

CONTACT RIEMANNIAN MANIFOLDS SATISFYING $R(\xi, X) \cdot R = 0$ AND $\xi \in (k, \mu)$ -NULLITY DISTRIBUTION

By

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Summary. This paper deals with a classification of the semisymmetric contact metric manifolds, and the contact manifolds satisfying $R(\xi, X) \cdot S = 0$, where S is the Ricci tensor, under the condition that the characteristic vector field ξ belongs to the (k, μ) -nullity distribution.

1. Introduction

It is well known that there exist contact Riemannian manifolds $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ for which the curvature tensor R in the direction of the characteristic vector field ξ satisfies $R_{XY}\xi = 0$, for any tangent vector fields X, Y of M^{2n+1} . The tangent sphere bundle of a flat Riemannian manifold, for example, admits such a structure [2]. Applying a D -homothetic deformation [13] on M^{2n+1} with $R_{XY}\xi = 0$, we find a new class of contact metric manifolds satisfying the relation

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY); \quad k, \mu \in R \quad (1.1)$$

where $2h$ is the Lie derivative of φ with respect to ξ . An interesting property of this class is that the type of (1.1) is invariant under a D -homothetic deformation. The purpose of this paper is the classification of the contact Riemannian manifolds satisfying either

$$\text{i) } R(\xi, X) \cdot R = 0 \quad \text{or} \quad \text{ii) } R(\xi, X) \cdot S = 0, \quad (1.2)$$

under the condition that the characteristic vector field ξ belongs to the (k, μ) -nullity distribution i.e. under the condition (1.1). This paper is organized as follows: In the second section we give some definitions and known results. In the third section we consider semisymmetric contact metric manifolds $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ with characteristic vector field belonging to the (k, μ) -

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nullity distribution (see section two for definitions). The main result we prove is included in Theorem 3.4 and it is an improvement of Perrone's recent result Theorem 3.1 [9] and Takahashi's result, Theorem 3 [11]. If S is the Ricci tensor of $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ then the condition (1.2i) implies the condition (1.2ii). So, it is meaningful to undertake the study of manifolds satisfying (1.2ii). The two conditions (1.2i) and (1.2ii) are equivalent if $\dim M=3$. In the fourth paragraph we consider a contact Riemannian manifold M^{2n+1} satisfying the conditions (1.1), (1.2ii) and $2n+1>3$ and prove that such a manifold is either locally isometric to the product $E^{n+1}(0) \times S^n(4)$, or to an Einstein-Sasakian, or to an η -Einstein and it is a generalization of Perrone's Theorem 4.1 [9].

2. Preliminaries and known results

Let M be a $(2n+1)$ -dimensional contact manifolds with contact form η i.e. $\eta \wedge (d\eta)^n \neq 0$. It is well known that a contact manifold admits a vector field ξ , called the *characteristic vector field*, such that $\eta(\xi)=1$ and $d\eta(\xi, X)=0$ for every $X \in \chi(M)$. Moreover, M admits a Riemannian metric g and a tensor field φ of type $(1, 1)$ such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad g(X, \xi) = \eta(X), \quad g(X, \varphi Y) = d\eta(X, Y) \quad (2.1)$$

We then say that (φ, ξ, η, g) is a *contact metric structure*. As a consequence of the relations (2.1) one has

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \varphi\xi = 0, \quad \eta\varphi = 0 \quad (2.2)$$

Denoting by \mathcal{L} and R Lie differentiation and the curvature tensor respectively, we define the operators l and h by

$$lX = R(X, \xi)\xi, \quad hX = \frac{1}{2}(\mathcal{L}_\xi \varphi)X \quad (2.3)$$

The $(1, 1)$ tensors h and l are self-adjoint and satisfy

$$h\xi = 0, \quad l\xi = 0, \quad \text{tr } h = \text{tr } h\varphi = 0, \quad h\varphi = -\varphi h \quad (2.4)$$

Since now h anticommutes with φ , if X is an eigenvector of h corresponding to the eigenvalue λ , then φX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. If ∇ is the Riemannian connection of g , then

$$\text{i) } \nabla_X \xi = -\varphi X - \varphi hX, \quad \text{ii) } \nabla_\xi \varphi = 0, \quad \text{iii) } \varphi l\varphi - l = 2(h^2 + \varphi^2) \quad (2.5)$$

A contact metric manifold for which ξ is a Killing vector field is called *K-contact manifold*. It is well known that a contact manifold is *K-contact* if and only if $h=0$. Moreover on a *K-contact* manifold it is valid $R(X, \xi)\xi = X - \eta(X)\xi$. A contact metric manifold is said to be a *Sasakian manifold* if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \quad (2.6)$$

in which case

$$i) \nabla_X \xi = -\varphi X, \quad ii) R(X, Y)\xi = \eta(Y)X - \eta(X)Y \quad (2.7)$$

Note that a Sasakian manifold is K -contact, but the converse holds only if $\dim M = 3$.

A contact manifold is said to be η -Einstein if

$$Q = aId + b\eta \otimes \xi \quad (2.8)$$

where Q is the Ricci operator and a, b are smooth functions on M^{2n+1} .

The sectional curvature $K(\xi, X)$ of a plane section spanned by ξ and a vector X orthogonal to ξ is called a ξ -sectional curvature while the sectional curvature $K(X, \varphi X)$ is called a φ -sectional curvature.

The (k, μ) -Nullity distribution of a contact metric manifold $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ for the pair $(k, \mu) \in \mathbb{R}^2$, is a distribution

$$N(k, \mu): p \rightarrow N_p(k, \mu) = \{Z \in T_p M \mid R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] \\ + \mu[g(Y, Z)hX - g(X, Z)hY]\}.$$

So, if the characteristic vector field ξ belongs to the (k, μ) -nullity distribution we have:

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \quad (2.9)$$

Finally if a Riemannian manifold (M, g) is locally symmetric, then its curvature tensor R satisfies

$$R(X, Y) \cdot R = 0 \quad (2.10)$$

for all tangent vectors X, Y where the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M . Any Riemannian manifold satisfying (2.10) is called *semi-symmetric* space.

In the next paragraphs we will use the following Theorem of Blair:

Theorem 2.1 [2]. Let $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ be a contact metric manifold with $R(X, Y)\xi = 0$ for all vector fields X, Y . Then M^{2n+1} is locally the product of a flat $(n+1)$ -dimensional manifold and an n -dimensional manifold of positive constant curvature equal to 4.

If S is the Ricci tensor of a Riemannian manifold (M, g) , then the condition (2.10) implies in particular

$$R(X, Y) \cdot S = 0, \quad (2.11)$$

for any tangent vectors X, Y where the endomorphism R acts on S as a

derivation. The Ricci operator Q is the symmetric endomorphism on the tangent space given by

$$g(QX, Y) = S(X, Y). \quad (2.12)$$

Another result ([5]) which we will use later is the following

Lemma 2.1. *In any contact metric manifold $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ with ξ belonging to the (k, μ) -nullity distribution the Ricci operator Q is given by*

$$QX = [2(n-1) - n\mu]X + [2(n-1) + \mu]hX + [2(1-n) + n(2k + \mu)]\eta(X)\xi, \quad n \geq 1 \quad (2.13)$$

for any $X \in \chi(M)$.

3. Semisymmetric contact Riemannian manifolds and (k, μ) -nullity distribution

Assume that $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ be a semisymmetric contact Riemannian manifold with ξ belonging to the (k, μ) -nullity distribution. Then (2.9) and (2.10) hold. Moreover, (2.9) implies that

$$R(\xi, X)Y = k(g(X, Y)\xi - \eta(Y)X) + \mu(g(hX, Y)\xi - \eta(Y)hX) \quad (3.1)$$

In the following we will use the next result of Takahashi [11].

Theorem 3.1. *A Sasakian manifold satisfying $R(X, Y) \cdot R = 0$ for all tangent vector X, Y is of constant curvature 1.*

On the other hand D. Blair, T. Koufogiorgos and the author treated the condition of (k, μ) -nullity distribution on a contact manifold and got the following theorem.

Theorem 3.2 [5]. *Let $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ be a contact manifold with ξ belonging to the (k, μ) -nullity distribution. If $k < 1$ then for any X orthogonal to ξ ;*

- 1) *The ξ -sectional curvature $K(X, \xi)$ is given by*

$$K(X, \xi) = k + \mu g(hX, X) = \begin{cases} k + \lambda\mu, & \text{if } X \in D(\lambda) \\ k - \lambda\mu, & \text{if } X \in D(-\lambda) \end{cases}$$

- 2) *The sectional curvature of a plane section $\{X, Y\}$ normal to ξ is given by*

$$K(X, Y) = \begin{cases} \text{i) } 2(1+\lambda) - \mu, & \text{if } X, Y \in D(\lambda), \\ \text{ii) } -(k+\mu)(g(X, \varphi Y))^2 & \text{for any unit vectors } X \in D(\lambda), Y \in D(-\lambda), \\ \text{iii) } 2(1-\lambda) - \mu, & \text{if } X, Y \in D(-\lambda), n > 1. \end{cases}$$

We state and prove now the following theorem.

Theorem 3.3. *Let $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ be a semisymmetric contact metric manifold with ξ belonging to the (k, μ) -nullity distribution. Then for any $X, Y \in \chi(M)$ we have:*

$$\begin{aligned} & \{kg(X, R_{XY}Y) + \mu g(hX, R_{XY}Y) - [kg(X, X) + \mu g(hX, X)][kg(Y, Y) \\ & + \mu g(hY, Y)] + [kg(X, Y) + \mu g(hX, Y)]^2\} \xi \\ & - \{k\eta(R_{XY}Y) + 2k\eta(Y)[kg(X, Y) + \mu g(hX, Y)]\} X \\ & - \{\mu\eta(R_{XY}Y) + 2\mu\eta(Y)[kg(X, Y) + \mu g(hX, Y)]\} hX \\ & + \{k\eta(Y)[kg(X, X) + \mu g(hX, X)] + k\eta(X)[kg(X, Y) + \mu g(hX, Y)]\} Y \\ & + \{\mu\eta(Y)[kg(X, X) + \mu g(hX, X)] + \mu\eta(X)[kg(X, Y) + \mu g(hX, Y)]\} hY \\ & + k\eta(X)R_{XY}Y + \mu\eta(X)R_{hX, Y}Y + \mu\eta(Y)R_{X, hX}Y \\ & + \mu\eta(Y)R_{XY}hX + k\eta(Y)R_{XY}X = 0 \end{aligned} \quad (3.2)$$

Proof. Since the manifold M is semisymmetric we will have $R(X, Y) \cdot R = 0$ or $(R(\xi, X) \cdot R)(X, Y)Y = 0$ for any $X, Y \in \chi(M)$ and ξ being the characteristic vector field. This last equation may also be written equivalently as:

$$\begin{aligned} & R(\xi, X)R(X, Y)Y - R(R_{\xi X}X, Y)Y - R(X, R_{\xi X}Y)Y \\ & - R(X, Y)R_{\xi X}Y = 0 \end{aligned} \quad (3.3)$$

Using equation (3.1) one easily gets:

$$\begin{aligned} & R(\xi, X)R(X, Y)Y = [kg(X, R_{XY}Y) + \mu g(hX, R_{XY}Y)] \xi \\ & - k\eta(R_{XY}Y)X - \mu\eta(R_{XY}Y)hX \end{aligned} \quad (1)$$

Using also, the equation (3.1) we get

$$\begin{aligned} & R(R_{\xi X}X, Y)Y = [kg(X, X) + \mu g(hX, X)][kg(Y, Y) + \mu g(hY, Y)] \xi \\ & - k\eta(Y)[kg(X, X) + \mu g(hX, X)]Y - \mu\eta(Y)[kg(X, X) \\ & + \mu g(hX, X)]hY - k\eta(X)R_{XY}Y - \mu\eta(X)R_{hX, Y}Y, \end{aligned} \quad (2)$$

$$\begin{aligned} & R(X, R_{\xi X}Y)Y = -[kg(X, Y) + \mu g(hX, Y)]^2 \xi + k\eta(Y)[kg(X, Y) \\ & + \mu g(hX, Y)]X + \mu\eta(Y)[kg(X, Y) + \mu g(hX, Y)]hX - \mu\eta(Y)R_{X, hX}Y \end{aligned} \quad (3)$$

and

$$\begin{aligned} & R(X, Y)R_{\xi X}Y = k\eta(Y)[kg(X, Y) + \mu g(hX, Y)]X + \mu\eta(Y)[kg(X, Y) \\ & + \mu g(hX, Y)]hX - k\eta(X)[kg(X, Y) + \mu g(hX, Y)]Y - \mu\eta(X)[kg(X, Y) \\ & + \mu g(hX, Y)]hY - k\eta(Y)R_{XY}X - \mu\eta(Y)R_{XY}hX \end{aligned} \quad (4)$$

If we now substitute the relations (1), (2), (3) and (4) to (3.3) we get the required result.

From the above theorem we have the following corollary.

Corollary 3.1. *For any unit vector fields $X, Y \in \mathcal{X}(M)$ such that $\eta(X) = \eta(Y) = 0$ and $g(X, Y) = 0$, where M is a semisymmetric contact manifold and ξ belongs to the (k, μ) -nullity distribution, we have:*

$$\begin{aligned} & \{[kg(X, R_{XY}Y) + \mu g(hX, R_{XY}Y)] - [k + \mu g(hX, X)][k + \mu g(hY, Y)] \\ & + \mu^2 g^2(hX, Y)\} \xi - kg(\xi, R_{XY}Y)X - \mu g(\xi, R_{XY}Y)hX = 0 \end{aligned} \quad (3.4)$$

Proof. This is an immediate consequence of (3.2).

Next, we state and prove the main result.

Theorem 3.4. *Let $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ be a semisymmetric contact metric manifold with ξ belonging to the (k, μ) -nullity distribution. Then*

- a) *If $\dim M > 3$, M^{2n+1} is either*
 - (i) *A Sasakian manifold of constant sectional curvature 1, or*
 - (ii) *Locally isometric to the product of a flat $(n+1)$ -dimensional Euclidean manifold and an n -dimensional manifold of constant curvature 4.*
- b) *If $\dim M = 3$, M^3 is either*
 - 1) *A Sasakian manifold of constant sectional curvature $+1$, or*
 - 2) *Locally isometric either to*
 - i) *A flat manifold ($\mu = 0, \lambda = 1$), or to*
 - ii) *$SU(2)$, ($\mu = 0, 0 \leq \lambda < 1$)*

Proof. a) (i). If $k=1$ then $\lambda=0$ and hence $h=0$. Therefore the manifold is K -contact. Moreover, the equation (2.9) is reduced to

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

and therefore, the semi-symmetric manifold M^{2n+1} is a Sasakian manifold. Applying now Theorem 3.1 [11], we conclude that M^{2n+1} is of constant curvature $+1$.

a) (ii). It is known, Tanno [5], that if $k < 1$ then M^{2n+1} admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$ defined by the eigenspaces of h , where $\lambda = \sqrt{1-k}$.

Suppose that X, Y are orthonormal vectors of the distribution $D(\lambda)$. Then from Theorem 3.2 we have

$$K(X, Y) = 2(1+\lambda) - \mu. \quad (1)$$

On the other hand, applying the equation (3.4) with $hX = \lambda X$, $hY = \lambda Y$ we get

$$[(k+\lambda\mu)K(X, Y) - (k+\lambda\mu)^2]\xi - (k+\lambda\mu)g(\xi, R_{XY}Y)X = 0 \quad (2)$$

Taking the inner products with ξ we get

$$\text{i) } K(X, Y) = k + \lambda\mu, \text{ or ii) } k = -\lambda\mu \quad (3)$$

Comparing the equations (1) and (3i) we get

$$\mu = 1 + \lambda \quad (4)$$

Suppose now that $X, Y \in D(-\lambda)$ and $g(X, Y) = 0$, then from Theorem 3.2 we have

$$K(X, Y) = 2(1 - \lambda) - \mu \quad (5)$$

On the other hand, applying the relation (3.4), for $hX = -\lambda X$, $hY = -\lambda Y$ we get

$$[(k - \lambda\mu)K(X, Y) - (k - \lambda\mu)^2]\xi - (k - \lambda\mu)g(\xi, R_{XY}Y)X = 0.$$

Taking the inner product with ξ we get

$$\text{i) } K(X, Y) = k - \lambda\mu, \text{ or ii) } k = \lambda\mu \quad (6)$$

Comparing now the relations (5) and (6i) we have

$$\text{i) } \mu = 1 - \lambda, \text{ or ii) } \lambda = 1 \quad (7)$$

Suppose now that $X \in D(\lambda)$, $Y \in D(-\lambda)$. Then using Theorem 3.2 we have

$$K(X, Y) = -(k + \mu)[g(X, \varphi Y)]^2 \quad (8)$$

On the other hand equation (3.4), for $hX = \lambda X$, $hY = -\lambda Y$, is reduced to

$$[(k + \lambda\mu)K(X, Y) - (k - \lambda\mu)(k + \lambda\mu)]\xi - (k + \lambda\mu)g(\xi, R_{XY}Y)X = 0$$

from which taking the inner products with ξ we have

$$\text{i) } K(X, Y) = k - \lambda\mu \text{ or } k = -\lambda\mu \quad (9)$$

while if $X \in D(-\lambda)$ and $Y \in D(\lambda)$ we similarly prove that

$$\text{ii) } K(X, Y) = k + \lambda\mu \text{ or } k = \lambda\mu \quad (10)$$

By the combination now of the equations (3ii), (4), (6ii), (7), (9) and (10) we establish the following five systems among the unknowns k , λ and μ , the remainder being inconsistent (give a contradiction).

1. $\{\mu = 1 + \lambda, \mu = 1 - \lambda, \lambda = 0\}$
2. $\{k = -\lambda\mu, \mu = 1 - \lambda, \mu = 0, \lambda > 0\}$
3. $\{k = -\lambda\mu, \lambda = 1, \mu = 0\}$
4. $\{k = \lambda\mu, k = -\lambda\mu, \mu = 0, \lambda > 0\}$
5. $\{\mu = 1 + \lambda, k = \lambda\mu, k \neq 0\}$

From the first system we get easily $\mu = 1$ and since $\lambda^2 = 1 - k$ we have $k = 1$, which is a contradiction, since we required that $k < 1$. The systems now 2, 3 and 4 have as the only solution $k = 0$, $\lambda = 1$, $\mu = 0$. Hence $R_{XY}\xi = 0$ for any X ,

$Y \in \mathcal{X}(M)$. Therefore, applying Theorem 2.1, we conclude that the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$. The last system gives $k=3/4$, $\lambda=1/2$, $\mu=3/2$ which are not acceptable since from (9) and (10) we get a contradiction from (8). Thus the proof of a) (ii) is also complete.

b) Suppose now that $\dim M=3$. Then from the combination of the equations (9) and (10) we get four systems with respect to the k, λ, μ and the sectional curvature $K(X, Y)$, from which we have the following possibilities:

1) $\lambda=0, \mu \neq 0$, which leads to M^3 being a Sasakian manifold of constant sectional curvature $+1$,

2) $k=\lambda\mu$ or $k=-\lambda\mu$ and $K(X, Y)=0$, and therefore M^3 is a flat manifold, and

3) $\lambda \geq 0, \mu=0, K(X, Y)=k$.

It is now, well known [5] that, if M is contact metric manifold for which ξ belongs to the (k, μ) -nullity distribution then

$$Q\varphi - \varphi Q = 2[2(n-1) + \mu]h\varphi \quad (3.5)$$

where Q is the Ricci operator. For $n=1, \mu=0$ we easily get $Q\varphi = \varphi Q$, therefore [4], M^3 has to be either

i) A flat manifold ($\mu=0, \lambda=1$), or

ii) $SL(2, \mathbf{R})$ ($\mu=0, \lambda>1$), or

iii) $SU(2)$ ($\mu=0, 0 \leq \lambda < 1$).

It remains now to be examined which of those manifolds are semisymmetric.

First, any flat manifold is locally symmetric and hence semisymmetric. Next, we will exhibit the contact metric structure on these Lie groups, such that (2.9) to be satisfied. Consider the general Lie algebra structure on these manifolds [7]:

$$[e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad [e_1, e_2] = c_3 e_3 \quad (3.6)$$

Let $\{\omega_i\}$ be the dual 1-forms of the vector fields $\{e_i\}$. Then using (3.6) we have $d\omega_1(e_2, e_3) = -d\omega_1(e_3, e_2) = -c_1/2 \neq 0$ and $d\omega_1(e_i, e_j) = 0$ for $(i, j) \neq (2, 3), (3, 2)$. It is easy to check that ω_1 is a contact form and e_1 is the characteristic vector field. Defining a Riemannian metric g by $g(e_i, e_j) = \delta_{ij}$, then φ has the same matrix as $d\omega_1$, with respect to the basis $\{e_i\}$, since $d\omega_1(e_i, e_j) = g(e_i, \varphi e_j)$. Moreover, in order g to be an associated metric we must have $\varphi^2 = -id + \omega_1 \otimes e_1$. So, $(\varphi, e_1, \omega_1, g)$ is a contact metric structure if we get $c_1=2$. The unique Riemannian connection ∇ corresponding to g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ & - g(Y, [X, Z]) + g(Z, [X, Y]) \end{aligned}$$

So, using (3.6) and $c_1=2$ we have

$$\begin{aligned}
\nabla_{e_1}e_1 &= \nabla_{e_2}e_2 = \nabla_{e_3}e_3 = 0, & \nabla_{e_2}e_1 &= \frac{1}{2}(c_2 - c_3 - 2)e_3, & \nabla_{e_1}e_2 &= \frac{1}{2}(c_2 + c_3 - 2)e_3, \\
\nabla_{e_3}e_2 &= \frac{1}{2}(c_3 - c_2 - 2)e_1, & \nabla_{e_2}e_3 &= \frac{1}{2}(c_3 - c_2 + 2)e_1, & \nabla_{e_3}e_1 &= \frac{1}{2}(2 + c_2 - c_3)e_2, \\
\nabla_{e_1}e_3 &= \frac{1}{2}(2 - c_2 - c_3)e_2
\end{aligned} \tag{3.7}$$

The $\{e_i\}$ are eigenvectors of h with corresponding eigenvalues $\{0, \lambda, -\lambda\}$ where λ is given by

$$\lambda = \frac{c_3 - c_2}{2} \tag{3.8}$$

Moreover, by direct computations,

$$\begin{aligned}
R(e_1, e_2)e_1 &= -\nu e_2, & R(e_1, e_2)e_2 &= \nu e_1, & R(e_1, e_3)e_1 &= -\tau e_3, \\
R(e_1, e_2)e_3 &= R(e_2, e_3)e_1 = R(e_3, e_1)e_2 = 0, \\
R(e_2, e_3)e_3 &= \rho e_2, & R(e_2, e_3)e_2 &= -\rho e_3, & R(e_1, e_3)e_3 &= \tau e_1,
\end{aligned} \tag{3.9}$$

where :

$$\begin{aligned}
\nu &= \frac{1}{4}[(2 - c_2 + c_3)(2 - c_2 - c_3) + 2c_3(2 + c_2 - c_3)] \\
\rho &= \frac{1}{4}[(2 + c_2 - c_3)(c_2 - c_3 - 2) + 4(c_2 + c_3 - 2)] \\
\tau &= \frac{1}{4}[(c_2 + c_3 - 2)(c_3 - c_2 - 2) + 2c_2(2 - c_2 + c_3)]
\end{aligned} \tag{3.10}$$

Using now the equations (3.9) we have :

$$\begin{aligned}
\text{i) } (R_{e_1e_2} \cdot R)(e_2, e_3)e_3 &= \nu(\rho - \tau)e_1, & \text{ii) } (R_{e_1e_2} \cdot R)(e_3, e_2)e_2 &= 0, \\
\text{iii) } (R_{e_2e_3} \cdot R)(e_3, e_1)e_1 &= \rho(\tau - \nu)e_2, & \text{iv) } (R_{e_2e_3} \cdot R)(e_1, e_3)e_3 &= 0, \\
\text{v) } (R_{e_3e_1} \cdot R)(e_1, e_2)e_2 &= \tau(\nu - \rho)e_3, & \text{vi) } (R_{e_3e_1} \cdot R)(e_2, e_1)e_1 &= 0,
\end{aligned} \tag{3.11}$$

where we use both the notations $R(X, Y)$ and R_{XY} . In fact, we prove one of those, say the first :

$$\begin{aligned}
&(R_{e_1e_2} \cdot R)(e_2, e_3)e_3 \\
&= R_{e_1e_2}R(e_2, e_3)e_3 - R(R_{e_1e_2}e_2, e_3)e_3 - R(e_2, R_{e_1e_2}e_3)e_3 - R(e_2, e_3)R_{e_1e_2}e_3 \\
&= R_{e_1e_2}\rho e_2 - R(\nu e_1, e_3)e_3 - R(e_3, 0)e_3 \\
&= \rho \nu e_1 - \nu \cdot \tau e_1 = \nu(\rho - \tau)e_1.
\end{aligned}$$

The proofs of the remainder are similar, so we omit these. Therefore, in order for the manifold to be semisymmetric, it is necessary and sufficient

$$\nu(\rho-\tau)=0, \quad \rho(\tau-\nu)=0, \quad \tau(\nu-\rho)=0 \quad (3.12)$$

This system of equations is equivalent to the following five systems:

1. $\{\nu=0, \rho=0, \tau=0\}$, 2. $\{\nu=\rho=0\}$, 3. $\{\tau=\nu=0\}$,
4. $\{\tau=\rho=0\}$, 5. $\{\tau=\nu=\rho\}$.

This system now is equivalent to the following systems

$$\text{i) } \nu=\rho=\tau=0, \quad \text{ii) } \tau=\nu=\rho,$$

since each of the systems 2, 3 and 4 has as solution set either $\{c_2=2, c_3=0\}$ or $\{c_3=2, c_2=0\}$ and therefore the third unknown from (3.10) equals zero as well. Hence, the solutions of the system (i) are $\{c_2=2, c_3=0\}$ or $\{c_2=0, c_3=2\}$. The second equation (ii), is equivalent to the following equations

$$\{(c_2-2)(c_2-c_3+2)=0, (c_2-c_3)(c_2+c_3-2)=0, (c_3-2)(c_3-c_2+2)=0\}$$

It is now easy to find out that the solutions of this system are $c_2=c_3=2$ or $c_2=0, c_3=2$ or $c_2=2, c_3=0$. Therefore, the solution sets of the system (3.12) are

$$\text{i) } c_2=c_3=2>0, \quad \text{ii) } c_2=0, c_3=2, \quad \text{iii) } c_2=2, c_3=0.$$

If the first case holds then according to [4] we conclude that M^3 is locally isometric to the three sphere $SU(2)$. If the second case holds then according to the classification Theorem 3 of [5], M^3 must be the group $E(2)$ of the right motions of the Euclidean 2-space. But this case must be excluded since it requires $\mu \neq 0$, while we have $\mu=0$. If the third case occurs then, according to the equation (3.8) we get $\lambda=-1$ which is impossible, since $\lambda \geq 0$, and the proof of the Theorem is complete.

4. Contact Riemannian manifolds with $R(\xi, X) \cdot S=0$ and $\xi \in (k, \mu)$ -nullity distribution

Let $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ be a contact Riemannian manifold of dimension $2n+1 > 3$. Tanno [13] proved that if $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ is an Einstein manifold and ξ belongs to the k -nullity distribution, then M is a Sasakian manifold. Perrone [9] generalizing this result proved that if M is a contact Riemannian manifold with $R(X, \xi) \cdot S=0$, where S is the Ricci tensor, and ξ belongs to the k -nullity distribution, where k is some function on M , then M is either an Einstein-Sasakian manifold or the product $E^{n+1}(0) \times S^n(4)$. In this section we extend the latest Perrone's result substituting the k -nullity distribution with the (k, μ) -nullity distribution. More precisely we have the following theorem.

Theorem 4.1. *Let $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ be a contact Riemannian manifold*

such that

- i) $R(\xi, X) \cdot S = 0$, where S is the Ricci tensor, and
- ii) $R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$, $\forall (k, \mu) \in \mathbb{R}^2$.

Then the manifold is either

- (i) locally isometric to $E^{n+1}(0) \times S^n(4)$, or
- (ii) an Einstein-Sasakian manifold, or
- (iii) an η -Einstein manifold if $k^2 + \mu^2(k-1) \neq 0$.

Proof. (i) If $k=0, \mu=0$ then we have $R_{XY}\xi=0$ for any tangent vector fields X, Y and hence [2], the manifold is locally isometric to $E^{n+1}(0) \times S^n(4)$.

(ii) Let $k \neq 0$, then from the first hypothesis we have:

$$0 = (R(\xi, X) \cdot S)(Y, Z) = R(\xi, X)S(Y, Z) - S(R(\xi, X)Y, Z) - S(Y, R(\xi, X)Z),$$

from which

$$S(R(\xi, X)Y, Z) = -S(Y, R(\xi, X)Z), \quad \forall X, Y, Z \in \chi(M) \quad (4.1)$$

From this equation, setting $Z = \xi$ we get

$$S(R(\xi, X)Y, \xi) = -S(Y, R(\xi, X)\xi) \quad (4.2)$$

Using now equation (2.13) we get

$$\begin{aligned} S(X, Y) = g(QX, Y) &= [2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y) \\ &\quad + [2(1-n) + n(2k + \mu)]\eta(X)\eta(Y) \end{aligned} \quad (4.3)$$

from which

$$S(X, \xi) = 2nk\eta(X), \quad \forall X \in \chi(M) \quad (4.4)$$

Equation now (4.2) by means of (4.4) and $lX = R(X, \xi)\xi$, gives

$$S(lX, Y) = 2nk g(lX, Y), \quad \forall X, Y \in \chi(M). \quad (4.5)$$

But using the second hypothesis

$$lX = R(X, \xi)\xi = k(X - \eta(X)\xi) + \mu hX \quad (4.6)$$

and equation (4.5) is reduced to

$$\mu S(hX, Y) + k S(X, Y) = 2nk^2 g(X, Y) + 2nk\mu g(hX, Y) \quad (4.7)$$

If $\mu=0$ then since $k \neq 0$, we get that the manifold is Einstein and using the Theorem 5.2 of [13], we conclude that M is a Sasakian manifold.

(iii) Suppose now that $k \neq 0, \mu \neq 0$. Then, using the equation (4.3) and $h^2 = (k-1)\varphi^2$, $k \leq 1$ [5], we have

$$\begin{aligned} S(hX, Y) &= [2(n-1) - n\mu]g(hX, Y) - (k-1)[2(n-1) + \mu]g(X, Y) \\ &\quad + (k-1)[2(n-1) + \mu]\eta(X)\eta(Y) \end{aligned} \quad (4.8)$$

Therefore, equation (4.7) by means of this expression gives:

$$\begin{aligned} & \{2nk\mu - k[2(n-1) + \mu] - \mu[2(n-1) - n\mu]\} g(hX, Y) \\ &= \{k[2(n-1) - n\mu] - \mu(k-1)[2(n-1) + \mu] - 2nk^2\} g(X, Y) \\ &+ \{k[2(1-n) + n(2k + \mu)] + \mu(k-1)[2(n-1) + \mu]\} \eta(X)\eta(Y) \end{aligned} \quad (4.9)$$

From this equation now we can get $g(hX, Y)$ in terms of $g(X, Y)$ and $\eta(X)\eta(Y)$, since $k^2 + \mu^2(k-1) \neq 0$. In fact, we have

$$g(hX, Y) = Ag(X, Y) + B(\eta X)\eta(Y) \quad (4.10)$$

where

$$\begin{aligned} A &= \frac{k[2(n-1) - n\mu] - \mu(k-1)[2(n-1) + \mu] - 2nk^2}{2nk\mu - k[2(n-1) + \mu] - \mu[2(n-1) - n\mu]}, \\ B &= \frac{k[2(1-n) + n(2k + \mu)] + \mu(k-1)[2(n-1) + \mu]}{2nk\mu - k[2(n-1) + \mu] - \mu[2(n-1) - n\mu]} \end{aligned} \quad (4.11)$$

So, by the equation (4.3) by means of (4.10) and (4.11) takes the form

$$\begin{aligned} S(X, Y) &= \{2(n-1) - \mu n + A[2(n-1) + \mu]\} g(X, Y) \\ &+ \{B[2(1-n) + \mu] + [2(1-n) + n(2k + \mu)]\} \eta(X)\eta(Y) \end{aligned} \quad (4.12)$$

from which, we conclude easily that the Ricci operator Q is of the form (2.8) and therefore $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ is an η -Einstein manifold and the proof is complete.

If the Ricci tensor S is parallel, then the condition $R(\xi, X)S=0$ is satisfied. Hence, we have the following corollary of the above theorem:

Corollary 4.1. *If $[M^{2n+1}, (\varphi, \xi, \eta, g)]$ is a contact Riemannian manifold, the Ricci tensor of which is parallel and ξ belongs to the (k, μ) -nullity distribution, then M is either*

- (i) *locally isometric to the product $E^{n+1}(0) \times S^n(4)$, or*
- (ii) *an Einstein-Sasakian manifold, or*
- (iii) *an η -Einstein manifold.*

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