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AN LIL FOR RANDOM WALKS WITH TIME STATIONARY RANDOM DISTRIBUTION FUNCTION

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Summary. Let $\mathcal{F}$ be a family of distribution functions and let ν be a stationary ergodic probability measure on $\mathcal{F}_1^\infty=\prod_{i=1}^\infty \mathcal{F}$ of copies of $\mathcal{F}$. Now for each $\omega=(F_1^\omega, F_2^\omega, \cdots)\in\mathcal{F}_1^\infty$, we define a probability measure P_ω on $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$ so that $P_\omega=\prod_{i=1}^\infty \mathcal{F}_i^\omega$, Let $X_n: \mathcal{R}_1^\infty\to\mathcal{R}$ be the coordinate functions $X_n(x)=x_n$, $x=(x_n)$. In this paper we study LIL for partial sums of $\{X_n\}$ with respect to P_ω and as a special case of above model we also study LIL for interchangeable process.

1. Introduction

Let $\mathcal{F}$ be a set of distributions on $\mathcal{R}^1$ with the topology of weak convergence, and let $\mathcal{A}$ be the σ-field generated by the open sets. We denote by $\mathcal{F}_1^\infty$ the space consisting of all infinite sequence $(F_1, F_2, \cdots)$, $F_n\in\mathcal{F}$, and $\mathcal{R}_1^\infty$ the space consisting of all infinite sequences $(x_1, x_2, \cdots)$ of real numbers. Take the σ-field $\mathcal{A}_1^\infty$ to be the smallest σ-field of subsets of $\mathcal{F}_1^\infty$ containing all finite-dimensional rectangles and take $\mathcal{B}_1^\infty$ to be the Borel σ-field of $\mathcal{R}_1^\infty$. Let $\omega=(F_1^\omega, F_2^\omega, \cdots)$ be the coordinate process in $\mathcal{F}_1^\infty$ and ν its distribution on $\mathcal{A}_1^\infty$. Let θ be the coordinate shift: $\theta^k(\omega)=\omega'$ with $F_n^{\omega'}=F_{n+k}^\omega$, $k=1, 2, \cdots$. On $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$ we also define the shift transformation $\sigma: \mathcal{R}_1^\infty\to\mathcal{R}_1^\infty$ by $\sigma(x_1, x_2, \cdots)=(x_2, x_3, \cdots)$. ν is called stationary if for every $A\in\mathcal{A}_1^\infty$, $\nu(\theta^{-1}(A))=\nu(A)$ and we let π be its marginal distribution. Let $\mathcal{G}$ be the σ-field of invariant sets in $\mathcal{B}_1^\infty$, that is, $\mathcal{G}=\{B\mid\sigma^{-1}(B)=B, B\in\mathcal{B}_1^\infty\}$. For each ω define a probability measure P_ω on $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$ so that $P_\omega=\prod_{i=1}^\infty F_i^\omega$. A monotone class argument shows that $P_\omega(B)$, $B\in\mathcal{B}_1^\infty$, is $\mathcal{A}_1^\infty$-measurable as a function of ω. So we can define a new probability measure P such that $P(B)=\int P_\omega(B)\nu(d\omega)$. Define the process $\{X_n\}$ on $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$ such that $X_n(x_1, x_2, \cdots)=x_n$ and set $S_n=X_1+X_2+\cdots+X_n$. By the definition of P_ω, $\{X_n\}$ are independent with respect to P_ω and we also note that $\{X_n\}$ is a sequence

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of independent and identically distributed random variables when $\mathcal{F}$ has just one element. The purpose of this paper is to study LIL for partial sums of X_n with respect to P_ω and as an application we apply this result to interchangeable processes. The following propositions are important basic tools throughout this paper.

Proposition 1. *If ν is stationary, then $\{X_n\}$ is a stationary process with respect to P.*

Proof. Let $f(\omega)=P_\omega(B)$, $B\in\mathcal{B}_1^\infty$, then f is a measurable function of ω. Then

$$P(B)=\int P_\omega(B)\nu(d\omega)=\int f(\omega)\nu(d\omega)=\int f(\theta(\omega))\nu(d\omega)$$

$$=\int P_{\theta(\omega)}(B)\nu(d\omega)=\int P_\omega(\sigma^{-1}(B))\nu(d\omega)=P(\sigma^{-1}(B)).$$

Proposition 2. *If ν is ergodic, then $\{X_n\}$ is ergodic with respect to P.*

Proof. Let $C\in\mathcal{B}_1^\infty$ be an invariant set, i.e. $\sigma^{-1}(C)=C$ and let $f(\omega)=P_\omega(C)$, then

$$f(\omega)=P_\omega(C)=P_\omega(\sigma^{-1}(C))=P_{\theta(\omega)}(C)=f(\theta(\omega)).$$

This implies f is an invariant random variable, hence it is a.s. constant, since ν is ergodic. By Proposition 6.32 [1] and Kolmogorov zero-one law, $f(\omega)=0$ or 1, then $P_\omega(C)=0$ ν-a.e. ω or 1 ν-a.e. ω. Hence $P(C)=0$ or 1, therefore $\{X_n\}$ is ergodic with respect to P.

Proposition 3. *Let $A\subset\mathcal{R}_1^\infty$ be measurable. Then $P_\omega(A)=1$ for ν-a.e. ω if and only if $P(A)=1$.*

Proof. The proof follows directly from the definition of P.

2. Results and Proofs

As a generalization of the Hartman-Wintner theorem, we first prove the following theorem

Theorem 1. *Let $\mathcal{F}=\left\{F\,|\int x\,dF(x)=0\right\}$ and let ν be stationary and ergodic with $\iint x^2 dF(x)\pi(dF)=1$. Then we have*

$$P_\omega\left\{\limsup\frac{S_n}{(2n\log\log n)^{1/2}}=1\right\}=1,\quad \nu\text{-a.e. } \omega.$$

To prove above theorem we need the following lemma.

Lemma 1. *Let* $\mathcal{F}=\left\{F \mid \int x dF(x)=0\right\}$ *and* ν *stationary with* $\iint |x| dF(x)\pi(dF) < \infty$. *Then* $\{X_n\}$ *with respect to* P *satisfies*

$$E[X_i | X_1, X_2, \cdots, X_{i-1}]=0 \quad a.s. \ \textit{for all } i \geqq 2.$$

Proof. By the assumption, $E|X_1|<\infty$ and hence $E[X_i|X_1, X_2, \cdots, X_{i-1}]$ exists for all $i\geqq 2$. Now let $A\in\sigma(X_1, X_2, \cdots, X_{i-1})$ and let $\{(X_1, X_2, \cdots, X_{i-1})\in B\}=A$ for some $i-1$ dimensional cylinder set B. Then we have

$$\begin{aligned}\int_A E[X_i|X_1, X_2, \cdots, X_{i-1}]dP &= \int_A X_i dP = \int 1_A X_i dP \\ &= \iint 1_B(x_1, \cdots, x_{i-1}) x_i dF_1^\omega(x_1) \cdots dF_{i-1}^\omega(x_{i-1})\nu(d\omega) \\ &= \int\left(\int 1_B(x_1, \cdots, x_{i-1}) dF_1^\omega(x_1)\cdots dF_{i-1}^\omega(x_{i-1})\right)\left(\int x_i dF_i^\omega(x_i)\right)\nu(d\omega) \\ &= 0,\end{aligned}$$

the last equality holding since $\int x dF(x)=0$ for all $F\in\mathcal{F}$. This proves the lemma.

Proof of Theorem 1. By Propositions 1, 2 and Lemma 1, $\{X_n\}$ is a stationary and ergodic process with respect to P such that $E[X_i|X_1, X_2, \cdots, X_{i-1}]=0$ a.s. for all $i\geqq 2$ and by assumption $EX_1^2=\iint x^2 dF(x)\pi(dF)=1$. Now applying Stout's result [5], we have

$$P\left\{\limsup \frac{S_n}{(2n \log\log n)^{1/2}}=1\right\}=1.$$

Hence by Proposition 3,

$$P_\omega\left\{\limsup \frac{S_n}{(2n \log\log n)^{1/2}}=1\right\}=1, \quad \nu\text{-a.e. } \omega.$$

What if the ergodicity assumption in above theorem is dropped? In this paper we obtain one possible answer for this question, that is, we need to impose one extra conditon on $\mathcal{F}$. As an application we apply this result to interchangeable processes.

Theorem 2. *Let* $\mathcal{F}=\left\{F \mid \int x dF(x)=0 \text{ and } \int x^2 dF(x)=1\right\}$ *and let* ν *be stationary. Then we have*

$$P_\omega\left\{\limsup \frac{S_n}{(2n \log\log n)^{1/2}}\right\}=1, \quad \nu\text{-}a.e.\ \omega.$$

Proof. By the ergodic decomposition theorem [6, Theorem 5.2.16], there is a probability measure $\rho\nu$ on $M_1^{\theta}(\mathcal{F}_1^{\infty})$, the space of stationary probability measures on $\mathcal{F}_1^{\infty}$, with the properties that $\rho\nu(EM_1^{\theta}(\mathcal{F}_1^{\infty}))=1$, where $EM_1^{\theta}(\mathcal{F}_1^{\infty})$ is the set of ergodic elements of $M_1^{\theta}(\mathcal{F}_1^{\infty})$, and

$$\nu=\int_{M_1^{\theta}(\mathcal{F}_1^{\infty})} R\rho\nu(dR)$$

holds. For every $R\in EM_1^{\theta}(\mathcal{F}_1^{\infty})$, we have that

$$\iint x^2 dF_1^{\omega}(x)R(d\omega)=1,$$

since

$$\int x^2 dF_1^{\omega}(x)=1 \qquad \text{for any } \omega.$$

Then by above theorem, we have for any $R\in EM_1^{\theta}(\mathcal{F}_1^{\infty})$

$$\int P_{\omega}\left\{\limsup \frac{S_n}{(2n\log\log n)^{1/2}}=1\right\}R(d\omega)=1.$$

Now

$$\int P_{\omega}\left\{\limsup \frac{S_n}{(2n\log\log n)^{1/2}}=1\right\}\nu(d\omega)$$
$$=\iint P_{\omega}\left\{\limsup \frac{S_n}{(2n\log\log n)^{1/2}}=1\right\}R(d\omega)\rho\nu(dR)=1,$$

which is equivalent to

$$P_{\omega}\left\{\limsup \frac{S_n}{(2n\log\log n)^{1/2}}=1\right\}=1, \quad \nu\text{-a.e. } \omega,$$

which completes the proof.

Next we consider a special case of the model in introduction. Let $\nu\{\omega\,|\,F_i^{\omega}=F_j^{\omega}$ for all $i\neq j\}=1$. Then clearly ν is stationary and hence we see from Theorem 2 and the definition of P that

$$P\left\{\limsup \frac{S_n}{(2n\log\log n)^{1/2}}=1\right\}=1$$

if

$$\nu\left\{\omega\,\middle|\,\int x\,dF_1^{\omega}(x)=0 \text{ and } \int x^2 dF_1^{\omega}(x)=1\right\}=1.$$

We shall show that this condition is also necessary. Since $\{X_n\}$ is independent and identically distributed with respect to P_{ω} ν-a.e. ω,

$$P_\omega\left\{\limsup \frac{S_n}{(2n \log\log n)^{1/2}} = 1\right\} = 1, \quad \nu\text{-a.e. } \omega$$

implies

$$\int x\, dF_1^\omega(x) = 0 \quad \text{and} \quad \int x^2 dF_1^\omega(x) = 1 \quad \nu\text{-a.e. } \omega$$

by Martikainen theorem [4]. Hence

$$\nu\left\{\omega \mid \int x\, dF_1^\omega((x) = 0 \quad \text{and} \quad \int x^2 dF_1^\omega(x) = 1\right\} = 1.$$

We summarize in

Lemma 2. *Let* $\nu\{\omega \mid F_i^\omega = F_j^\omega$ *for all* $i \neq j\} = 1$. *Then*

$$P\left\{\limsup \frac{S_n}{(2n \log\log n)^{1/2}} = 1\right\} = 1$$

if and only if

$$\nu\left\{\omega \mid \int x\, dF_1^\omega(x) = 0 \quad \text{and} \quad \int x^2 dF_1^\omega(x) = 1\right\} = 1.$$

A random variables $\{Y_n, n \geqq 1\}$ on $(\Omega, \mathcal{F}, \tilde{P})$ are called interchangeable if the joint distribution of every finite subset of k of these random variables depends only on k and not the particular subset, $k \geqq 1$. According to Theorem 7.3.2 [2] the random variables $\{Y_n, n \geqq 1\}$ are conditionally i.i.d. given the σ-field $\mathcal{G}$ of permutable events and according to Corollary 7.3.5 [2] there is a regular conditional distribution, say P^ω, for $Y = (Y_1, Y_2, \cdots)$ given $\mathcal{G}$ such that for each $\omega \in \Omega$ the coordinate random variables $\{X_n, n \geqq 1\}$ of the probability space $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty, P^\omega)$ are i.i.d.. Now suppose $EY_1 = 0$ and $EY_1^2 = 1$. Then, moreover, we see that

$$Cov(Y_1, Y_2) = 0 = Cov(Y_1^2, Y_2^2)$$

is equivalent to

$$E[Y_1 \mid \mathcal{G}] = 0 \quad \text{and} \quad E[Y_1^2 \mid \mathcal{G}] = 1 \qquad \text{a.s. [2, p. 310].}$$

Consequently, for almost all ω

$$\int_{-\infty}^{\infty} \xi_1 dP^\omega = 0, \qquad \int_{-\infty}^{\infty} \xi_1^2 dP^\omega = 1,$$

and thus, by Lemma 2 we have the following theorem.

Therem 3. *Let* $\{Y_n, n \geqq 1\}$ *be an interchangeable process with mean zero and variance one. Then the law of the iterated logarithm holds for the process if* *and only if*

$$Cov(Y_1, Y_2)=0=Cov(Y_1^2, Y_2^2).$$

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