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ON A PROBLEM OF MILLER AND MOCANU

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Abstract. Let $f(z)$ be analytic in $D=\{z: |z|<1\}$ and $f(0)=0$. It is shown that if

$$\left|\frac{f''(z)}{f'(z)}\right|<\frac{\sqrt{\alpha}}{2} \qquad (z\in D),$$

where $\sqrt{\alpha}/2=2.5159\cdots$, then $f(z)$ is starlike in D.

1. Introduction

Suppose that f is analytic in the unit disc $D=\{z: |z|<1\}$ with $f(0)=0$. It was shown in [1] that if $|f''(z)/f'(z)|<2$ then f is starlike in D. On the other hand, $f_0(z)=e^{\lambda z}-1$ is starlike if and only if $|\lambda|<M_1=2.8329\cdots$, [2]. Miller and Mocanu [1] posed the question of finding the maximum value of M for which $|f''(z)/f'(z)|<M$ implies f satarlike in D. Clearly $2\leqq M\leqq M_1$ and recently Nunokawa et al [3] have improved the lower bound of M to $13\sqrt{2}/8=2.298\cdots$. In this paper we further improve this lower bound to $2.5159\cdots$.

2. Results

We prove

Theorem. *Let f be analytic in D with $f(0)=0$, then if*

$$\left|\frac{f''(z)}{f'(z)}\right|<\frac{\sqrt{\alpha}}{2} \qquad (z\in D), \tag{1}$$

f is starlike in D, where α is the minimum value of

$$\phi(t)=\frac{6}{5+4t}\left(14+\frac{21}{2}t+8t^2+2t^3\right)+\frac{157}{16}+\frac{39}{4}t+3t^2 \qquad (-1\leqq t\leqq 1).$$

Proof. Let p be analytic in D with $p(0)=1$. Then if $p(z)=zf'(z)/f(z)$, the inequality (1) is equivalent to

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$$\left| p(z)+\frac{zp'(z)}{p(z)}-1 \right| < \frac{\sqrt{\alpha}}{2}. \tag{2}$$

Thus we need to show that (2) implies Re $p(z)>0$ for $z\in D$.

Write

$$p(z)=(1+\omega(z))^3, \tag{3}$$

so that ω is analytic in D and $\omega(0)=0$. Then

$$p(z)+\frac{zp'(z)}{p(z)}-1=\frac{3z\omega'(z)}{1+\omega(z)}+\omega(z)[3+3\omega(z)+\omega(z)^2].$$

Suppose now that there exists $z_0\in D$ such that

$$\max_{|z|\leqq|z_0|}|\omega(z)|=|\omega(z_0)|=1/2,$$

then from the Clunie-Jack Lemma, $z_0\omega'(z_0)=k\omega(z_0)$ for $k\geqq 1$. Thus

$$\left| p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}-1 \right| = |\omega(z_0)| \left| \frac{3k}{1+\omega(z_0)}+3+3\omega(z_0)+\omega(z_0)^2 \right|.$$

Now write $\omega(z_0)=e^{i\theta}/2$. Then with $t=\cos\theta$ and since $k\geqq 1$,

$$\begin{aligned}4\left| p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}-1 \right|^2 &= \left| \frac{6k(2+e^{-i\theta})}{5+4t}+3+3e^{i\theta}/2+e^{2i\theta}/4 \right|^2 \\ &= \frac{36k^2}{5+4t}+\frac{6k}{5+4t}[12+12t+4(2t^2-1)+(4t^3-3t)/2] \\ &\quad +181/16+9t+3(2t^2-1)/2+3(4t^3-3t)/4 \\ &\geqq \frac{6}{5+4t}[14+21t/2+8t^2+2t^3]+157/16+39t/4+3t^2 \\ &=\phi(t) \text{ say.}\end{aligned}$$

A simple calculation shows that $\phi(t)$ is minimum when $t=-0.26933\cdots$, and so

$$\left| p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}-1 \right| \geqq \frac{\sqrt{\alpha}}{2}$$

which contradicts (2). Thus $|\omega(z)|<1/2$ for all $z\in D$ and so from (3) it follows that Re $p(z)>0$ for $z\in D$ and the proof is complete.

Remark. The function $\phi(t)$ given in Theorem take its minimum value when $t_0=-0.26933\cdots$. Then we have $\alpha=\phi(t_0)=25.319011\cdots$, or $\sqrt{\alpha}/2=2.5159\cdots$. From this fact, we see that our result is the improvement of the theorem by Nunokawa et al [3].

We remark that by choosing $p(z)=(1+\omega(z))^n$ and $|\omega(z_0)|=\sin(\pi/2n)$, for $n\geqq 1$, the above method will give different estimates for M. However when

$n \geqq 4$ the computations become much more complicated and will be the subject of a subsequent paper.

References

[1] S.S. Miller and P.T. Mocanu, On some classes of first-order differential subordinations, *Michigan Math. J.*, **32** (1985), 185-195.
[2] P.T. Mocanu, Asupra razei de stelaritate a functiilor univalente, *Stud. Cerc. Mat. (Cluj)*, **11** (1960), 337-341.
[3] M. Nunokawa, S. Owa, H. Saitoh and K. Ohtake, An improvement of sufficient conditions for starlike functions, *Proc. Japan Acad.*, **66** (1990), 312-314.

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