YOKOHAMA MATHEMATICAL
JOURNAL VOL. 40, 1993

SYMPLECTIC TRIPLE SYSTEMS AND QUATERNION-KÄHLER MANIFOLDS

Dedicated to Professor Hisao Nakagawa on his sixtieth birthday

By

OSAMI YASUKURA*

(Received May 18, 1992)

Abstract. A quaternionic version of Schur's theorem on space forms is proved as a corollary of the construction of semi-symplectic triple system on each tangent space, which is well-defined because the Riemannian curvature tensor is invariant by the action of $\boldsymbol{Sp}(1)$.

Introduction

Besse [4] gave a survey on a quaternion-Kähler manifold in an elementary way as an example of Einstein manifolds. This paper was made under the influence of this book, in which Salamon and Bérard Bergery's twistor construction was reviewed by an elementary way. On the other hand, Ishihara [7] gave an elementary proof of Berger's result: A locally quaternion-Kähler manifold is Einstein. A quaternionic symmetric space M of rank one which is isometric to a quaternionic projective space $\boldsymbol{HP}_n$, a quaternionic hyperbolic space $\boldsymbol{HH}_n$ or a quaternionic Euclidean space $\boldsymbol{H}^n$ ($n \geqq 1$) with the standard metric (abbrev. a *quaternionic space form*) has constant Q-sectional curvature $\rho(p)$, i.e. each Q-section $\boldsymbol{H}(X)$ has a Q-sectional curvature $\rho(X)$ and $\rho(X)$ is constant $\rho(p)$ for all tangent vectors $X \in T_pM$ at $p \in M$ (See Definition 3.7). Conversely, Ishihara [7] proved, by tensor calculus, that a locally quaternion-Kähler manifold M with non-zero scalar curvature having constant Q-sectional curvature $\rho(p)$ at all $p \in M$ is locally isometric to a quaternionic space form. But the condition that $\rho(X)$ is constant $\rho(p)$ for all tangent vectors X at $p \in M$ is not necessary. It is sufficient to assume that *M has Q-sectional curvatures $\rho(X)$ at all $p \in M$.* Since the Riemannian curvature tensor is invariant by the action of $\boldsymbol{Sp}(1)$ (cf. Lemma 2.1), it is equivalent to Alekseevskii [1]'s condition: The sectional curvature

---

* Partially supported by the Grant-in-Aid for Encouragement of Young Scientists (No. 03740054), Ministry of Education, Science and Culture.

1991 Mathematics Subject Classification: Primary 53C35.

Key words and phrases: symplectic triple systems, quaternion-Kähler, twistor space.

$K(X, AX)$ does not depend on the choice of $A \in \boldsymbol{Sp}(1)$ for each $X \in T_pM$ at all $p \in M$. In this paper, the following theorem is proved, which is considered as a generalization of Schur's theorem for $\boldsymbol{HP}_1 = S^4$, $\boldsymbol{HH}_1 = H^4$, $\boldsymbol{H}^1 = \boldsymbol{R}^4$ to quaternionic space forms.

Main Theorem. *A locally quaternion-Kähler manifold M of dimension $4n \geqq 4$ having Q-sectional curvatures $\rho(X)$ at all points has the Riemannian universal covering space which is isometrically immersed into a quaternionic space form of $4n$-dimension. In particular, M has a constant Q-sectional curvature ρ, i.e.*

$$\rho(X) = \rho(p) = \rho, \text{ a constant on the tangent bundle } TM.$$

In the case of $\rho(X) \neq 0$, M is a quaternion-Kähler manifold. If moreover M is camplete and $\rho(X) > 0$, then M is isometric to $\boldsymbol{HP}_n$.

The first half part of the above theorem is known by Alekseevkii [1] describing the decomposition of the space of curvature-like tensors by the restricted holonomy group $Sp(n)\boldsymbol{Sp}(1)$ only with a sketched proof (cf. Besse [4; p. 405, *l*. 9]). This paper gives a direct proof to the above theorem by constructing a semi-symplectic triple system with respect to the Riemannian curvature tensor on each tangent space T_pM, which is a generalization of Yamaguti-Asano's construction of complex simple Lie algebras of rank $\geqq 2$ or Freudenthal's construction of E_8 (cf. [2], [3], [14], [17]). It is appeared that the condition of having Q-sectional curvatures $\rho(X)$ is equivalent to the vanishing condition of the q-polynomial on the corresponding semi-symplectic triple systems (cf. Lemma 3.8). This paper also reviews some basic results on the geometry of locally quaternion-Kähler manifolds and locally symmetric spaces.

In § 1, we give the definition of a locally quaternion-Kähler manifold and examine its linear holonomy group. It is observed that an irreducible locally quaternion-Kähler manifold M of dimension $\geqq 8$ is also quaternion-Kähler, if it is not locally isometric to a Grassmann manifold $SO_{n+4}/S(O_n \times O_4)$ or its noncompact dual (cf. Theorem 1.4). In particular, such M is orientable. Then a Berger's theorem [4; Theorem 14.43] on the characterization on a quaternionic projective space by sectional curvatures is relaxed on the assumption of orientability. In § 2, a semi-symplectic triple system on each tangent space of a locally quaternion-Kähler manifold is constructed by the Riemannian curvature tensor and the quaternionic structure. It is well-defined, because the Riemannian curvature tensor is invariant by the action of $\boldsymbol{Sp}(1)$ (cf. Lemma 2.1), which is proved by means of Salamon and Bérard Bergery's pseudo-Kählerian twistor construction [10, 4] and O'Neill's formula [9] on a semi-Riemannian submersion. In § 3, it is observed (cf. Theorem 3.2) that a de Rham irreducible locally quaternion-Kähler manifold is locally-symmetric if and only if the corresponding semi-symplectic triple systems at all points are symplectic triple systems, by means of a theorem of Szabō [12] on Riemannian semi-symmetric spaces and Cartan-Ambrose-Hicks theorem (cf. Lemma 3.3). As a corollary, the above theorem is reduced to a theorem of Asano [17; Theorem 1.6] on the characterization of a symplectic triple system of type C_{n+1}.

The author is grateful to Prof. H. Asano, Prof. T. Takahashi, Prof. M. Itoh, Prof. H. Tasaki and Prof. M. Takeuchi for valuable communications.

1. Locally quaternion-Kähler manifolds

Let $\boldsymbol{H}:=\boldsymbol{R}1\oplus\boldsymbol{R}\boldsymbol{i}\oplus\boldsymbol{R}\boldsymbol{j}\oplus\boldsymbol{R}\boldsymbol{k}=\boldsymbol{R}^4$ be the real algebra of quaternion with the Hamilton's triple $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and the standard inner product g. The *conjugation* $\bar{p}$ of $p\in\boldsymbol{H}$ is defined as minus of the orthogonal reflection of p with respect to the pure imaginary part $\boldsymbol{H}_0:=\boldsymbol{R}\boldsymbol{i}\oplus\boldsymbol{R}\boldsymbol{j}\oplus\boldsymbol{R}\boldsymbol{k}$. The real numbers $\boldsymbol{R}:=\boldsymbol{R}1$ and the *complex numbers* $\boldsymbol{C}:=\boldsymbol{R}1\oplus\boldsymbol{R}\boldsymbol{i}$ are naturally embeded in $\boldsymbol{H}$. Then g is the real part of the *quaternionic inner product* $h(p_1, p_2):=\bar{p}_1p_2$. The conjugation and g, h are $c:=\boldsymbol{C}\otimes_{\boldsymbol{R}}1$ linearly extended on the complex quaternions $\boldsymbol{H}^c:=\boldsymbol{C}\otimes_{\boldsymbol{R}}\boldsymbol{H}$. For $K=\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$ (resp. $\boldsymbol{H}^c$), the real (resp. c-) vector space of $n\times m-K$-matrices is denoted as $K(n, m)$. Put $K^n:=K(n, 1)$, $K(n):=K(n, n)$. For $x_i=\begin{bmatrix} x_{1i} \\ \vdots \\ x_{ni}\end{bmatrix}\in K^n$ $(i=1, 2)$, denote

$$g(x_1, x_2):=\sum_{i=1}^{n} g(x_{i1}, x_{i2}) \quad\text{and}\quad h(x_1, x_2):=\sum_{i=1}^{n} h(x_{i1}, x_{i2}).$$

Then

$$h(x_1, x_2)=g(x_1, x_2)-\boldsymbol{i}g(x_1, x_2\boldsymbol{i})-\boldsymbol{j}g(x_1, x_2\boldsymbol{j})-\boldsymbol{k}g(x_1, x_2\boldsymbol{k}). \tag{1.0}$$

Denote the identity matrix $1=1_n$ in $K(n)$ and the general linear matrix group $GL(n, K):=\{A\in K(n);\ AB=BA=1_n$ for some $B\in K(n)\}$. For $F=\boldsymbol{R}$ or c, and a F-module V, denote

$$\mathfrak{gl}_F(V):=\{\varphi: V\to V;\ F\text{-morphism}\}$$

and

$$GL_F(V):=\{\alpha\in\mathfrak{gl}_F(V);\ \text{there is } \beta\in\mathfrak{gl}_F(V) \text{ such that } \alpha\circ\beta=\beta\circ\alpha=1_V\}.$$

For a Lie subgroup G of $GL_F(V)$, denote the *identity connected component* of G as G^o. Then $GL(n, K)\times GL(m, K)$ is naturally immersed in $GL_F(K(n, m))$ as

$$\iota_{K(n,m)}: GL(n, K)\times GL(m, K) \longrightarrow GL_F(K(n, m))$$

where

$$\iota_{K(n,m)}(A, B)X = AXB^{-1} \qquad \text{for } X \in K(n, m).$$

The symplectic group in $\boldsymbol{H}(n)$, the complex symplectic group in $\boldsymbol{H}^c(n)$, the symplectic group of $\boldsymbol{H}^n$ and the complex symplectic group of $(\boldsymbol{H}^c)^n$ are defined as follows:

$$Sp(n) := \{A \in \boldsymbol{H}(n);\ {}^t\bar{A}A = 1_n\}, \quad Sp(n)^c := \{A \in \boldsymbol{H}^c(n);\ {}^t\bar{A}A = 1_n\},$$

$$Sp_n := \{\alpha \in GL_{\boldsymbol{R}}(\boldsymbol{H}(n));\ h(\alpha x, \alpha y) = h(x, y)\ (x, y \in \boldsymbol{H}^n)\},$$

$$Sp_n{}^c := \{\alpha \in GL_c(\boldsymbol{H}^c(n));\ h(\alpha x, \alpha y) = h(x, y)\ (x, y \in (\boldsymbol{H}^c)^n)\}.$$

Then $Sp_n = \iota_{\boldsymbol{H}^n}(Sp(n) \times \{1\})$ and $Sp_n{}^c = \iota_{(\boldsymbol{H}^c)^n}(Sp(n)^c \times \{1\})$. Denote also

$$\boldsymbol{Sp}_1 := \iota_{\boldsymbol{H}^n}(\{1\} \times Sp(1)), \quad \boldsymbol{Sp}_1{}^c := \iota_{(\boldsymbol{H}^c)^n}(\{1\} \times Sp(1)^c),$$

$I := \iota_{\boldsymbol{H}^n}(1, \boldsymbol{i})$, $J := \iota_{\boldsymbol{H}^n}(1, \boldsymbol{j})$ and $K := \iota_{\boldsymbol{H}^n}(1, \boldsymbol{k})$. The orthogonal group on $\boldsymbol{H}^n$ and the special orthogonal group $\boldsymbol{H}^n$ is defined as

$$O_{4n} := \{\alpha \in GL_{\boldsymbol{R}}(\boldsymbol{H}^n);\ g(\alpha x, \alpha y) = g(x, y)\ (x, y \in \boldsymbol{H}^n)\},$$

$$SO_{4n} := O_{4n} \cap SL_C((\boldsymbol{H}^n)^c);$$

$$SL_C((\boldsymbol{H}^n)^c) := \{\alpha \in GL_C((\boldsymbol{H}^n)^c);\ \det \alpha = 1\}.$$

Then $Sp_n\boldsymbol{Sp}_1 = \iota_{\boldsymbol{H}^n}(Sp(n) \times Sp(1)) \subset SO_{4n} \subset O_{4n}$ is $\boldsymbol{R}$-irreducible on $\boldsymbol{H}^n$ and also c-irreducible on $(\boldsymbol{H}^n)^c$. On the other hand, Sp_n is $\boldsymbol{R}$-irreducible on $\boldsymbol{H}^n$ but c-reducible on $(\boldsymbol{H}^n)^c$.

Example 1.1. (1) For $U(n) := \{A \in \boldsymbol{C}(n);\ {}^t\bar{A}A = 1_n\} \subset Sp(n)$ and $SU(n) := \{A \in U(n);\ \det A = 1\}$, the linear group

$$U_n\boldsymbol{SU}_2 := \iota_{C(n,2)}(U(n) \times SU(2))$$

is $\boldsymbol{R}$-linearly equivariant to the linear group

$$U_n\boldsymbol{Sp}_1 := \iota_{\boldsymbol{H}^n}(U(n) \times Sp(1))$$

by

$$k_n : \boldsymbol{C}(n, 2) \longrightarrow \boldsymbol{H}^n;\ (x_{i1}, x_{i2}) \longrightarrow (x_{i1} + x_{i2}\boldsymbol{j})$$

because of the equivariant isomorphism

$$\mathrm{Ad}(k_n) : U_n \times SU_2 \longrightarrow U_n \times Sp_1;\ \left(A, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\right) \longrightarrow (A, a + b\boldsymbol{j}).$$

(2) Put $O(n) := \{A \in \boldsymbol{R}(n);\ {}^tAA = 1_n\}$, $SO(n) := \{A \in O(n);\ \det A = 1\}$ and $SO(n)Sp(1) := \{Aq \in GL(n, \boldsymbol{H});\ A \in SO(n),\ q \in Sp(1)\} \subset Sp(n)$. Then $SO_n\boldsymbol{SO}_4 := \iota_{\boldsymbol{R}(n,4)}(SO(n) \times SO(4))$ is $\boldsymbol{R}$-linearly equivariant to $(SO_nSp_1)\boldsymbol{Sp}_1 := \iota_{\boldsymbol{H}^n}((SO(n)Sp(1)) \times Sp(1))$ by

$$h_n : \boldsymbol{R}(n, 4) \longrightarrow \boldsymbol{H}^n;\ (x_{i1}, x_{i2}, x_{i3}, x_{i4}) \longrightarrow (x_{i1} + x_{i2}\boldsymbol{i} + x_{i3}\boldsymbol{j} + x_{i4}\boldsymbol{k}),$$

because of $h_1^{-1}(\iota_{H^1}(Sp(1)\times Sp(1)))h_1=\iota_{R(1,4)}(1\times SO(4))$.

Lemma 1.2. (1) *Let G be a Lie subgroup of O_{4n} such that $Sp_n\boldsymbol{Sp}_1\subset G$. Then $G=Sp_n\boldsymbol{Sp}_1$, SO_{4n} or O_{4n}.*

(2) *The normalizer of $\boldsymbol{Sp}_1$ in O_{4n} is $Sp_n\boldsymbol{Sp}_1$.*

(3) *The normalizer of Sp_n in O_{4n} is $Sp_n\boldsymbol{Sp}_1$.*

Proof. (1) (cf. Gray [6]) Since $Sp_n\boldsymbol{Sp}_1$ is transitive on the unit sphere S^{4n-1} and c-irreducible on $(\boldsymbol{H}^n)^c$, so is G^0. Since $G^0\subset SO_{4n}$, G^0 is semisimple and compact. Thus the only possibility for G^0 is a compact connected $\boldsymbol{R}$-irreducible linear group of cohomogeneity one on $\boldsymbol{R}^{4n}$, which is linearly equivalent to $Sp_n\boldsymbol{Sp}_1$, SO_{4n}, $Spin_7$, $Spin_9$, U_{2n}, $Sp_n\boldsymbol{U}_1(:=\iota_{H^n}(Sp(n)\times U(1)))$, SU_{2n}, or Sp_n (cf. [16; Theorem 4.8]). By dimension, Sp_n and $Sp_n\boldsymbol{U}_1$ (resp. $Spin_9$) cannot contain $Sp_n\boldsymbol{Sp}_1$ (resp. $Sp_4\boldsymbol{Sp}_1$). We rule out (U_{2n} and) SU_{2n}, because of the c-reducibility on $(\boldsymbol{H}^n)^c$. Finally, the inclusion $Sp_2\boldsymbol{Sp}_1\subset Spin_7$ is impossible since the minimal dimensional representation of $Sp_2\boldsymbol{Sp}_1$ is of degree 8> degree 7 of the vector representation of $Spin_7$ (cf. [16]). We conclude that $G^0=Sp_n\boldsymbol{Sp}_1$ or SO_{4n}. If $G^0=SO_{4n}$, then $G=SO_{4n}$ or O_{4n}. Assume $G^0=Sp_n\boldsymbol{Sp}_1$. Then G normalizes $Sp_n\boldsymbol{Sp}_1$. If $n=1$, then $G^0=Sp_1\boldsymbol{Sp}_1=SO_4$, hence $G=SO_4$ or O_4. Assume $n\geqq 2$. To complete the proof, it suffices to show that $Sp_n\boldsymbol{Sp}_1$ is its own normalizer in O_{4n}. Let $A\in O_{4n}$ normalize $Sp_n\boldsymbol{Sp}_1$. Since $Sp_n\boldsymbol{Sp}_1$ has no outer automorphism for $n\geqq 2$, there exists $B\in Sp_n\boldsymbol{Sp}_1$ such that $A\circ B^{-1}$ is in the centralizer of $Sp_n\boldsymbol{Sp}_1$. Since $Sp_n\boldsymbol{Sp}_1$ is c-irreducible on $(\boldsymbol{H}^n)^c$, by Schur's lemma, $A\circ B^{-1}=\pm 1\in Sp_n\boldsymbol{Sp}_1$ on $\boldsymbol{H}^n$. Thus $A\in Sp_n\boldsymbol{Sp}_1$. This completes the proof.

(2) (resp. (3)) Let G be the normalizer of $\boldsymbol{Sp}_1$ (resp. Sp_n) in O_{4n}. Then $Sp_n\boldsymbol{Sp}_1\subset G\subset O_{4n}$. By (1), $G=Sp_n\boldsymbol{Sp}_1$, SO_{4n} or O_{4n}. If $n\geqq 2$, then SO_{4n} and O_{4n} are simple Lie groups and that $\boldsymbol{Sp}_1$ (resp. Sp_n) is their proper subgroup. Thus the central simple SO_{4n} and O_{4n} cannot be the normalizer. Hence, the normalizer of $\boldsymbol{Sp}_1$ (resp. Sp_n) in O_{4n} is $Sp_n\boldsymbol{Sp}_1$. If $n=1$, then $Sp_1\boldsymbol{Sp}_1=SO_4$ and the conjugation $^{-}\in O_4\setminus SO_4$ changes the right $\boldsymbol{Sp}_1$ and the left Sp_1. Thus the normalizer of $\boldsymbol{Sp}_1$ or Sp_1 in O_4 equals to $SO_4=Sp_1\boldsymbol{Sp}_1$. Q. E. D.

Throughout this paper, M denotes a *connected smooth manifold of dimension* $4n$ *with a Riemannian metric* g. The holonomy group of (M, g) at a point $p\in M$ is denoted by $\mathcal{H}ol_p$ with the linear Lie algebra $\mathfrak{hol}_p$. By definition, (M^{4n}, g) is called a *quaternion-Kähler* (resp. *hyperkählerian*) manifold, if there exists a point $p\in M$ and a linear isometry $\iota_p:\boldsymbol{H}^n\to T_pM$ such that $\iota_p^*\mathcal{H}ol_p:=\iota_p^{-1}(\mathcal{H}ol_p)\iota_p\subset Sp_n\boldsymbol{Sp}_1$ (resp. Sp_n). A Riemannian manifold is called a *locally quaternion-Kähler* (resp. *locally hyperkählerian*) manifold, if the Riemannian universal covering manifold is quaternion-Kähler (resp. hyperkählerian).

Proposition 1.3. *The restricted holonomy group* $\mathcal{H}ol_p^0$ *of an irreducible locally quaternion-Kähler manifold of dimension* $\geqq 8$ *is orthogonal linearly equivariant to one of the below:*

$$Sp_n,\ Sp_n\boldsymbol{Sp}_1,\ U_n\boldsymbol{Sp}_1,\ SO_n\boldsymbol{SO}_4\,.$$

$$(3\Lambda_1SU_2)\boldsymbol{Sp}_1,\ (\Lambda_3Sp_3)\boldsymbol{Sp}_1,\ (\Lambda_3SU_6)\boldsymbol{Sp}_1,\ Ss_{12}\boldsymbol{Sp}_1 \text{ or } E_7\boldsymbol{Sp}_1,$$

where $3\Lambda_1SU_2$ $(\subset Sp_2)$ *is the image of the representation of* SU_2 *on the space* $S^3\boldsymbol{C}^2$ *of all complex symmetric 3-forms on* $\boldsymbol{C}^2$, Λ_3SU_6 $(\subset Sp_{10})$ *is the image of the representation of* SU_6 *on the space* $\Lambda^3\boldsymbol{C}^6$ *of all complex alternating 3-forms on* $\boldsymbol{C}^6$, Λ_3Sp_3 $(\subset Sp_7)$ *is the image of the representation on a 14-dimensional* $\boldsymbol{C}$-*subspace* $\boldsymbol{C}^{14}$ *in* $\Lambda^3\boldsymbol{C}^6$, Ss_{12} $(\subset Sp_{16})$ *is the image of the half-spin representation of* $Spin_{12}$, *and* E_7 $(\subset Sp_{28})$ *is the image of the 56-dimensional irreducible* $\boldsymbol{C}$-*representation of the connected and simply connected compact simple Lie group* E_7 *of type* E_7.

Proof. By the classification of the restricted holonomy group of an irreducible Riemannian manifold (cf. [4; 10.90-95, 10.66, 14.47]), it is orthogonal linearly equivariant to Sp_n (cf. [16]) or the restricted holonomy group of a quaternionic symmetric space of compact type. The latter is known as above (e. g. [18] for the detail). Q. E. D.

The oriented Grassmann manifold $SO_{n+4}/(SO_n\times SO_4)$ and its non-compact dual are simply-connected and quaternion-Kähler. On the other hand, the Grassmann manifold $SO_{n+4}/S(O_n\times O_4)$ is not globally quaternion-Kähler, because a compact quaternion-Kähler manifold with positive scalar curvature is simply connected (cf. [10], [4; 14.83]).

Theorem 1.4. *A de Rham irreducible locally quaternion-Kähler manifold* M *of dimension* $\geqq 8$ *is quaternion-Kähler and orientable, except when* M *is locally isometric to a Grassmann manifold* $SO_{n+4}/S(O_n\times O_4)$ *or its noncompact dual.*

Proof. Assume that M is not Ricci flat (Then M is automatically irreducible by [4; 14.45(b)]). By [4; Lemma 14.46] and Proposition 1.3, there exists a linear isometry $\iota_p: \boldsymbol{H}^n\to T_pM$ at one point $p\in M$ such that $\iota_p^*\mathcal{H}ol_p^0=Sp_n\boldsymbol{Sp}_1$, $U_n\boldsymbol{Sp}_1$, $E_7\boldsymbol{Sp}_1$, $Spin_{12}\boldsymbol{Sp}_1$, $(\Lambda_3SU_6)\boldsymbol{Sp}_1$, $(\Lambda_3Sp_3)\boldsymbol{Sp}_1$ or $(3\Lambda_1Sp_1)\boldsymbol{Sp}_1$. Then $\iota_p^*\mathcal{H}ol_p$ normalizes $\boldsymbol{Sp}_1$ being the only normal 3-dimensional subgroup whose representation on the tangent space T_pM is $\boldsymbol{R}$-linearly equivalent to n-direct sum of the 4-dimensional $\boldsymbol{R}$-irreducible representation. Hence $\iota_p^*\mathcal{H}ol_p\subset Sp_n\boldsymbol{Sp}_1$ by Lemma 1.2(2). Assume that M is Ricci-flat. Then M is locally hyperkählerian and $\iota_p^*\mathcal{H}ol_p^0=Sp_n$ by Proposition 1.3 and [4; Lemma 14.40] (cf. Proposition 1.3). Hence $\iota_p^*\mathcal{H}ol_p$ is contained in the normalizer of Sp_n. So $\iota_p^*\mathcal{H}ol_p\subset Sp_n\boldsymbol{Sp}_1$ by

Lemma 1.2(3). Since $Sp_n\boldsymbol{Sp}_1\subset SO_{4n}$, M is orientable. Q.E.D.

Then the Berger's result [4; Theorem 14.43] can be relaxed as follows:

Theorem 1.5. *A complete locally quaternion-Kähler manifold of dimension* $\geqq 8$ *with positive sectional curvature is isometric to a quaternionic projectice space.*

Proof. Since the Riemannian manifold with positive sectional curvature is irreducible, the restricted holonomy group is $Sp_n\boldsymbol{Sp}_1$ by Proposition 1.3. By Theorem 1.4, the manifold is orientable. By Myers theorem, it is compact (cf. Salamon [11; p. 103, *l*. 24]). By Berger [4; Theorem 14.43], a compact orientable locally quaternion-Kähler manifold of dimension $\geqq 8$ with positive sectionaal curvature is isometric to a quaternionic projective space. Q. E. D.

Remark 1.6. *For even* $n=2m\geqq 2$, *any de Rham irreducible, locally quaternion-Kähler manifold of dimension* $4n=8m$ ($\geqq 8$), *is orientable.*

Proof. For a semisimple compact connected Lie subgroup G of SO_{4n} in $GL_c((\boldsymbol{H}^n)^c)$, we denote the linear Lie algebra of G as L, the Dynkin diagram of G as D, the normalizer and the centralizer of G in O_{4n} as N and C, respectively. Then we have that

$$N/C\subset \mathrm{Aut}\,(G)\subset \mathrm{Aut}\,(L)=\mathrm{Int}\,(L)\,\mathrm{Aut}\,(D)\ \text{(semi-direct product)},$$

where the action of $\mathrm{Int}\,(L)$ on G is induced by the adjoint action of G. Let $\bar{C}$ be the centralizer of G in $GL_c((\boldsymbol{H}^n)^c)$. If there exists a subset A in $GL_c((\boldsymbol{H}^n)^c)$ whose conjugate action preserves G and represents all elements of $\mathrm{Aut}(G)$, then

$$N\subset (A\bar{C})\cap O_{4n}\,. \tag{1.7}$$

If moreover there exists a subset E of $GL_c((\boldsymbol{H}^n)^c)$ whose conjugate action preserves D and represents all elements of $\mathrm{Aut}(D)$, then

$$N\subset (GE\bar{C})\cap O_{4n}\,. \tag{1.8}$$

By Theorem 1.4, it is sufficient to prove the case when the restricted holonomy group is $SO_n\boldsymbol{SO}_4$. Then the holonomy group is contained in the normalizer N of $G:=SO_n\boldsymbol{SO}_4$ in O_{4n}. We show that

$$N\subset SO_{4n}$$

for the case of $n=2m$, a even number $\geqq 2$. If $n\neq 2$, then

$$\bar{C}=\{\lambda 1\,;\,\lambda\in c\}\quad\text{and}\quad C=\{\pm 1\}\subset SO_{4n}\,. \tag{1.9}$$

Assume $n\neq 2, 4$. Then $\mathrm{Aut}(G)=\mathrm{Aut}(SO_n)\times\mathrm{Aut}(\boldsymbol{SO}_4)=$the conjugate action A' of $A:=O_n\boldsymbol{O}_4$, where $\boldsymbol{O}_4:=\iota_{R(n,4)}(\{1\}\times O(4)\subset SO(4n)$ for even n and $\boldsymbol{O}_n:=\iota_{R(n,4)}O(n)\times\{1\})\subset SO_{4n}$. Hence, a Riemannian manifold M^{4n} of even $n\geqq 6$ with the restricted holonomy group $SO_n\boldsymbol{SO}_4$ is orientable.

When $n=4$, then $\mathrm{Aut}(G)=\mathrm{Aut}(SO(4)SO(4))\subset\mathrm{Aut}(su_2\oplus su_2\oplus su_2\oplus su_2)=\mathrm{Int}(L)S_4$, where $S_4=$the permutation group of our su_2's. Since $(SO(4)SO(4), (\boldsymbol{R}^{4,4})^c)\cong(SU_2\times SU_2\times SU_2\times SU_2, \boldsymbol{C}^2\otimes\boldsymbol{C}^2\otimes\boldsymbol{C}^2\otimes\boldsymbol{C}^2)$, S_4 is realized as the group of all permutations of these four subspaces $\boldsymbol{C}^2$. The elements $b_{12}:=\mathrm{diag}(-1, 1, 1, 1)\mathbf{1}$, $b_{34}:=1\,\mathbf{diag}(-\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$, and $b_{13,24}:=$the transpose on $\boldsymbol{R}^{4,4}$ of $O_4\boldsymbol{O}_4\subset SO_{16}$ give the permutation of first two $\boldsymbol{C}^2$, the permutation of the last two $\boldsymbol{C}^2$, and the permutation between the first two $\boldsymbol{C}^2$ and the last two $\boldsymbol{C}^2$, respectively. They generate a subgroup B of SO_{16}, which gives a subgroup B' of S_4 with 8 elements. Then $N\supset GBC$ and the conjugate action gives an inclusion $N/GBC\subset S_4/B'$, which consists of prime-3 elements. If $N=GBC$, then $N\subset SO_{16}$. If $N\neq GBC$, then there is $f\in N\setminus GBC$, whose conjugate action $f'\in S_4\setminus B'$ on G generates S_4/B'. Hence $B'\cup f'B'$ generates S_4. In particular, any element of S_4 comes from some $f\in N\subset O_{16}$ as f'. Hence, for any two $\boldsymbol{C}^2$, there is $f\in N$ which translates them to the first two $\boldsymbol{C}^2$. Since the permutation of the first two $\boldsymbol{C}^2$ is given by $b_{12}\in SO_{16}\cap N$, a permutation of any two $\boldsymbol{C}^2$ is given by $f^{-1}b_{12}f\in SO_{16}\cap N$. Since S_4 is generated by permutations of any two $\boldsymbol{C}^2$, S_4 is given by the conjugate action of some $E\subset SO_{16}\cap N$. By (1.8), (1.9)' $N\subset SO_{16}$.

When $n=2$, $C=SO_2\cdot\mathbf{1}$. Because of $\mathrm{Aut}(G)=\mathrm{Aut}(SO_2)\times\mathrm{Aut}(\boldsymbol{SO}_4)=\boldsymbol{Z}\times\boldsymbol{O}_4/\{\pm\mathbf{1}\}$ where $\boldsymbol{Z}=$(the set of n-power actions of integers n), $N'=N/C=\{\pm 1\}\times\boldsymbol{O}_4/\{\pm\mathbf{1}\}=O_2/C\times\boldsymbol{O}_4/\{\pm\mathbf{1}\}$. So $N=O_2\boldsymbol{O}_4\subset SO_8$. Q. E. D.

2. Semi-symplectic triple systems

By Yamaguti-Asano [14], Asano [2], [3], and Yasukura [17], there is a one-to-one functor $\boldsymbol{a}$ from the category $\mathcal{B}$ of all complex simple symplectic triple systems up to isomorphism to the category $\mathcal{C}$ of all complex simple Lie algebras of rank $\geqq 2$ up to isomorphism, as a generalization of the Freudenthal's construction of a complex simple Lie algebra of the exceptional type E_8. On the other hand, by Wolf [19], there is a one-to-one functor $\boldsymbol{b}$ from $\mathcal{C}$ to the category $\mathcal{A}$ of all quaternionic symmetric spaces of compact type up to homothety (cf. [4; 14E, D], [13]). By Tasaki [13], the morphisms of $\mathcal{B}$ are Lie algebra homomorphisms of index 1. Then there is a one-to-one functor $\boldsymbol{c}$ from $\mathcal{A}$ to $\mathcal{B}$ (cf. [18]). In this section, using the twistor construction of Salamon [10; Theorem 6.1] and Bérard Bergery [4; 14.80, 14.86], a complex semi-symplectic triple system $\boldsymbol{c}_p(M)$ is constructed for each locally quaternion-Kähler manifold M of dimension $\geqq 8$ with a base point p by the Riemannian curvature tensor R_p.

Let (M, g) be a locally quaternion-Kähler manifold of dimension $4n \geqq 8$. Since M is connected, at each point $p \in M$, there is a linear isometry $\iota_p: \boldsymbol{H}^n \rightarrow T_pM$ such that $\iota_p^* \mathcal{H}ol_p^0(M) \subset Sp_n \boldsymbol{Sp}_1$. Put $\boldsymbol{Sp}(1) = \boldsymbol{Sp}(1)_p = \iota_p^{*-1} \boldsymbol{Sp}_1$ with the linear Lie algebra $\boldsymbol{sp}(1) = \boldsymbol{sp}(1)_p$, and their intersection $\boldsymbol{Z}_p := \boldsymbol{Sp}(1) \cap \boldsymbol{sp}(1) = \{\alpha \in \boldsymbol{Sp}(1);\ \alpha^2 = -1 \text{ on } T_pM\}$ ($\cong \boldsymbol{C}P_1$, i.e. the complex projective line) which is normalized by $\mathcal{H}ol_p^0(M)$. In general, a $\mathcal{H}ol_p^0(M)$-normalized set, e.g. $\boldsymbol{Z}_p$, is not unique, if M is locally isometric to a Grassmann manifold $SO_{n+4}/S(O_n \times O_4)$, its non-compact dual, or a de Rham reducible hyperkählerian manifold. Denote also $Sp(n) = Sp(n)_p = \iota_p^{*-1} Sp_n$ with the linear Lie algebra $sp(n) = sp(n)_p$, which commutes with the Hamilton's triple $I := \iota_p^{*-1}(I)$, $J := \iota_p^{*-1}(J)$, $K := \iota_p^{*-1}(K) \in \boldsymbol{Z}_p$. Let D, R_p be the covariant derivative of the Levi-Civita connection and the Riemannian curvature tensor at p of (M, g), respectively.

Lemma 2.1. *If $\alpha \in \boldsymbol{Sp}(1)$ and $A \in \boldsymbol{sp}(1)$, then, for $X, Y, Z \in T_pM$, one has the following equations:*

$$\alpha(R_p(X, Y)Z) = R_p(\alpha X, \alpha Y)(\alpha Z),$$

$$A(R_p(X, Y)Z) = R_p(AX, Y)Z + R_p(X, AY)Z + R_p(X, Y)AZ.$$

Proof. The second equation follows from first equation. We show the first equation for all $\alpha \in \boldsymbol{Z}_p$. Then the first equation for $\alpha \in \boldsymbol{Sp}(1)$ follows because $\boldsymbol{Z}_p$ generates the group $\boldsymbol{Sp}(1)$. Since $\boldsymbol{sp}(1)$ is normalized by $\mathfrak{hol}_p$, there exists alternating 2-forms α, β, γ on T_pM such that

(a) $$[R_p(X, Y), I] = \gamma(X, Y)J - \beta(X, Y)K,$$

(b) $$[R_p(X, Y), J] = -\gamma(X, Y)I + \alpha(X, Y)K,$$

(c) $$[R_p(X, Y), K] = \beta(X, Y)I - \alpha(X, Y)J,$$

for any vectors $X, Y \in T_pM$. Applying (c) to a vector field Z and evaluate the result with JZ by g, we get

(d) $$\alpha(X, Y)|Z|^2 = g(R_p(X, Y)Z, IZ) + g(R_p(X, Y)JZ, KZ),$$

whose computations induce for the *Ricci tensor* $r(X, Y)$ that

(e) $$n\alpha(X, IY) + \beta(X, JY) + \gamma(X, KY) = r(X, Y).$$

If $\dim M = 4n \geqq 8$, then the same computations for β and γ induce that

$$\alpha(X, Y) = \frac{2}{n+2} r(IX, Y),$$

$$\beta(X, Y)=\frac{2}{n+2}r(JX, Y),$$

$$\gamma(X, Y)=\frac{2}{n+2}\gamma(KX, Y),$$

and $r(X, Y)={}_s g_p(X, Y)$ for some ${}_s\in\boldsymbol{R}$, the *scalar curvature* of M (cf. [7], [4; 14.40]). In the case of ${}_s=0$: $[R_p(X, Y), I]=[R_p(X, Y), J]=[R_p(X, Y), K]=0$. Hence, for any $\alpha\in\boldsymbol{Z}_p$, $[R_p(X, Y), \alpha]=0$. Then $g(R_p(\alpha X, \alpha Y)Z, W)=g(R_p(Z, W)\alpha X, \alpha Y)=g(\alpha(R_p(Z, W)X), \alpha Y)=g(R_p(Z, W)X, Y)=g(R_p(X, Y)Z, W)$. So that $R_p(\alpha X, \alpha Y)=R_p(X, Y)$. Hence, $\alpha(R_p(X, Y)Z)=R_p(\alpha X, \alpha Y)\alpha Z$.

In the case of ${}_s>0$ (resp. ${}_s<0$): Let U be a simply connected neighborhood of p. Then, $\mathcal{H}ol_p(U)=\mathcal{H}ol_p^0(M)$ normalizes $\boldsymbol{Z}_p$. Define the twistor space $\boldsymbol{Z}_U$ as a parallel subbundle of End(TU) by parallel translations of $\boldsymbol{Z}_p$. By [4; 14.68, 14.71, 14.80, 14.86b], $\boldsymbol{Z}_U$ admits an almost complex structure J and a J-invariant semi-Riemannian metric $\tilde{g}$ such that

(0) the natural projection $\pi:\boldsymbol{Z}_U\to U$ is a semi-Riemannian submersion (see [9; p. 212] for the definition),

(1) $\tilde{D}J=0$ for the Levi-Civita connection $\tilde{D}$ of $(\boldsymbol{Z}_U, \tilde{g})$,

(2) the vertical projection $\mathcal{V}$ of π satisfies $\mathcal{V}J=J\mathcal{V}$, and

(3) $d\pi(J_z\tilde{X})=z(d\pi(\tilde{X}))$ for all $\tilde{X}\in T_z\boldsymbol{Z}_U$.

Let $\tilde{X}, \tilde{Y}, \tilde{Z}\in T\boldsymbol{Z}_U$. By (1), $\tilde{D}_{\tilde{X}}\circ J=J\circ\tilde{D}_{\tilde{X}}$, so the curvature transformation $\tilde{R}(\tilde{X}, \tilde{Y})$ of $\tilde{D}$ also commutes with J. Hence, $\tilde{R}(J\tilde{X}, J\tilde{Y})=\tilde{R}(\tilde{X}, \tilde{Y})$ and $J(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z})=\tilde{R}(J\tilde{X}, J\tilde{Y})(J\tilde{Z})$. Put $\tilde{K}(\tilde{X}, \tilde{Y}):=\tilde{g}(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{X}, \tilde{Y})$. Then J preserves $\tilde{K}(\tilde{X}, \tilde{Y})$:

$$\tilde{K}(J\tilde{X}, J\tilde{Y})=\tilde{K}(\tilde{X}, \tilde{Y}).$$

For any horizontal vector fields $\tilde{X}, \tilde{Y}$ on $\boldsymbol{Z}_U$, by the condition (2),

$$\frac{1}{2}\mathcal{V}([\tilde{X}, J\tilde{Y}])=\mathcal{V}(\tilde{D}_{\tilde{X}}(J\tilde{Y}))=\mathcal{V}(J\tilde{D}_{\tilde{X}}\tilde{Y})=J(\mathcal{V}(\tilde{D}_{\tilde{X}}\tilde{Y}))=\frac{1}{2}J(\mathcal{V}([\tilde{X}, \tilde{Y}])).$$

Hence,

$$\mathcal{V}([J\tilde{X}, J\tilde{Y}])=-\mathcal{V}([\tilde{X}, \tilde{Y}]).$$

Put $K(X, Y):=g(R(X, Y)X, Y)$ for $X, Y\in TM$. By the O'Neill's formula (cf. [9; p. 213], [4; 9.24, (9.29c)]):

$$\tilde{K}(\tilde{X}, \tilde{Y})=K(d\pi\tilde{X}, d\pi\tilde{Y})-\frac{3}{4}\tilde{g}(\mathcal{V}[\tilde{X}, \tilde{Y}], \mathcal{V}[\tilde{X}, \tilde{Y}]),$$

and the condition (3), one has that

$$K(z(d\pi\tilde{X}), z(d\pi\tilde{Y}))=K(d\pi(J_z\tilde{X}), d\pi(J_z\tilde{Y}))=K(d\pi\tilde{X}, d\pi\tilde{Y})$$

at each $z\in\boldsymbol{Z}_U$. Then the proof of the first equation is completed. Q. E. D.

Using this lemma, one can define an algebraic system $\boldsymbol{c}_p(M)$ on a subspace of the complexification $T_p^cM:=\boldsymbol{C}\underset{\boldsymbol{R}}{\otimes}T_pM$ at each $p\in M$ where $T^c{}_pM$ is identified with $(\boldsymbol{H}^c)^n$ by the complex linear extension $\iota^c{}_p$ of ι_p. Then $\iota^c{}_p{}^*(\mathcal{H}ol_p{}^0)\subset Sp_n{}^c\boldsymbol{Sp}_1{}^c$. Denote $\boldsymbol{Sp}(1)^c=\boldsymbol{Sp}(1)^c{}_p=(\iota_p{}^{c-1})^*\boldsymbol{Sp}_1{}^c$ and $Sp(n)^c=Sp(n)^c{}_p=(\iota_p{}^{c-1})^*Sp_n{}^c$ with the linear Lie algebras $\boldsymbol{sp}(1)^c=\boldsymbol{sp}(1)^c{}_p$ and $sp(n(^c=sp(n)^c{}_p$, respectively

Definition 2.2. Fix a base point $p\in M$. Denote R, g the $\boldsymbol{C}$-multilinear extension of R_p, g_p on T_p^cM, respectively. Fix a $\boldsymbol{C}$-basis H, E_+, E_- of $\boldsymbol{sp}(1)^c$ such as

(2.2.1) $[H, E_\pm]=\pm 2E_\pm,\quad [E_+, E_-]=H$ and $H^2=1$ on T_p^cM,

For example,

(2.2.1′) $$H:=-iI,\quad E_+:=\frac{1}{2}(iJ+K),\quad E_-:=\frac{1}{2}(iJ-K).$$

Denote $T_p^\pm M:=\{x\in T_p^cM;\ Hx=\pm x\}$. For x, y, $z\in T_p^+M$, a $\boldsymbol{C}$-trilinear product $[xyz]$ and a skew-symmetric $\boldsymbol{C}$-bilinear form $\langle x, y\rangle$ are defined as follows:

(2.2.2) $$L(x, y):=\frac{1}{2}\{R(x, E_-y)+R(y, E_-x)\}\quad(\in sp(n)^c),$$

(2.2.3) $$[xyz]:=L(x, y)z\quad(\in T_p^*M),$$

(2.2.4) $$R(x, y)=:2\langle x, y\rangle E_+\quad(\in \boldsymbol{sp}(1)^c).$$

Then, denote $\boldsymbol{c}_p(M):=(T_p^*M, [xyz], \langle x, y\rangle)$.

Theorem 2.3. (1) *$\boldsymbol{c}_p(M)$ is well-defined for a locally quaternion-Kähler manifold (M, g) with a base point $p\in M$ and a linear isometry $\iota_p:\boldsymbol{H}^n\to T_pM$ such that $\iota_p^*(\mathcal{H}ol_p)\subset Sp_n\boldsymbol{Sp}_1$ of dimensioon $4n\geqq 8$ and any scalar curvature s*

(2) *$\boldsymbol{c}_p(M)=(T_p^+M, [xyz], \langle x, y\rangle)$ is a semi-symplectic triple system:*

(ST. 0) $\langle[xyz], w\rangle+\langle z, [xyw]\rangle=0$,

(ST. 1) $[xyz]=[yxz]$,

(ST. 2) $[xyz]-[xzy]=\langle x, z\rangle y-\langle x, y\rangle z+2\langle y, z\rangle x$.

(3) *If $s\neq 0$, then $\langle,\rangle$ is non-degenerate and $\boldsymbol{c}_p(M)$ is simple. If $s=0$, then $\langle,\rangle$ is vanishing and $(T_p^+M, [xyz])$ is a symmetric triple system: $[xyz]=[yxz]=[xzy]$.*

To prove the theorem, we prepare the following proposition:

Proposition 2.4. *Let x, $y\in T_p^+M$. Then*

(2.4.1) $E_-(T_p^-M)=E_+(T_p^+M)=0$.

(2.4.2) $E_+E_-=1$ *on* T_p^+M, $E_-E_+=1$ *on* T_p^-M.

(2.4.3) $E_-(T_p^+M)=T_p^-M$, $E_+(T_p^-M)=T_p^+M$.

(2.4.4) $g(T_p^+M, T_p^+M)=g(T_p^-M, T_p^-M)=0$.

(2.4.5) $g(x, E_-y)$ *is a* $Sp(n)$*-invariant symplectic form.*

(2.4.6) $R(T_p^+M, T_p^+M)\subset CE_+\subset \boldsymbol{sp}(1)^c$, $R(T_p^-M, T_p^-M)\subset CE_-\subset \boldsymbol{sp}(1)^c$.

(2.4.7) $L(T_p^+M, T_p^+M)\subset sp(n)^c$.

(2.4.8) $L(x, y)=-\langle x, y\rangle H+R(y, E_-x)=\langle x, y\rangle H+R(x, E_-y)$.

(2.4.9) $R(E_-x, E_-y)=-2\langle x, y\rangle E_-$.

(2.4.10) *There is a constant* $\lambda\in C$ *such that* $\langle x, y\rangle=\lambda g(x, E_-y)$.

Proof. 0) For $x\in T_p^{\pm}M$, $HE_{\mp}x=([H, E_{\mp}]+E_{\mp}H)x=(\mp 2\pm 1)E_{\mp}x=\mp E_{\mp}x$. So that $E_-(T_p^+M)\subset T_p^-M$ and $E_+(T_p^-M)\subset T_p^+M$. 1) For $x\in T_p^{\pm}M$, $HE_{\pm}x=([H, E_{\pm}]+E_{\pm}H)x=(\pm 2\pm 1)E_{\pm}x=\pm 3E_{\pm}x$. Since $H^2=1$ on T_p^cM, $E_{\pm}x=0$. 2) follows from 1) and $E_+E_-=H+E_-E_+$. 3) follows from 0) and 2). Since H is skew-symmetric with respect to g, (4) follows. (5) Since E_- is skew-symmetric with respect to g, $g(x, E_-y)$ is skew-symmetric. Since g is non-degenerate on T_p^cM, $g(x, E_-y)$ is non-degenerate on T_p^+M by 3) and 4). Since $Sp(n)$ commutes with E_- and preserves g, $g(x, E_-y)$ is $Sp(n)$-invariant. 6) For $x, y\in T_p^{\pm}M$, $[H, R(x, y)]=R(Hx, y)+R(x, Hy)=\pm 2R(x, y)\in[\boldsymbol{sp}(1)^c, \mathfrak{hol}_p^c]\subset \boldsymbol{sp}(1)^c$ by Lemma 2.1. Hence $R(x, y)\in CE_{\pm}$. By 2) and Lemma 2.1, for $x, y\in T_p^+M$, $[H, L(x, y)]=[E_{\pm}, L(x, y)]=0$. Then 7) follows from the definition of $Sp(n)^c$ and the equation (1.0) in §1. 8) By Lemma 2.1, $-2\langle x, y\rangle H=[E_-, R(x, y)]=R(E_-x, y)+R(x, E_-y)=R(x, E_-y)-R(y, E_-x)$. Since $2L(x, y)=R(x, E_-y)+R(y, E_-x)$, we have that

$$2L(x, y)-2\langle x, y\rangle H=2R(x, E_-y) \text{ and } -2L(x, y)-2\langle x, y\rangle H=-2R(y, E_-x).$$

9) By Def. 2.2 and Lemma 2.1, $R(E_-x, E_-y)=1/2[E_-, [E_-, R(x, y)]]=-2\langle x, y\rangle E_-$. (10) By 6), for $u, v\in T_p^+M$ and $\alpha\in Sp(n)$, $\alpha\circ R(u, v)=R(u, v)\circ\alpha$ on T_p^cM. So that $g(R(\alpha u, \alpha v)z, w)=g(R(z, w)\alpha u, \alpha v)=g(\alpha(R(z, w)u), \alpha v)=g(R(z, w)u, v)=g(R(u, v)z, w)$ for $z, w\in T_p^cM$. Hence $R(\alpha u, \alpha v)=R(u, v)$ on T_p^cM, i.e. $\langle\alpha u, \alpha v\rangle=\langle u, v\rangle$. Since $Sp(n)$ is C-irreducible on T_p^+M, we get the result by 5). Q.E.D.

Proof of Theorem 2.3. (1) For $x, y, z\in T_p^+M$, $H[xyz]=1/2\{(R(Hx, E_-y)+R(Hy, E_-x)+R(x, HE_-y)+R(y, HE_-x))z+(R(x, E_-y)+R(y, E_-x))(Hz)\}=[xyz]$ by Lemma 2.1 and (2.4.3). Hence, $[xyz]\in T_p^+M$. By (2.4.6), $\langle x, y\rangle$ is also well-defined. If H', E'_+, E'_- is another basis of $\boldsymbol{sp}(1)^c$ satisfying (2.2.1), then the C-linear transformation φ on $\boldsymbol{sp}(1)^c$ such that $\varphi(H)=H'$, $\varphi(E_+)=E'_+$, $\varphi(E_-)$

$=E'_-$, is an automorphism of $\boldsymbol{sp}(1)^c$. Since all automorphisms of $\boldsymbol{sp}(1)^c$ are inner, there is $\alpha\in\boldsymbol{Sp}(1)^c$ such that $\alpha\circ X\circ\alpha^{-1}=\varphi(X)$ for all $X\in\boldsymbol{sp}(1)^c$. Put $T_p^+{}'M:=\{X\in T_p^cM: H'X=X\}$. Then $\alpha(T_p^+M)=T_p^+{}'M$. Since $\alpha\in\boldsymbol{Sp}(1)^c$ preserves R (by Lemma 2.1), $[(\alpha x)(\alpha y)(\alpha z)]'=\alpha[xyz]$ and $\langle\alpha x, \alpha y\rangle'=\langle x, y\rangle$ for all $x, y, z\in T_p^+M$.

(2) (ST. 2): By (2.4.8), $[xyz]=-\langle x, y\rangle z+R(y, E_-x)z$. Hence

$$\begin{aligned}[xyz]-[xzy]&=-\langle x, y\rangle z+R(y, E_-x)z+\langle x, z\rangle y-R(z, E_-x)y\\&=-\langle x, y\rangle z+\langle x, z\rangle y-R(z, y)E_-x \text{ (by the Bianchi identity)}\\&=-\langle x, y\rangle z+\langle x, z\rangle y-2\langle z, y\rangle E_+E_-x\\&=\langle x, z\rangle y-\langle x, y\rangle z+2\langle y, z\rangle x \text{ (by (2.4.2)).}\end{aligned}$$

(ST. 1): Obvious from the definition of $[xyz]$.

(ST. 0): By (2.4.5), (2.4.7), (2.4.10) and (2.2.3), it is obtained.

(3) Assume $\lambda=0$ in (2.4.10). Let $x, y\in T_p^+M$. Then $R(x, y)=0$. By (2.4.9), $R(E_-x, E_-y)=0$. By (2.4.8), $R(x, E_-y)=L(x, y)\in sp(n)^c$. By (2.4.3), $\{\sum_i R(x_i, y_i)| x_i, y_i\in T_p^cM\}\subset sp(n)^c$. Denote τ the $\boldsymbol{C}$-conjugation on T_p^cM with respect to T_pM. Then $sp(n)=\{A\in sp(n)^c=sp(n)\oplus \boldsymbol{i}sp(n)\,;\,\tau A\tau=A\}$. And $\tau R(x, y)\tau=R(\tau x, \tau y)$ for $x, y\in T_p^cM$, since $R(T_pM, T_pM)T_pM\subset T_pM$ and the mapping $(x, y, z)\to R(x, y)z$ is $\boldsymbol{C}$-trilinear. Hence, $\{\sum_i R(X_i, Y_i)|X_i, Y_i\in T_pM\}\subset sp(n)$. Then $_s=0$ (cf. Proof of Lemma 2.1). Take the contraposition of the above: If $_s\neq 0$, then $\lambda\neq 0$ and $\langle x, y\rangle$ is non-degenerate. In this case, we show that $(T_p^+M, [xyz], \langle x, y\rangle)$ is simple (cf. [14; Theorem 2], [17], and Remark 2.6 (1) below): Put $\mathcal{V}:=\{x\in T_p^+M: \langle x, y\rangle=0 \text{ for all } y\in T_p^+M\}$. By the assumption, $\mathcal{V}=0$. Let $\mathcal{B}$ be any $\boldsymbol{C}$-subspace of T_p^+M such that $[T_p^+MT_p^+M\,\mathcal{B}]+[T_p^+M\,\mathcal{B}\,T_p^+M]+[\mathcal{B}\,T_p^+MT_p^+M]\subset\mathcal{B}$ i.e. $\mathcal{B}$ is a *tri-ideal* of $\boldsymbol{c}_p(M)$. Putting $z\in\mathcal{B}$ in (ST. 2), we have that $\langle x, z\rangle y+2\langle y, z\rangle x\in\mathcal{B}$. Changing x and y, we have that $\langle y, z\rangle x+2\langle x, z\rangle y\in\mathcal{B}$. Then $3\langle y, z\rangle x\in\mathcal{B}$, i.e. $\langle T_p^+M, \mathcal{B}\rangle T_p^+M\subset\mathcal{B}$. If $\mathcal{B}\neq T_pM$, then $\langle T_p^+M, \mathcal{B}\rangle=0$, i.e. $\mathcal{B}\subset\mathcal{V}=0$. So that $\boldsymbol{c}_p(M)$ has no non-trivial tri-ideal, i.e. $\boldsymbol{c}_p(M)$ is simple. In this case, $[T_p^+M\ T_p^+M\ T_p^+M]\neq\{0\}$ since $\dim_{\boldsymbol{C}}T_p^+M\geqq 2$. Assume $_s=0$. By the proof of Lemma 2.1, $R(X, Y)\in sp(n)$ for $X, Y\in T_pM$. Hence, $\langle x, y\rangle E_+=R(x, y)\in sp(n)^c\cap\boldsymbol{sp}(1)^c=0$ for $x, y\in T_p^+M$, i.e. $\langle x, y\rangle\equiv 0$. By (ST. 1) and (ST. 2), $[xyz]=[yxz]=[xzy]$. Q. E. D.

For a semi-symplectic triple system with the vanishing skew-symmetric form, there are some remarks:

Remark 2.5. (1) If V is a vector space over $\boldsymbol{C}$, not necessary of even dimension, with the tri-linear product $[xyz]\equiv 0$ and the skew-symmetric form $\langle x, y\rangle\equiv 0$, then $(V, [xyz], \langle x, y\rangle)$ satisfies (ST. 0), 1), and 2). So that not all of semi-symplectic triple systems are obtained as $\boldsymbol{c}_p(M)$ from a locally quaternion-

Kähler manifold M, because the dimension of M is even.

(2) If $(V, \{xyz\}, \langle , \rangle')$ is a simple Freudenthal triple system (see [8] for the definition and the classification), then another triple system $(V, [xyz], \langle x, y\rangle)$ on the same V defined by $[xyz]\equiv\{xyz\}$ and $\langle x, y\rangle\equiv 0$ is a simple semi-symplectic triple system, whose skew-symmetric form is vanishing and degenerate. The author does not know whether there is a hyperkählerian manifold M whose semi-symplectic triple system $\boldsymbol{c}_p(M)$ is isomorphic to the above.

Then, [17] should be modified as follows.

Remark 2.6. (1) In the assumption of [17; Proposition 1.2, Theorem 1.6, Theorem 2.6], it should be assumed that the skew-symmetric from $\langle x, y\rangle$ is not identically zero. Then they are proved as [14; Theorem 2, Theorem 4]. If $\langle x, y\rangle\equiv 0$, then counter examples are constructed from complex simple Freudenthal triple systems as Remark 2.5 (2).

(2) (Due to Prof. H. Asano) [17; Proposition 2.3, (4)] should be modified as

(4) $[\mathcal{G}_{-1}, \mathcal{G}_{-1}]=\mathcal{G}_{-2}$, $[\mathcal{G}_1, \mathcal{G}_1]=\mathcal{G}_2$, and $[\mathcal{G}_{-1}, \mathcal{G}_1]=\mathcal{G}_0$.

And the proof should also be modified as follows.

(4) It is easily verified that $\mathcal{G}':=[\mathcal{G}_{-1}, \mathcal{G}_{-1}]\oplus\mathcal{G}_{-1}\oplus[\mathcal{G}_{-1}, \mathcal{G}_1]\oplus\mathcal{G}_1\oplus[\mathcal{G}_1, \mathcal{G}_1]$ is an ideal of $\mathcal{G}$.

(3) [17; Proposition 2.3, (2)] should be modified as follows:

(2) Any Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}_0$ is a Cartan subalgebra of $\mathcal{G}$ containing K.

Due to [8; Proof of Lemma 1], the proof should also be modified:

(2) $\mathcal{H}$ is a maximal nilpotent subalgebra of $\mathcal{G}_0$ which coincides with its normalizer in $\mathcal{G}_0$. Since $[K, \mathcal{H}]=0$, $K\in\mathcal{H}$. Since $\operatorname{ad}K|\mathcal{G}_i=i1_{\mathcal{G}_i}$, $\mathcal{H}$ is a maximal nilpotent subalgebra of $\mathcal{G}$ which coincides with its normalizer in $\mathcal{G}$.

3. Locally quaternionic space forms

The following condition is necessary for a Riemannian manifold to be locally symmetric.

Definition 3.1 (Szabó [12]). A Riemannian manifold (M, g) is called *semi-symetric* if, for all $x, y, u, v\in T_p^cM$, the Riemannian curvature tensor R satisfies that

$$\text{(LT. 3)}\qquad [R(x, y), R(u, v)]=R(R(x, y)u, v)+R(u, R(x, y)v).$$

i.e. T_p^cM with the tri-linear product $(x, y, z)\to R(x, y)z$ is a Lie triple system over $\boldsymbol{C}$.

The above condition is also sufficient for a de Rham irreducible quaternion-

Kähler manifold to be locally symmetric by Szabó [12; Prop. 5.2].

Theorem 3.2. (1) *For a locally quaternion-Kähler manifold* (M, g) *of dimension* $4n \geqq 8$, *the following three conditions are equivalent*:
(i) (M, g) *is semi-symmetric,*
(ii) *For each point* $p \in M$, *there exists a linear isometry* $\iota_p : \boldsymbol{H}^n \to T_pM$ *such that* $\iota_p^*(\mathcal{H}ol_p) \subset Sp_n \boldsymbol{Sp}_1$ *and the corresponding semi-symplectic triple system* $\boldsymbol{c}_p(M)$ *satsfies*

$$\text{(ST. 3)} \qquad [xy[uvw]] = [[xyu]vw] + [u[xyv]w] + [uv[xyw]],$$

i.e. $\boldsymbol{c}_p(M)$ *is a symplectic triple system over* $\boldsymbol{C}$.
(iii) *For each point* $p \in M$ *and any linear isometry* $\iota : \boldsymbol{H}^n \to T_pM$ *such that* $\iota_p^*(\mathcal{H}ol_p) \subset Sp_n \boldsymbol{Sp}_1$ *and the corresponding semi-symplectic triple system* $\boldsymbol{c}_p(M)$ *is a symplectic triple system over* $\boldsymbol{C}$.

(2) *In this case, if moreover* (M, g) *is de Rham irreducible or compact, then* (M, g) *is locally symmetric.*

Proof. (1) (iii)→(ii): Trivial. (ii)→(i): By (2.4.6), $R(x, y)$, $R(E_-x, E^-y) \in \boldsymbol{sp}(\boldsymbol{1})^c$ for $x, y \in T_p^+M$. By Lemma 2.1, (LT. 3) is always satisfied for $(x, y) \in (T_p^+M \times T_p^+M) \cup (T_p^-M \times T_p^-M)$. Assume (ii). Let $x, y, u, v \in T_p^+M$. Then (ST. 3) is equivalent to

$$\text{(ST. 3')} \qquad [L(x, y), L(u, v)] = L(L(x, y)u, v) + L(u, L(x, y)v),$$

i. e. $[L(x, y), R(u, E_-v) + R(v, E_-u)]$

$= R(L(x, y)u, E_-v) + R(u, E_-L(x, y)v) + R(L(x, y)v, E_-u) + R(v, E_-L(x, y)u).$

On the other hand, Lemma 2.1, (2.4.7) and (ST. 0) induce

$$[L(x,y), R(u, E_-v) - R(v, E_-u)] = [L(x,y), [E_-, R(u, v)]] \subset [L(x.y), \boldsymbol{sp}_r(\boldsymbol{1})^c] = 0$$

and $R(L(x,y)u, E_-v) + R(u, E_-L(x,y)v) - R(L(x,y)v, E_-u) - R(v, E_-L(x, y)u)$

$$= [E_-, R(L(x,y)u, v)] + [E_-, R(u, L(x,y)v)]$$

$$= -2\{\langle L(x,y)u, v\rangle + \langle u, L(x,y)v\rangle\} H = 0.$$

Summing up the above three equations,

$$[L(x, y), R(u, E_-v)] = R(L(x, y)u, E_-v) + R(u, E_-L(x, y)v)$$

$$= R(L(x, y)u, E_-v) + R(u, L(x, y)E_-v).$$

By (2.4.8), we have that

$$[R(x, E_-y), R(u, E_-v)] = R(R(x, E_-y)u, E_-v) + R(u, R(x, E_-y)E_-v),$$

i. e. (LT. 3) is satisfied for $(x, y), (u, v) \in T_p^+M \times T_p^-M$.

By (ST. 0),

$$[L(x, y), R(u, v)]=R(L(x, y)u, v)+R(u, L(x, y)v)=0$$

and

$$[L(x, y), R(E_{-}u, E_{-}v)]=R(L(x, y(E_{-}u, E_{-}a)+R(E_{-}u, L(x, y)E_{-}v)=0.$$

By (2,4,8), we have that

$$[R(x, E_{-}y), R(u, v)]=R(R(x, E_{-}y)u, v)+R(u, R(x, E_{-}y)v),$$

$$[R(x, E_{-}y), R(E_{-}u, E_{-}v)]=R(R(x, E_{-}y)E_{-}u, E_{-}v)+R(E_{-}u, R(x, E_{-}y)E_{-}v),$$

i.e. (LT. 3) is satisfied for $(x, y) \in T_p^+M \times T_p^-M$, $(u, v) \in (T_p^+M \times T_p^+M) \cup (T_p^-M \times T_p^-M)$.
Hence (LT. 3) is satisfied for all $x, y, u, v \in T_p^cM$.

(i)→(iii): Assume (i). By (2.4.7), we have $[L(x, y), E_{-}]=0$ and

$$4[L(x, y), L(u, v)]=[R(x, E_{-}y)+R(y, E_{-}x), R(u, E_{-}v)+R(v, E_{-}u)]$$
$$=2\{R(L(x,y)u, E_{-}v)+R(u, L(x,y)E_{-}v)+R(L(x,y)v, E_{-}u)+R(v, L(x,y)E_{-}u)\}$$
$$=2\{R(L(x,y)u, E_{-}v)+R(u, E_{-}L(x,y)v)+R(L(x,y)v, E_{-}u)+R(v, E_{-}L(x,y)u)\}$$
$$=4\{L(L(x, y)u, v)+L(u, L(x, y)v)\}.$$

(2) By Szabó [12; Proposition 5.2], a de Rham irreducible semi-symmetric Riemannian manifold with the restricted holonomy group $\mathcal{H}ol_p^0 \neq SO(4n)$, $U(2n)$ is locally symmetric. By Lichnerowicz [15; p. 22, Theorem 6.1], a compact semi-symmetric Riemannian manifold with the parallel Ricci tensor is locally-symmetric. Hence (2) is obtained by (1)(i). Q. E. D.

To study an equivalence problem of a locally symmetric locally quaternion-Kähler manifold, *Ambrose's theorem* is reviewed in the below (Lemma 3.3). Let (M, g) and $(\bar{M}, \bar{g})$ be connected Riemannian manifolds of the same dimension. A *broken geodesic* is a continuous curve $\gamma: [0, l] \to M$ such that $\gamma | [t_i, t_{i+1}]$ is a smooth geodesic for $0=t_0<t_1< \cdots <t_n<t_{n+1}=l$ Set ${}_i\gamma=\gamma | [0, t_i]$, and define v_i as the initial velocity vector of the smooth curve $(\gamma | [t_i, t_{i+1}])(t)=\exp_{\gamma(t_i)}((t-t_i)v_i)$. Assume that $\bar{M}$ is complete. For a linear isometry $I: T_pM \to T_{\bar{p}}\bar{M}$ at $p \in M$, $\bar{p} \in \bar{M}$ and a broken geodesic $\gamma: [0, l] \to M$ such that $\gamma(0)=p$, we define a broken geodesic $\bar{\gamma}: [0, l] \to \bar{M}$ such that $\bar{\gamma}(0)=\bar{p}$ and a linear isometry $I_\gamma: T_{\gamma(l)}M \to T_{\bar{\gamma}(l)}\bar{M}$ as follows: Set ${}_1\bar{\gamma}(t)=\exp_{\gamma(0)}t\Phi(v_0)$ $(t \in [0, t_1])$. Assume ${}_i\bar{\gamma}$ has already been defined on $[0, t_i]$. Set ${}_{i+1}\bar{\gamma} | [0, t_i]={}_i\bar{\gamma}$ and $({}_{i+1}\bar{\gamma} | [t_i, t_{i+1}])(t)=\exp_{{}_i\bar{\gamma}(t_i)}t(P_{{}_i\bar{\gamma}} \circ I \circ P_{{}_i\gamma}{}^{-1}(v_i))$ where P_c denotes the parallel translation along a curve c. Then define $\bar{\gamma}(t)={}_{n+1}\bar{\gamma}(t)$ for $t \in [0, l]$, and $I_\gamma==P_{\bar{\gamma}} \circ I \circ P_\gamma{}^{-1}$. Remark that M is not assumed to be complete in the above definition.

Lemma 3.3. (1) *Let (M, g) and $(\bar{M}, \bar{g})$ be the same dimensional connected Riemannian manifold with the Riemannian curvature tensors R and $\bar{R}$, respectively. Assume M is simply connected and $\bar{M}$ is complete. Suppose that there exists $p \in M$, $\bar{p} \in \bar{M}$ and a linear isometry $I: T_pM \to T_{\bar{p}}\bar{M}$ such that*

$$(*) \qquad I_\gamma(R(X, Y)Z) = \bar{R}(I_\gamma(X), I_\gamma(Y))I_\gamma(Z) \quad (X, Y, Z \in T_pM)$$

for any broken geodesic γ starting from p of M. Then there is an isometric immersion $\Phi: M \to \bar{M}$ such that

$$\text{(i)} \quad \Phi(\gamma(l)) = \bar{\gamma}(l), \qquad \text{(ii)} \quad \Phi_{*\gamma(l)} = I_\gamma.$$

(2) *If moreover M is complete, then Φ is a covering map.*

Proof. A map $\Phi: M \to \bar{M}$ defined by (i) is well-defined (cf. [5; Proof (1), (2), (3) of Theorem 1.36]). Let $q \in M$ be arbitrary. Since M is connected, it is also path-connected and there is a broken geodesic $\gamma: [0, l] \to M$ such that $\gamma(0) = p$ and $\gamma(l) = q$. Because of $(I_\gamma)_c = I_{\gamma \cup c}$ for any geodesic c starting from $\gamma(l)$, $(I_\gamma)_c$ satisfies $(*)$ and $\varphi = \exp_{\bar{\gamma}(l)} \circ I_\gamma \circ \exp_{\gamma(l)}^{-1}$ is an isometry between normal neighborhoods $B_r(\gamma(l))$ and $B_r(\bar{\gamma}(l))$. By the definition of Φ, $\varphi = \Phi | B_r(p)$, $\Phi_{*\gamma(l)} = I_\gamma$ and Φ is an isometric immersion, i.e. (1). (2) because of [5; Lemma 1.32]. Q. E. D.

In Cheeger-Ebin [5; Proof (1), (2), (3) of Theorem 1.36], it is not necessary that $\bar{\gamma}_0$, $\bar{\gamma}_1$ in (1), $\bar{\gamma}_1(t_{i+2})$, $\bar{\gamma}_1(t_{i+1})$, $\bar{\gamma}_0(t_{i+1})$, $\bar{\gamma}_0(t_i)$ in (2), and $\bar{\gamma}_{s,j+1}(t_{i+2})$, $\bar{\gamma}_{s,j+1}(t_{i+1})$, $\bar{\gamma}_{s,j}(t_{i+1})$, $\bar{\gamma}_{s,j}(t_i)$ in (3) are contained in the corresponding *normal* coordinate neighborhoods for $\varphi = \exp_{\bar{p}} \circ I \circ \exp_p^{-1} | B_r(p)$ in (1) being an isometric immersion (cf. [5; Lemma 1.35]). So one can omit the argument of [5; p. 40, l. 7-l. 10]. Hence the proof of [5; Theorem 1.36] becomes more simple.

Definition 3.4. For a semi-symplectic triple triple system $(V, [xyz], \langle x, y \rangle)$ over $\boldsymbol{C}$, we define a polynormal q of homogeneous degree 4 as

$$q(x) = \langle [xxx], x \rangle \qquad \text{for } x \in V.$$

We give examples of M such that the *q-polynomial* for $\boldsymbol{c}_p(M)$ is vanished at each $p \in M$:

Example 3.5 (*Symplectic triple systems of Quaternionic space forms*).

(1) Let M be a $4n$-dimensional Euclidean space $\boldsymbol{H}^n$ with the standard metric $g(x, y)$, on which $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \in \boldsymbol{H}$ act on the right. Then $(\boldsymbol{H}^n; g; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ defines a hyperkähler manifold (cf. [4; 14.10]). Since the Riemannian curvature tensor R is zero, $\boldsymbol{c}_0(\boldsymbol{H}^n)$ is isomorphic to $(\boldsymbol{C}^{2n}, [xyz] \equiv 0, \langle x, y \rangle \equiv 0)$. Hence, $q = 0$.

(2) Let M be a quaternionic projective space $\boldsymbol{H}P_n = Sp_{n+1}/(Sp_n \times Sp_1)$ or a quaternionic hyperbolic space $\boldsymbol{H}H_n = Sp_{n+1}^1/(Sp_n \times Sp_1)$ with the standard metric.

The complexified tangent space $T_o^c M$ at the origin o is identified with

$$U:=\left\{(x_1, x_2, y_1, y_2)=\begin{bmatrix} 0 & x_1 & 0 & x_2 \\ -{}^t y_2 & 0 & {}^t x_2 & 0 \\ 0 & y_1 & 0 & y_2 \\ {}^t y_1 & 0 & -{}^t x_1 & 0 \end{bmatrix};\ {}^t x_1,\ {}^t x_2 \in \boldsymbol{C}^n\right\}$$

in $sp_{n+1}(\boldsymbol{C}) := \{X \in \mathrm{gl}_{2(n+1)}(\boldsymbol{C});\ {}^t X J_{n+1} + J_{n+1} X = 0\}$ where $J_{n+1} := \begin{bmatrix} 0 & 1_{n+1} \\ -1_{n+1} & 0 \end{bmatrix}$.

Put

$$W:=\left\{\begin{bmatrix} a & 0 & c & 0 \\ 0 & A & 0 & C \\ b & 0 & -a & 0 \\ 0 & B & 0 & -{}^t A \end{bmatrix};\ \begin{bmatrix} a & c \\ b & -a \end{bmatrix} \in sp_1(\boldsymbol{C}),\ \begin{bmatrix} A & C \\ B & -{}^t A \end{bmatrix} \in sp_n(\boldsymbol{C})\right\},$$

$$h:=\begin{bmatrix} 1 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & -1 & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{bmatrix}, \quad e_+:=\begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix},$$

$e_- := {}^t e_+ \in W \subset sp_{n+1}(\boldsymbol{C}) = U \oplus W$. Then $H = \mathrm{ad}(h)$, $E_+ = \mathrm{ad}(e_+)$, $E_- = \mathrm{ad}(e_-)$ on $T_o^c M$. So that $T_o^+ M = \{(x_1, x_2, 0, 0) \in T_o^c M\} \cong \{x := (x_1, x_2);\ {}^t x \in \boldsymbol{C}^{2n}\}$ with $\langle x, y\rangle = -x_1 {}^t y_2 + x_2 {}^t y_1$ and $[xyz] = \langle x, z\rangle y + \langle y, z\rangle x$. Hence, $q = 0$.

If the scalar curvature is non-zero, then the converse is true up to local isometry:

Theorem 3.6. (1) *Let M be a locally quaternion-Kähler manifold of dimension $4n \geq 8$ with non-zero scalar curvature. Suppose that there is a family $\iota: M \to \mathcal{F}rame(M)$; $p \to \iota_p: \boldsymbol{H}^n \to T_p M$ of linear isometries such that $\iota_p^*(\mathcal{H}ol_p^0) \subset Sp_n \boldsymbol{Sp}_1$ and that the q-polynomial for $\boldsymbol{c}_p(M)$ is vanished. Then there is an isometric immersion Φ from the Riemannian universal covering manifold $\tilde{M}$ of M into a $4n$-dimensional quaternionic projective space $\boldsymbol{HP}_n$ or a $4n$-dimensional quaternionic hyperbolic space $\boldsymbol{HH}_n$.*

(2) *If moreover M is complete, then $\Phi: \tilde{M} \to \boldsymbol{HP}_n$ or $\boldsymbol{HH}_n$ is a covering map.*

Proof. Since the scalar curvature of M is non-zero, $\langle x, y\rangle$ of $\boldsymbol{c}_p(M)$ is non-degenerate by Theorem 2.3 (3). Due to Asano [17; Theorem 1.6], $\boldsymbol{c}_p(M)$ is isomorphic to a simple symplectic triple system of type C_{n+1} in Example 3.5(2) by polarizations of the equation $q(x) \equiv 0$ (cf. Remark 2.6 (1)). By Theorem 3.2, M is locally symmetric. Assume that the scalar curvature of M is positive (resp. negative). On $\boldsymbol{HP}_n$ (resp. $\boldsymbol{HH}_n$), by homothety, take the standard metric g such that the scalar curvature is same as M's. Denote $T_o := T_o\boldsymbol{HP}_n$ (resp. $T_o\boldsymbol{HH}_n$). Let $f: T_p^+M \to T_o^+$ be an isomorphism between symplectic triple systems. Putting $f(E_-x)=E_-(fx)$ ($x \in T_p^+M$), f can be extended to an isomorphism $f: T_p^cM \to T_o^c$ such as $f(R_p(x, y)z)=R_o(fx, fy)(fz)$ ($x, y, z \in T_p^cM$) because of (2.4.8) and [17; 1.4. Def.]. Since they are Einstein with the same scalar curvature, $g_o(fx, fy)=g_p(x, y)$ ($x, y \in T_p^cM$). Since $Sp(n)^c$ acts transitively on the space

$$\{(x_1, \cdots, x_n) \in ((\boldsymbol{H}^n)^c)^n \,;\, h(x_i, x_j)={}^t\bar{x}_i x_j=\delta_{ij}\}$$

of all quaternion-unitary basis of the right $\boldsymbol{H}^c$-module $(\boldsymbol{H}^c)^n$, it is appeared that the linear holonomy action of $Sp(n)^c$ is transitive on the set $\mathcal{H}$ of all $\boldsymbol{Sp}(1)$-invariant real $4n$-dimensional subspaces of T_o^c on which g_o is positive definite, by the equation (1.0) in § 1. By the definition of f, any $\alpha \in \boldsymbol{Sp}(1)$ and f are commutative. Then $f(T_pM)$ is an element of $\mathcal{H}$. So there exists $\beta \in Sp(n)^c$ such that $\beta(f(T_pM))=T_o \in \mathcal{H}$. Since $Sp(n)^c$ preseves R_o and g_o, $I := \beta \circ f : T_pM \to T_o$ satisfies that $I(R_p(X, Y)Z)=R_o(IX, IY)(IZ)$ and $g_p(X, Y)=g_o(IX, IY)$ ($X, Y, Z \in T_pM$). Since $\tilde{M}$ and $\boldsymbol{HP}_n$ (resp. $\boldsymbol{HH}_n$) are symmetric, the result follows from Cartan-Ambrose-Hicks theorem (cf. Lemma 3.3). Q. E. D.

Let M be a locally quaternion-Kähler manifold of dimension $\geqq 4$ with a family $\iota: M \to Frame(M): p \to \iota_p : \boldsymbol{H}^n \to T_pM$ of linear isometries such that $\iota_p^*(\mathcal{H}ol_p^0) \subset Sp_n\boldsymbol{Sp}_1$. A *Q-section at a point* $p \in M$ is defined as a four dimensional $\boldsymbol{Sp}(1)$-invariant subspace of T_pM. Then each Q-section at p has a form $\boldsymbol{H}(X) := \boldsymbol{R}X \oplus \boldsymbol{R}IX \oplus \boldsymbol{R}JX \oplus \boldsymbol{R}KX$ for some non-zero $X \in T_pM$. Note that $\boldsymbol{H}(X_1) \cap \boldsymbol{H}(X_2) = \{0\}$ if $\boldsymbol{H}(X_1) \neq \boldsymbol{H}(X_2)$. And that

$$(*) \qquad T_pM = \bigcup_{i=1}^{n} \boldsymbol{H}(X_i) \quad \text{(a disjoint union of } Q\text{-sections)}$$

Definition 3.7 (cf. Ishihara [7], Alekseevskii [1], Lemma 2.1).

(0) A Q-section $\boldsymbol{H}(X)$ is called to *have a Q-sectional curvature* $\rho(X)$, if the sectional curvature of M is a constant $\rho(X)$ on $\boldsymbol{H}(X)$, i. e.

$$g_p((R_p(X, \alpha X)X, \alpha X)=g_p(X, X)^2\rho(X) \qquad \text{for all } \alpha \in \boldsymbol{Z}_p\,.$$

(1) For $p \in M$, (M, p) is called to *have Q-sectional curvatures* $\rho(X)$ at $p \in M$, if all Q-sections $\boldsymbol{H}(X)$ at the point $p \in M$ have Q-sectional curvatures $\rho(X)$.

(2) M is called to *have constant Q-sectional curvature* $\rho(p)$ at $p \in M$, if M has the same Q-sectional curvatures $\rho(X)$ for all Q-sections $\boldsymbol{H}(X)$ at the point $p \in M$:

$$\rho(X)=\rho(p)\,.$$

(3) M is called to *have a constant Q-sectional curvature* ρ, if M has constant Q-sectional curvature $\rho(p)$ at all $p\in M$ which is constant ρ of $p\in M$:

$$\rho(p)=\rho\ .$$

By the condition (*), it is not trivial in Definition 3.7 that the condition (1) induces the condition (2). On the other hand, the main theorem claims that the conditions (1), (2) for all $p\in M$, and (3) in Definition 3.7 are all equivalent.

Proof of Main Theorem. Suppose that $\dim M=4$. Then the conditions (1) and (2) in Definition 3.7 are equivalent to that the sectional curvature depends only on the point, i.e. that M is a Riemannian manifold of constant sectional curvature. By Schur's theorem, M has a constant (Q-) sectional curvature ρ and the assertion holds by Cartan-Ambrose-Hicks theorem (cf. Lemma 3.3).

Assume that $\dim M\geqq 8$: In this case, we first show the following lemma, which compares Q-sectional curvature and q-polynomial. Put

$$K(X, Y, Z, W):=g(R(X, Y)Z, W) \quad\text{and}\quad K(X, Y):=K(X, Y, X, Y).$$

Lemma 3.8. (1) *The following three conditions are equivalent:*
(i) *M has a constant Q-sectional curvature ρ.*
(ii) *M has Q-sectional curvatures $\rho(X)$.*
(iii) *$K(x, E_{-}x)=0$ for all $x\in T_p^{+}M$ at each $p\in M$.*
(2) *If the scalar curvature $s\neq 0$, then they are also equivalent to*
(iv) *$q(x)\equiv 0$ for $x\in T_p^{+}M$ at each $p\in M$.*

Proof of the Lemma: (i)→(ii): trivial by the definition. (i)→(ii): Let M has Q-sectional curvatures $\rho(X)$ at $p\in M$. Then

$$\text{(a)}\quad \rho(X)=K(X, JX)=K(X, KX)\ .$$

which equals also to $(1/2)K(X, (K+J)X)$ since $|(K+J)X|^2=2|X|^2$. So

$$\text{(b)}\quad K(X, JX, X, KX)=0\ .$$

In Lemma 2.1 (d), putting $(X, Y, Z)\rightarrow(X, IX, X)$, we get

$$\frac{2}{n+2}r(X, X)|X|^2=K(X, IX)+K(X, IX, JX, KX)\ .$$

By the assumption (ii), the mapping: $X\rightarrow K(X, IX)$ is $\boldsymbol{Sp}(1)$-invariant. Hence, the mapping: $X\rightarrow K(X, IX, JX, KX)$ is also $\boldsymbol{Sp}(1)$-invariant. For $\alpha\in\boldsymbol{Sp}(1)$, $K(X, IX, JX, KX)=K(\alpha X, I\alpha X, J\alpha X, K\alpha X)=g(\alpha^{-1}R(\alpha X, I\alpha X)J\alpha X, \alpha^{-1}K\alpha X)=K(X, \alpha^{-1}I\alpha X, \alpha^{-1}J\alpha X, \alpha^{-1}K\alpha X)$. Since there is $\alpha\in\boldsymbol{Sp}(1)$ such that $\alpha^{-1}(I, J, K)\alpha=(J, K, I)$ (resp. (K, I, J)), we get

$$\text{(c)}\quad K(X, IX, JX, KX)=K(X, JX, KX, IX)=K(X, KX, IX, JX)\ .$$

Suppose that $H=-iI$, $E_+=(1/2)(iJ+K)$, $E_-=(1/2)(iJ-K)$. Then the $\boldsymbol{C}$-conjugation τ on T_p^cM with respect to T_pM satisfies that $\tau(T_p^-M)=T_p^+M$. For each $x\in T_p^+M$, denote $x_-:=E_-x\in T_p^-M$, then $E_+x_-=x$. Hence, $X:=\tau(x_-)+x_-\in T_pM$ and $E_+X=x$. So that $K(x, E_-x)=K(E_+X, E_-E_+X)=(1/4)K((iJ+K)X, (iI-1_{T_pM})X)$, whose expanded form is easily shown to be zero by means of (a), (b), (c), and Lemma 2.1. If (H', E'_+, E'_-) is another one, then there is $\alpha\in \boldsymbol{Sp}(1)^c$ such that $\alpha(H, E_+, E_-)\alpha^{-1}=(H', E'_+, E'_-)$. In this case, $\alpha(T_p^+M)=T_p^{+\prime}M$ and $K(\alpha x, E'_-\alpha x)=K(x, \alpha^{-1}E'_-\alpha x)=K(x, E_-x)=0$. (iii)→(iv): is trivial at each point $p\in M$ by (2.4.10). (iii)→(i) when $s=0$: If $s=0$, then $g([xyz], E_-w)$ is symmetric with respect to x, y, z, w by Proposition 2.4. In this case, if $K(x, E_-x)\equiv g([xxx], E_-x)=0$, then $g([xyz], E_-w)\equiv 0$ and $[xyz]\equiv 0$. Since $\langle x, y\rangle\equiv 0$, we get $R=0$, i.e. M is locally flat. In particular, M has a constant Q-sectional curvature 0. (iii)→(0) when $s\neq 0$: If $s\neq 0$ and $q\equiv 0$, then M is locally isometric to $\boldsymbol{HP}_n$ or $\boldsymbol{HH}_n$ by Theorem 3.6. In this case, $\mathcal{H}ol_p^0(M)=Sp(n)\boldsymbol{Sp}(1)$ preserves R. Take $X\in T_pM$, $Y, Z\in \boldsymbol{H}(X)$ such that $|X|=|Y|=|Z|=1$ and $g(Y, Z)=0$. There is $\alpha\in Sp(n)$ such as $\alpha(Y)=X$. Then $\alpha(Z)=a_1IX+a_2JX+a_3KX$ for some $a_i\in\boldsymbol{R}$ such that $\sum_i a_i^2=1$. There is $\beta\in\boldsymbol{Sp}(1)$ (resp. $\gamma\in Sp(n)$) such that $\beta^{-1}\circ(a_1I+a_2J+a_3K)\circ\beta=I$ (resp. $\gamma X=\beta X$). So that $K(Y, Z)=K(\alpha Y, \alpha Z)=K(X, (a_1I+a_2J+a_3K)X)=K(\beta^{-1}\gamma X, \beta^{-1}\gamma(a_1I+a_2J+a_3K)X)=K(X, \beta^{-1}(a_1I+a_2J+a_3K)\gamma X)=K(X, IX)$. Hence M has Q-sectional curvatures $\rho(X)$ at each $p\in M$. Since $Sp(n)$ is transitive on the tangent hyper sphere and preserves $\rho(X)$, M has constant Q-sectional curvature $\rho(p)$. Since M is locally homogeneous, for $p, q\in M$, there is a linear isometry $f: T_pM\to T_qM$ preserving R. Hence $f^*(\mathcal{H}ol_q^0)=\mathcal{H}ol_p^0$ and $f^*(\boldsymbol{Sp}(1)_q)=\boldsymbol{Sp}(1)_p$ as the unique normal three dimensional subgroup in $\mathcal{H}ol^0(M)\cong Sp(n)\boldsymbol{Sp}(1)$. So $f(\boldsymbol{H}(X))=\boldsymbol{H}(f(X))$. Hence, $\rho(p)=\rho(X)=\rho(fX)=\rho(q)$ is a constant ρ on M. Q.E.D.

End of the proof of Main Theorem Suppose that the scalar curvature $s=0$. By (2.4.7) and Theorem 2.3 (3), $[xyz]$ and $g([xyz], E_-w)$ are symmetric with respect to $x, y, z, w\in T_p^+M$. If M has Q-sectional curvatures $\rho(X)$ at each point, then, by Lemma 3.8 (1), for all $x\in T_p^+M$ at each $p\in M$, $g(R(x, E_-x)x, E_-x)=0$, i.e. $g([xxx], E_-x)=0$. Hence $g([xyz], E_-w)=0$ and $[xyz]=0$ for all $x, y, z\in T_p^+M$. Since $\langle x, y\rangle=0$, we have $R=0$, i.e. M is locally flat. By Cartan-Ambrose-Hicks theorem (cf. Lemma 3.3), the assertion holds. Conversely, if M is locally isometric to a $4n$-dimensional Euclidean space, then $R=0$ and M has a constant Q-sectional curvature 0. Assume $s\neq 0$. Suppose M has Q-sectional curvatures $\rho(X)$ at each point. By Lemma 3.8, $q=0$ for $\boldsymbol{c}_p(M)$ at each $p\in M$. By Theorem 3.6, the assertion follows. In this case, M has a constant Q-sectional curvature $\rho\neq 0$ by Lemma 3.8, and M is quaternion-Kähler by Theorem 1.4. Q.E.D.

References

[1] D.V. Alekseevskii, Riemannian spaces with exceptional holonomy groups (in Russian), *Funktional Anal. i Priložen*, 2 (2) (1968), 1-10; (in English), *Functional Anal. Appl.*, 2 (1968), 97-105.

[2] H. Asano, On triple systems (in Japanese), *Yokohama City Univ. Ronso, Ser. Natural Sci.*, 27 (1975), 7-31.

[3] H. Asano, Symplectic triple systems and simple Lie algebras (in Japanese), *Kyoto Univ. RIMS Kokyuroku*, 308 (1977), 41-54.

[4] A.L. Besse, *Einstein Manifolds*, Springer-Verlag, 1987.

[5] J. Cheeger and D.G. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland, 1975.

[6] A. Gray, A note on manifolds whose holonomy group is a subgroup of $Sp(n)Sp(1)$, *Michigan Math. J.*, 16 (1965), 125-128.

[7] S. Ishihara, Quaternion Kählerian manifolds, *J. Diff. Geo.*, 9 (1974), 483-500.

[8] I.L. Kantor and I.M. Skopets, Some results on Freudenthal triple systems, *Sel. Math. Sov.*, 2 (1982), 293-305: Originally published in *Trudy Sem. Vektor. Tenzor. Anal.*, 18 (1978), 250-263.

[9] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, 1983.

[10] S. Salamon, Quaternionic Kähler manifolds, *Inv. Math.*, 67 (1982), 143-171.

[11] S. Salamon, *Riemannian Geometry and Holonomy Groups*, Longman, 1989.

[12] Z.I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X.Y)\cdot R=0$, II: Global versions, *Geom. Dedicata*, 19 (1985), 65-108.

[13] H. Tasaki, Quaternionic submanifolds of quaternionic symmetric spaces, *Tohoku Math. J.*, 38 (1986), 513-538.

[14] K. Yamaguti and H. Asano, On Freudenthal's construction on exceptional Lie algebras, *Proc. Japan Acad.*, 51 (1975), 253-258.

[15] K. Yano, *The Theory of Lie Derivatives and Its Applications*, North-Holland, Amsterdam, 1955.

[16] O. Yasukura, A classification of orthogonal transformation groups of low cohomogeneity, *Tsukuba J. Math.*, 10-2 (1986), 299-326.

[17] O. Yasukura, Symplectic triple systems and graded Lie algebras, *International Symposium on Nonassociative Algebras and Related Topics*, Hiroshima, Japan 1990, ed. by K. Yamaguti & N. Kawamoto, World Scientific 1991, 345-351.

[18] O. Yasukura, Symplectic triple systems and quaternionic symmetric spaces, *in preparation*.

[19] J.A. Wolf, Complex homogeneous contact manifolds and quaternionic symmetric spaces, *J. of Math. and Mech.* 14 (1965), 1033-1047.

Department of Mathematics
Yokohama City University
Yokohama, 236 Japan

Present Address:
Department of Mathematics
Fukui University
Fukui, 910 Japan