $[HP^\infty \times S^n, S^n]$ for the calculation of $\mathrm{End}(HP^\infty \times S^n)$. We prepare the next lemma.

Lemma 1.2. *The homotopy set* $[\Sigma^i HP^\infty, \mathrm{Map}(S^n, S^n; k)^\wedge]$ *is* 0 *for all* $i \geqq 0$.

Proof. $\mathrm{Map}(S^n, S^n; k)$ is a nilpotent space. Since $[\Sigma^i HP^\infty, (S^n)^\wedge]$ is 0 by Theorem 3.1 of [2] for all $i \geqq 0$,

$$[\Sigma^i HP^\infty, \mathrm{Map}(S^n, S^n; k)^\wedge] = [\Sigma^i HP^\infty, \mathrm{Map}(S^n, (S^n)^\wedge; \hat{e}k)] = 0$$

by Theorem C of [9].

We calculate the homotopy set $[HP^\infty \times S^n, S^n]$ for $n>0$. For $n=1$, we have $[HP^\infty \times S^n, S^n] = Z$. Since S^n is simply connected for $n>1$, it is sufficient to calculate the free homotopy set $[HP^\infty \times S^n, S^n]_{\text{free}}$. The free homotopy set $[HP^\infty, \mathrm{Map}(S^n, S^n; k)]_{\text{free}}$ is equal to the based homotopy set $[HP^\infty, \mathrm{Map}(S^n, S^n; k)]$, because the fundamental group of $\mathrm{Map}(S^n, S^n; k)$ acts trivially on the free part of the homotopy group at least modulo torsion group. We can also prove it by the naturality of the forgetful functor from the based homotopy set to the free homotopy set. Hence we get the next result by Lemma 1.2 and Theorems C, D of [9].

Proposition 1.3. *The homotopy set* $G(n, k) = [HP^\infty, \mathrm{Map}(S^n, S^n; k)]$ *is given as follows:*

$$G(n, k) = \begin{cases} \hat{Z}/Z & n=4i+1>1 \text{ and any } k, \text{ or } n=4i+2>2 \text{ and } k=0, \\ 0 & \text{otherwise.} \end{cases}$$

The above proposition is also restated as follows. A map $x: HP^\infty \times S^n \to S^n$ is evaluated by the restriction maps $x \mid HP^\infty \times \{*\}$ and $x \mid \{*\} \times S^n$. We get also the next corollary (cf. [8]).

Corollary 1.4. *For $n \neq 4i+2$ or $n=2$, a map $x: HP^{\infty} \times S^n \to S^n$ is classified by the restriction maps $x | HP^{\infty} \times \{*\}$ and $x | \{*\} \times S^n$. For $n=4i+2>2$, a map $x: HP^{\infty} \times S^n \to S^n$ with $x | \{*\} \times S^n = 0$ is a phantom map corresponding to $U^i V$ where U and V are the generators of $H^4(HP^{\infty}: \hat{Z}/Z)$ and $H^n(S^n; \hat{Z}/Z)$ respectively, and a map x with $x | \{*\} \times S^n \neq 0$ is classified by $x | \{*\} \times S^n: S^n \to S^n$.*

To get $[HP^{\infty} \times S^n, HP^{\infty}]$, we determine the homotopy type of $\mathrm{Map}(HP^{\infty}, HP^{\infty}; k)$, where the homotopy set $[HP^{\infty}, HP^{\infty}]$ is equal to the set $\{0, 1, \cdots, (\mathrm{odd})^2\}$ by [4].

Proposition 1.5. *The homotopy type of* $\mathrm{Map}(HP^{\infty}, HP^{\infty}; k)$ *is weak homotopy equivalent to HP^{∞} for $k=0$ and $K(Z/2, 1) \times K(\hat{Z}/Z, 3)$ for $k \neq 0$ where k is a map of degree k at dimension 4. Moreover the map $h: HP^{\infty} \to HP^{\infty}$ of degree h induces the h-times multiplication on the homotopy groups of* $\mathrm{Map}(HP^{\infty}, HP^{\infty}; k)$.

Proof. Since $\mathrm{Map}_*(HP^{\infty}, HP^{\infty}; 0)$ is weak contractible by Theorem 3.1 in [2], we get the result for $k=0$. For $k \neq 0$, $\mathrm{Map}(HP^{\infty}, (HP^{\infty})^{\wedge}; \hat{e}k)$ is weak homotopy equivalent to $K(Z/2Z, 1)$ by (2.8) and the proof of Corollary 3.5 of [1]. By considering fibrations $\mathrm{Map}_*(HP^{\infty}, HP^{\infty}; k) \to \mathrm{Map}(HP^{\infty}, HP^{\infty}; k) \to HP^{\infty}$ and $\mathrm{Map}_*(HP^{\infty}, (HP^{\infty})^{\wedge}; \hat{e}k) \to \mathrm{Map}(HP^{\infty}, (HP^{\infty})^{\wedge}; \hat{e}k) \to (HP^{\infty})^{\wedge}$, we see the result by Corollary 1.3 of [1] for $k \neq 0$.

Since a map $HP^{\infty} \times S^n \to HP^{\infty}$ is classified by the equality $[HP^{\infty} \times S^n, HP^{\infty}] = \bigcup_k [S^n, \mathrm{Map}(HP^{\infty}, HP^{\infty}; k)]$, we need the next proposition.

Proposition 1.6. *Let $H(n, k)$ be the homotopy set* $[S^n, \mathrm{Map}(HP^{\infty}, HP^{\infty}; k)]$. *Then, we have the next formulas:*

$$H(n, 0) = \pi_n(HP^{\infty}), \qquad H(1, k) = Z/2 \quad \text{for } k \neq 0,$$

$$H(3, k) = \hat{Z}/Z \quad \text{for } k \neq 0, \qquad H(n, k) = 0 \quad \text{for } k \neq 0 \text{ and } n \neq 1, 3.$$

2. Endomorphisms of $HP^{\infty} \times S^n$

In this section, we determine the homotopy set $\mathrm{End}(HP^{\infty} \times S^n)$ of endomorphisms of $HP^{\infty} \times S^n$ for all n. Using the results of section 1, we can describe the endomorphisms of $HP^{\infty} \times S^n$ by the matrix form. For self-map $x: HP^{\infty} \times S^n \to HP^{\infty} \times S^n$, we attach a 2×2 matrix

$$X(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are defined as in Theorem 0.1.

Using the above representation, we summarize the results of section 1.

Theorem 2.1. *The elements of the homotopy set* $\mathrm{End}(HP^\infty \times S^n)$ *of endomorphisms of* $HP^\infty \times S^n$ *are in one-to-one correspondence with* 2×2 *matrices defined above as follows,*

(1) *odd* $n \neq 1, 3$ *or* $n=4i$ *or* $n=2$

$$a=0 \text{ and } b\in \pi_n(HP^\infty), \text{ or}$$

$$a\neq 0 \text{ and } b=0, \begin{cases} c\in \hat{Z}/Z,\ d\in Z \text{ for } n=4i+1>1 \\ c=0,\ d\in Z \text{ for } n=4i+3>3,\ 4i \text{ or } 2 \end{cases}$$

(2) $n=4i+2>2$

$$a=0 \text{ and } b\in \pi_n(HP^\infty), \text{ or}$$

$$a\neq 0 \text{ and } b=0,\ c\in \hat{Z}/Z,\ d\in Z \text{ with } cd=0$$

(3) $n=1, 3$

$$a\in [HP^\infty, HP^\infty],\ b\in [S^n, \mathrm{Map}(HP^\infty, HP^\infty; a)]=\begin{cases} Z/2Z \text{ for } n=1, \\ \hat{Z}/Z \text{ for } n=3, \end{cases}$$

$$c=0,\ d\in [S^n, S^n]$$

From the result, we can easily determine the sets of self-homotopy equivalences and weak identities of $HP^\infty \times S^n$. The multiplications are given in section 3.

Theorem 2.2. *The elements of group* $\mathrm{Aut}(HP^\infty \times S^n)$ *of self-homotopy equivalences of* $HP^\infty \times S^n$ *are in one-to-one correspondence with* 2×2 *matrices having the following components.*

(1) $n=4i+1>1$,

$$a=1,\quad b=0,\quad c\in \hat{Z}/Z,\quad d=\pm 1$$

(2) $n=4i+3>3$ *or* $4i$ *or* $4i+2$,

$$a=1,\quad b=0,\quad c=0,\quad d=\pm 1$$

(3) $n=1$,

$$a=1,\quad b\in \{0, 1\},\quad c=0,\quad d=\pm 1$$

(4) $n=3$,

$$a=1,\quad b\in \hat{Z}/Z,\quad c=0,\quad d=\pm 1$$

3. The monoid structure of $\mathrm{End}(HP^\infty \times S^n)$

In this section, we determine the monoid structure of $\mathrm{End}(HP^\infty \times S^n)$ for all n whose product is given by the composition of maps.

Let $(S^{2n})_P$ be the homotopy fiber of Sullivan completion $\hat{e} : S^{2n} \to S^{2n\wedge}$. Since $(S^{2n})_Q$ is the homotopy fiber of the cup square map $K(Q, 2n) \to K(Q, 4n)$, we see that $(S^{2n})_P$ is the homotopy fiber of $K(\hat{Z}/Z, 2n-1) \to K(\hat{Z}/Z, 4n-1)$ by the arithmetic square. Since a map $d : S^{2n} \to S^{2n}$ of degree d induces degrees d and

d^2 on the homotopy groups $\pi_j(S^{2n})$ for $j=2n$ and $4n-1$ respectively, the map $\tilde{d}:(S^{2n})\rho\to(S^{2n})\rho$ induces degrees d and d^2 on the homotopy groups $\pi_j((S^{2n})\rho)$ for $j=2n-1$ and $4n-2$ respectively.

For $n=4i+1>1$, a map $g: HP^\infty\times S^n\to S^n$ is determined by $g': HP_\infty\vee S^n\to S^n$ and a phantom map $g|HP^\infty: HP^\infty\times\{*\}\to S^n$ is represented by an element U^i of cohomology group $H^{n-1}(HP^\infty\times\{*\}\,;\,\hat{Z}/Z)$.

For $n=4i+2>2$, a phantom map $g: HP^\infty\times S^n\to S^n$ with $g|\{*\}\times S^n=0$ is represented by $g': HP^\infty\times S^n\to(S^n)\rho$ with $\rho g'=g$. Hence a phantom map $f: HP^\infty\times S^n\to S^n$ is represented by an element $U^i\ V$ of cohomology group $H^{2n-2}(HP^\infty\times S^n\,;\,\hat{Z}/Z)$. We get $[HP^\infty\times(S^n)\rho,\ S^n]=[HP^\infty\times(S^n)\rho,\ (S^n)\rho]$ by using the fact $[HP^\infty\times(S^n)\rho,\ (S^n)\hat{}\,]=[(S^n)\rho,\ \mathrm{Map}(HP^\infty,\ (S^n)\hat{}\,)]=[(S^n)\rho,\ (S^n)\hat{}\,]=0$. And we have also the next commutative diagram.

$$\begin{array}{ccccc} HP^\infty & \longrightarrow & HP^\infty\times S^n & \longrightarrow & S^n \\ & & \uparrow{\scriptstyle Id\times\rho} & & \uparrow{\scriptstyle\rho} \\ & & HP^\infty\times(S^n)\rho & \longrightarrow & (S^n)\rho \end{array}$$

Using the method of section 3 of [8], we can get the next lemma, where we interpret $c=g=0$ for $n\neq 4i+1$ or $n=2$ and hence $ce^i+dg=0$.

Lemma 3.1. *Suppose n is odd and $n\neq 1$, 3 or $n=4i$ or $n=2$. Let $y: HP^\infty\to HP^\infty\times S^n$ and $x: HP^\infty\times S^n\to S^n$ be represented by ${}^t[e,\ g]$ and $[c,\ d]$ respectively. Then, the composition $xy=[c,\ d]^t[e,\ g]$ is equal to ce^i+dg.*

Proof. For odd $n\neq 1$, 3 or $n=4i$ or $n=2$, essential phantom maps appear only for $n=4i+1>1$. Maps c and g are represented as $c=\rho c'$ and $g=\rho g'$ respectively by Theorem 2.3 of [5]. A map d induces the map $\tilde{d}:(S^n)\rho\to(S^n)\rho$ and the maps c and g are determined by $c',\ g'\in H^{n-1}(HP^\infty;\ \hat{Z}/Z)$. We get the following equalities by respecting the above diagram:

$$xy=[c,\ d]^t[e,\ g]=\rho[c',\ \tilde{d}]^t[e,\ g']=\rho(c'e+\tilde{d}g')=ce^i+dg\,.$$

The other cases are easily proved by the fact $c=0$, $g=0$.

We describe the multiplication of matrices by using Lemma 3.1 and properties of maps $HP^\infty\to HP^\infty\times S^n\to HP^\infty$, $S^n\to HP^\infty\times S^n\to HP^\infty$ and $S^n\to HP^\infty\times S^n\to S^n$. Note that $HP^\infty\to S^n\to HP^\infty$ and $S^n\to HP^\infty\to S^n$ are 0-homotopic by [4].

Theorem 3.2. *For odd $n\neq 1$, 3 or $n=4i$ or $n=2$, the multiplication of matrices is given as follows,*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae & af+bh \\ ce^i+dg & dh \end{pmatrix}$$

where i is given for $n=4i+1$, and $c=g=0$ for the other cases.

Theorem 3.3. *For $n=4i+2>2$, the multiplication of endomorphisms of $HP^\infty \times S^n$ is given by the following formula:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae & af+bh \\ ce^i h+d^2 g & dh \end{pmatrix}$$

Proof. It is sufficient to determine the (2,1)-component. If $d=0$, we can factor $c=\rho' c' : HP^\infty \times S^n \to K(\hat{Z}/Z, 2n-2) \to S^n$. Since c' is an element of the cohomology group $H^{2n-2}(HP^\infty \times S^n ; \hat{Z}/Z)$, we get the result on cohomology level. If $h=0$, $d\neq 0$, then we have $c=0$ and lift $[0, d]: HP^\infty \times S^n \to S^n$ to $[0, \tilde{d}]$: $HP^\infty \times (S^n)\rho \to (S^n)\rho$ with $[0, d](Id \times \rho)=\rho[0, \tilde{d}]$ by the above remark. By factoring $h=\rho' h'$ where h' is an element of the cohomology group, we get the result on cohomology level. If $dh\neq 0$, then we have $c=g=0$ and we get the result easily.

From these results, we can easily get the next result.

Corollary 3.4. *For $n\neq 1$, 3, $4i+1$, the group* $\mathrm{Aut}(HP^\infty \times S^n)$ *and* $\mathrm{WI}(HP^\infty \times S^n)$ *are $Z/2Z$ and 0 respectively. For $n=4i+1>1$, there is a split exact sequence*

$$(*) \qquad 0 \longrightarrow \hat{Z}/Z \longrightarrow \mathrm{Aut}(HP^\infty \times S^n) \longrightarrow Z/2Z \longrightarrow 0,$$

where the generator of $Z/2Z$ acts as $-1: \hat{Z}/Z \to \hat{Z}/Z$. Moreover it holds $\mathrm{WI}(HP^\infty \times S^n)=\hat{Z}/Z$ *for $n=4i+1>1$.*

Now we determine $\mathrm{End}(HP^\infty \times S^n)$ and its monoid structure for $n=1$, 3. Let $x, y : HP^\infty \times S^n \to HP^\infty \times S^n$ be the maps represented by the following matrices defined in section 2 respectively,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} e & f \\ 0 & h \end{pmatrix}.$$

It is sufficient to determine the (1,2)-component. If $ae=0$, then the (1,2)-component of the composition xy is 0-homotopic. Because, we have $b=0$ or $f=0$ if $a=0$ or $e=0$ by Propositions 1.5 or 1.6. If $ae\neq 0$, then we consider the adjoint

$$(xy)^{\sim} : S^n \longrightarrow \mathrm{Map}(HP^\infty, HP^\infty \times S^n) \longrightarrow \mathrm{Map}(HP^\infty, HP^\infty \times S^n)$$

of xy which is

$$S^n \longrightarrow K(Z/2Z, 1) \times K(\hat{Z}/Z, 3) \times S^n \longrightarrow K(Z/2Z, 1) \times K(\hat{Z}/Z, 3) \times S^n$$

by Proposition 1.5. Considering

$$\tilde{x}=\mathrm{Map}(HP^\infty, x)(id)^\sim : S^n \longrightarrow \mathrm{Map}(HP^\infty, HP^\infty; 1)\times \mathrm{Map}(HP^\infty, S^n)$$
$$\longrightarrow \mathrm{Map}(HP^\infty, HP^\infty; a)\times \mathrm{Map}(HP^\infty, S^n),$$

we have

$$[a, b]: \mathrm{Map}(HP^\infty, HP^\infty; 1)\times \mathrm{Map}(HP^\infty, S^n) \longrightarrow \mathrm{Map}(HP^\infty, HP^\infty; a).$$

Hence we get

$$\mathrm{Map}(HP^\infty, x)=[a, be]: \mathrm{Map}(HP^\infty, HP^\infty; a)\times \mathrm{Map}(HP^\infty, S^n)$$
$$\longrightarrow \mathrm{Map}(HP^\infty, HP^\infty; ae).$$

The (1,2)-component of $(xy)^\sim$ is equal to

$$[a, be]^t[f, h]=af+beh: S^n \longrightarrow K(Z/2Z, 1)\times K(\hat{Z}/Z, 3).$$

Finally, we get the next result.

Theorem 3.5. *For $n=1, 3$, the monoid structure of* $\mathrm{End}(HP^\infty\times S^n)$ *is given by the following formula:*

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} e & f \\ 0 & h \end{pmatrix}=\begin{pmatrix} ae & af+beh \\ 0 & dh \end{pmatrix}$$

Corollary 3.6. *The groups* $\mathrm{Aut}(HP^\infty\times S^1)$ *and* $\mathrm{WI}(HP^\infty\times S^1)$ *are $Z/2Z\times Z/2Z$ and* 0 *respectively.* $\mathrm{WI}(HP^\infty\times S^3)$ *is equal to* 0 *and* $\mathrm{Aut}(HP^\infty\times S^3)$ *is given by the next group extension*

$$(**) \qquad 0 \longrightarrow \hat{Z}/Z \longrightarrow \mathrm{Aut}(HP^\infty\times S^n) \longrightarrow Z/2Z \longrightarrow 0.$$

Proof. Since $\mathrm{Aut}(HP^\infty\times S^1)$ is a group of order 4 by Theorem 2.2, it is $Z/2Z\times Z/2Z$ or $Z/4Z$. Since the 2-power x^2 of any matrix x is the identity, $\mathrm{Aut}(HP^\infty\times S^1)=Z/2Z\times Z/2Z$ and $\mathrm{WI}(HP^\infty\times S^n)=0$. For $n=3$, $\mathrm{Aut}(HP^\infty\times S^3)$ is given by the group extension (**) and $\mathrm{WI}(HP^\infty\times S^3)$ is equal to 0 by Theorem of [1] or Proposition 1.5.

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