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AN ITERATED LOGARITHM THEOREM FOR SOME STATIONARY LINEAR PROCESSES

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Abstract. In this paper a law of the iterated logarithm is obtained for partial sums of a stationary linear process generated by martingale differences.

1. Introduction and result

Let $(\Omega, \mathcal{F}, P)$ be a probability space with an ergodic one-to-one bimeasurable measure preserving transformation T. Let $\mathcal{F}_0$ be a sub-σ-field of $\mathcal{F}$ such that $\mathcal{F}_0 \subset T^{-1}\mathcal{F}_0$, and write $\mathcal{F}_k = T^{-k}\mathcal{F}_0$. Let H_k denote the Hilbert subspace of $L_2(\Omega, \mathcal{F}, P)$ comprising those functions measurable with respect to $\mathcal{F}_k$, and let $G_k = H_k \ominus H_{k-1}$. Let $\varepsilon_0 \in G_0$, then for each integer k, $\varepsilon_k = \varepsilon_0 \circ T^k \in G_k$, and hence $E(\varepsilon_k | \mathcal{F}_{k-1}) = 0$ a.s.. Thus $\{\varepsilon_k, \mathcal{F}_k, -\infty < k < \infty\}$ forms a (strictly) stationary ergodic martingale difference sequence with finite variance $\sigma^2 = E\varepsilon_0^2$.

Define a stationary ergodic linear process $\{X_k, -\infty < k < \infty\}$ by

$$X_k = \sum_{j=-\infty}^{\infty} \alpha_j \varepsilon_{k-j}, \qquad \sum_{j=-\infty}^{\infty} \alpha_j^2 < \infty .$$

Then its spectral density is

$$f(\lambda) = (2\pi)^{-1}\sigma^2 \left| \sum_{j=-\infty}^{\infty} \alpha_j e^{i\lambda j} \right|^2, \qquad -\pi \leqq \lambda \leqq \pi .$$

In this paper, we will only be concerned with large values of t whenever $\log t$ and $\log\log t$ are involved. Hence, in order to avoid cumbersome expressions, we adopt the convention that $\log t = 1$ for $0 < t \leqq e$ and $\log\log t = 1$ for $0 < t \leqq e^e$.

It is our object here to give the following iterated logarithm result.

Theorem. *Suppose that $E|\varepsilon_0|^{2+\delta} < \infty$ for some $\delta > 0$, and that there exists a constant K such that for all $n \geqq 1$,*

(c-1) $$\sum_{|j| \geq n} |\alpha_j| \leqq K(\log n)^{-(2+\delta)/\delta} .$$

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Then

$$\limsup_{n\to\infty} \sum_{i=1}^{n} X_i/(2n \log\log n)^{1/2} = (2\pi f(0))^{1/2} \qquad a.s.$$

2. Proof

For simplicity of notation put $r=2+\delta$, and also put $\theta=(2+\delta)/\delta$. Note first that the condition (c-1) contains the condition

(c-2) $$\sum_{|j|<n} |j|\,|\alpha_j| \leqq K_1 n(\log n)^{-\theta}$$

for all $n\geqq 1$ and some constant K_1. To see this, let $g(x)=K\theta x^{-1}(\log x)^{-(\theta+1)}$ for $x\geqq e$. For $e\leqq n$,

$$\int_n^\infty g(x)dx = K(\log n)^{-\theta}.$$

For $x\geqq e^{\theta+1}$,

$$g(x)\leqq K\theta x^{-1}\{(\log x)^{-\theta}-\theta(\log x)^{-(\theta+1)}\},$$

and hence for $e^{\theta+1}\leqq n_0<n$,

$$\int_{n_0}^{n} xg(x)dx \leqq K\theta n(\log n)^{-\theta}.$$

(c-2) then follows, since $|\alpha_j|=O(g(|j|))$.

We next prove the following lemma.

Lemma. *Under the condition* (c-1), $f(\lambda)$ *is continuous at* $\lambda=0$ *with*

$$\max_{|\lambda|\leq\lambda_0} |f(\lambda)-f(0)| \leqq B(\log \lambda_0^{-1})^{-\theta}$$

for all $0<\lambda_0\leqq\pi$ *and some constant* B.

Proof of Lemma. Let $j_\lambda=[|\lambda|^{-1}]+1$. We have for $0<|\lambda|\leqq\lambda_0$,

$$\begin{aligned}
&2\pi\sigma^{-2}|f(\lambda)-f(0)| \\
&\quad=\Big|\,|\textstyle\sum_{j=-\infty}^{\infty}\alpha_j e^{i\lambda j}|^2-(\sum_{j=-\infty}^{\infty}\alpha_j)^2\Big| \\
&\quad=|(\textstyle\sum_{j=-\infty}^{\infty}\alpha_j\cos\lambda j)^2+(\sum_{j=-\infty}^{\infty}\alpha_j\sin\lambda j)^2-(\sum_{j=-\infty}^{\infty}\alpha_j)^2| \\
&\quad\leqq|(\textstyle\sum_{j=-\infty}^{\infty}\alpha_j(1+\cos\lambda j))(\sum_{j=-\infty}^{\infty}\alpha_j(1-\cos\lambda j))|+(\sum_{j=-\infty}^{\infty}\alpha_j\sin\lambda j)^2 \\
&\quad\leqq 2\textstyle\sum_{j=-\infty}^{\infty}|\alpha_j|(\sum_{j=\infty}^{\infty}|\alpha_j|(1-\cos\lambda j)+\sum_{j=-\infty}^{\infty}|\alpha_j|\,|\sin\lambda j|) \\
&\quad\leqq 2\textstyle\sum_{j=-\infty}^{\infty}|\alpha_j|(2|\lambda|\sum_{|j|<j_\lambda}|j|\,|\alpha_j|+3\sum_{|j|\geq j_\lambda}|\alpha_j|) \\
&\quad\leqq B_1(|\lambda|j_\lambda+1)(\log j_\lambda)^{-\theta} \qquad \text{(by (c-1) and (c-2))} \\
&\quad\leqq 6B_1(\log|\lambda|^{-1})^{-\theta}\leqq 6B_1(\log\lambda_0^{-1})^{-\theta},
\end{aligned}$$

where B_1 is a constant not depending on λ. Here we used the elementary facts that $|\sin x|\leqq 1$, $1\pm\cos x\leqq 2$, $|\sin x|\leqq|x|$ and $1-\cos x\leqq|x|$.

Proof of Theorem. Let $Y_k=(\sum_{j=-\infty}^{\infty}\alpha_j)\varepsilon_k$, and let $T_n=\sum_{i=1}^{n}Y_i$ for $n\geqq 1$. Then, since $\{Y_k\}$ is a stationary ergodic martingale difference sequence, it follows from a law of the iterated logarithm for martingales due to Stout [5] that

$$\limsup_{n\to\infty} T_n/(2n\log\log n)^{1/2}=(2\pi f(0))^{1/2} \quad \text{a.s.}$$

Thus it suffices to show that

$$\lim_{n\to\infty} U_n/(n\log\log n)^{1/2}=0 \quad \text{a.s.},$$

where $U_n=S_n-T_n$ and $S_n=\sum_{i=1}^{n}X_i$.

It is known that if $f(\lambda)$ is continuous at $\lambda=0$, then $n^{-1}EU_n^2\to 0$ as $n\to\infty$ (see [1] pp. 134-135). We shall show that the condition (c-1) gives adequate bounds on EU_n^2. Since $ET_n^2=2\pi f(0)n$,

$$\begin{aligned} EU_n^2&=E(S_n-T_n)^2=ES_n^2-ES_nT_n+ET_n^2\\ &\leqq|ES_n^2-2\pi f(0)n|+2|ES_nT_n-2\pi f(0)n|. \end{aligned} \tag{1}$$

Applying the lemma to the proof of Theorem 18.2.1 of [3], we obtain that

$$\begin{aligned} |ES_n^2-2\pi f(0)n|&\leqq 2\pi n\max_{|\lambda|\leq n^{-1/4}}|f(\lambda)-f(0)|+O(n^{1/2})\\ &\leqq C_1 n(\log n)^{-\theta} \end{aligned} \tag{2}$$

for all $n\geqq 1$ and some constant C_1. Next notice that

$$\begin{aligned} ES_nT_n&=\textstyle\sum_{i=1}^{n}\sum_{j=1}^{n}E(X_iY_j)=\sum_{|j|<n}(n-|j|)E(Y_0X_j)\\ &=(\textstyle\sum_{j=-\infty}^{\infty}\alpha_j)\sigma^2\sum_{|j|<n}(n-|j|)\alpha_j\\ &=2\pi f(0)n-\sigma^2(\textstyle\sum_{j=-\infty}^{\infty}\alpha_j)(n\sum_{|j|\geq n}\alpha_j+\sum_{|j|<n}|j|\alpha_j). \end{aligned}$$

Hence, under the condition (c-1) (and hence under the condition (c-2)), there is a constant C_2 for which

$$\begin{aligned} |ES_nT_n-2\pi f(0)n|&\leqq\sigma^2|\textstyle\sum_{j=-\infty}^{\infty}\alpha_j|(n\sum_{|j|\geq n}|\alpha_j|+\sum_{|j|<n}|j|\,|\alpha_j|)\\ &\leqq C_2 n(\log n)^{-\theta} \end{aligned} \tag{3}$$

for all $n\geqq 1$. Combining the estimates (1), (2) and (3), we get

$$EU_n^2\leqq C_3 n(\log n)^{-\theta} \tag{4}$$

for all $n\geqq 1$, where $C_3=C_1+2C_2$.

Now we set for $n\geqq 1$,

$$U_n=\textstyle\sum_{i=1}^{n}Z_i=\sum_{j=-\infty}^{\infty}a_{nj}\varepsilon_j,$$

where $Z_i = X_i - Y_i$, $a_{nj} = \sum_{i=1-j}^{n-j} \alpha_i - \sum_{i=-\infty}^{\infty} \alpha_i$ if $1 \leqq j \leqq n$ and $a_{nj} = \sum_{i=1-j}^{n-j} \alpha_i$ otherwise. Then it follows from (4) that

$$\sum_{j=-\infty}^{\infty} a_{nj}^2 = \sigma^{-2} E U_n^2 \leqq C_3 \sigma^{-2} n (\log n)^{-\theta} \tag{5}$$

for all $n \geqq 1$. By Burkholder's and Hölder's inequalities, there exists a constant C_4 depending only on r such that

$$\begin{aligned} E\left|\sum_{j=p}^{q} a_{nj}\varepsilon_j\right|^r &\leqq C_4 E\left(\sum_{j=p}^{q} a_{nj}^2 \varepsilon_j^2\right)^{r/2} \\ &= C_4 E\left(\sum_{j=p}^{q} |a_{nj}|^{(2r-4)/r} |a_{nj}|^{4/r} \varepsilon_j^2\right)^{r/2} \\ &\leqq C_4 E\left\{\left(\sum_{j=p}^{q} a_{nj}^2\right)^{r/2-1} \sum_{j=p}^{q} a_{nj}^2 |\varepsilon_j|^r\right\} \\ &\leqq C_4 \left(\sum_{j=-\infty}^{\infty} a_{nj}^2\right)^{r/2} E|\varepsilon_0|^r . \end{aligned}$$

Hence by the Fatou lemma and (5), there is a constant C_5 such that

$$E|U_n|^r \leqq C_5 n^{r/2} (\log n)^{-r\theta/2} \tag{6}$$

for all $n \geqq 1$. Using this together with Markov's inequality we have for $\varepsilon > 0$,

$$P(|U_n| > \varepsilon n^{1/2}) \leqq C_5 \varepsilon^{-r} (\log n)^{-r\theta/2} \tag{7}$$

for all $n \geqq 1$.

Let $a = 1/\theta$, and let $n_k = [\exp k^a]$ for $k \geqq 1$. Then $\sum_{k=1}^{\infty} (\log n_k)^{-r\theta/2} < \infty$, since $ar\theta/2 = r/2 > 1$, and hence by (7),

$$\sum_{k=1}^{\infty} P(|U_{n_k}| > \varepsilon n_k^{1/2}) < \infty$$

for each $\varepsilon > 0$. Hence by the Borel-Cantelli lemma,

$$n_k^{-1/2} U_{n_k} \longrightarrow 0 \quad \text{a.s.} \qquad \text{as} \quad k \to \infty .$$

Let

$$M_k = \max_{n_k < n \leqq n_{k+1}} |U_n - U_{n_k}| / (n_k \log\log n_k)^{1/2}, \qquad k \geqq 1 .$$

For each $k \geqq 1$,

$$\begin{aligned} |U_n| / (n \log\log n)^{1/2} &\leqq |U_{n_k}| / (n_n \log\log n_k)^{1/2} + M_k \\ &\leqq n_k^{-1/2} |U_{n_k}| + M_k \end{aligned}$$

for all $n_k < n \leqq n_{k+1}$. Thus it suffices to show that $M_k \to 0$ a.s. as $k \to \infty$ to complete the proof.

Now, the sequence $\{Z_k\}$ is stationary. Hence from (6),

$$E\left|\sum_{i=b+1}^{b+n} Z_i\right|^r \leqq C_5 n^{r/2}$$

holds for all integers $b \geqq 0$ and $n \geqq 1$. By Serfling's maximal identity ([4], Theorem B), there exists a constant C_6 such that

$$E(\max_{1 \leqq m \leqq n} |U_n|^r) \leqq C_6 n^{r/2}$$

for all $n \geqq 1$. Noting that $(n_{k+1}-n_k)/n_k \leqq (a+o(1))k^{-(1-a)}$, we have

$$\begin{aligned} EM_k^r &= E(\max_{1 \leq m \leq n_{k+1}-n_k} |U_m|^r)/(n_k \log \log n_k)^{r/2}, \\ &\leqq C_6(n_{k+1}-n_k)^{r/2}/(n_k \log \log n_k)^{r/2} \\ &\leqq C_7 k^{-(1-a)r/2}(\log k)^{-r/2} \\ &= C_7 k^{-1}(\log k)^{-r/2} \end{aligned}$$

for all $k \geqq 1$ and some constant C_7. $\sum_{k=1}^{\infty} EM_k^r < \infty$, since $r/2 > 1$, and hence $M_k \to 0$ a.s. as desired.

3. Remarks

(i) The conclusion of the theorem continues to hold in the absence of $E|\varepsilon_0|^{2+\delta} < \infty$ for $\delta > 0$ if

$$\text{(c-3)} \qquad \sum_{n=1}^{\infty} \{(\sum_{j=n}^{\infty} \alpha_j)^2 + (\sum_{j=n}^{\infty} \alpha_{-j})^2\} < \infty,$$

see [2]. The condition (c-1) does not contain the condition (c-3). For example, $\alpha_{|j|} = (-1)^j j^{-1}$ for $j \geqq 1$. Nevertheless, the condition (c-1) covers a wide class of coefficients α's being out of the condition (c-3). Examples are $\alpha_{|j|} \sim Cj^{-\lambda}$ as $j \to \infty$ for some $1 < \lambda \leqq 3/2$, and $\alpha_{|j|} \sim Cj^{-1}(\log j)^{-\lambda}$ as $j \to \infty$ for some $\lambda \geqq 2(1+\delta)/\delta$.

(ii) Both of the results in this paper and of Heyde stated above remain valid even when T is not ergodic. In this case the α_j and $f(\lambda)$ can be defined as $\mathcal{J}$-measurable random variables, where $\mathcal{J} = \{A \in \mathcal{F} \mid T^{-1}A = A\}$. The proofs are based on the decomposition of non-ergodic invariant measures into ergodic components, see [6] and references therein.

References

[1] P. Hall and C.C. Heyde, *Martingale Limit Theory and Its Application*, Academic Press, New York, 1980.

[2] C.C. Heye, On the central limit theorem and iterated logarithm law for stationary processes, *Bull. Austral. Math. Soc.*, 12 (1975), 1-8.

[3] I.A. Ibragimov and Yu.V. Linnik, *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff, Groningen, 1971.

[4] R.J. Serfling, Moment inequalities for the maximum cumulative sum, *Ann. Math. Statist.*, 41 (1970), 1227-1234.

[5] W.F. Stout, The Hartman Wintner law of the iterated logarithm for martingales, *Ann. Math. Statist.*, 41 (1970), 2158-2160.

[6] R. Yokoyama and D. Volný, On the central limit theorem and law of iterated logarithm for stationary linear processes, in: P. Mandl and M. Hušková, eds., *Proc. of the Fourth Prague Symp. on Asymptotic Statistics*, Charles Univ., Prague, 1989, pp. 535-538.

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