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A SUFFICIENT CONDITION FOR A COMPACT HYPERSURFACE IN A SPHERE TO BE A SPHERE

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Abstract. In this paper a characterization of small spheres in a sphere is obtained in terms of a pinching relation on the Ricci curvature.

1. Introduction

Although many works have gone into the study of minimal hypersurfaces of a sphere, with a view to characterizing totally geodesic spheres (great spheres), less attention has been given to establishing sufficient conditions for a hypersurface to be a small sphere. Such a characterization was obtained by Nomizu and Smyth (see [5], Theorem 1, (ii)), by using the Gauss image of the hypersurface. Reilly [6] (see also Carter and West [2]) has shown that the result of Nomizu and Smyth is equivalent to a certain hypersurface immersion into Euclidean space. A characterization of small hyperspheres of a sphere was also obtained by Markvorsen [4]. More recently, Coghlan and Itokawa [3] used a pinch of the sectional curvature and the position of a hypersurface of the sphere to characterize a small sphere.

In this paper we consider a compact hypersurface M of S^{n+1} and a parallel unit vector field Z in $\boldsymbol{R}^{n+2}$. Denoting the tangential projection of Z on S^{n+1} by Z^T and the tangential projection of Z^T on M by t, we can write $Z^T=t+\rho N$, where N is the unit normal vector field to M in S^{n+1} and $\rho=\langle Z^T, N\rangle$ is a smooth function on M, usually referred to as the relative support function of the hypersurface M with respect to the vector field Z in $\boldsymbol{R}^{n+2}$. Here $\langle\cdot, \cdot\rangle$ is the Euclidean metric in $\boldsymbol{R}^{n+2}$. The main object of this paper is to prove

Theorem. *Let M be a compact, connected and orientable hypersurface of S^{n+1} with a relative support function ρ with respect to a parallel unit vector field Z in $\boldsymbol{R}^{n+2}$. If the Ricci curvature of M satisfies the pinching relation*

$$\rho^2(n-1)(n+2)<\rho^2 \operatorname{Ric}\leqq n-1,$$

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then ρ *is a constant and* $M=S^n(1/\rho^2)$.

A hypersurface of constant mean curvature in S^{n+1} is also considered and a sufficient condition for such a hypersurface to be a hypersphere is obtained.

2. Preliminaries

Let M be an orientable hypersurface of the unit sphere S^{n+1} in the Euclidean space $\boldsymbol{R}^{n+2}$ with center at the origin. M therefore has a unique global unit normal vector field N in S^{n+1}. For any pair of vector fields X and Y on M the Riemannian connections $\bar{\nabla}$ and ∇ on S^{n+1} and M, respectively, are related by

$$\bar{\nabla}_X Y=\nabla_X Y+g(AX, Y)N, \tag{2.1}$$

where g is the induced metric on M and A is the Weingarten map, which is a symmetric tensor field of type (1, 1) on M defined by

$$\bar{\nabla}_X N=-AX. \tag{2.2}$$

Fix a parallel unit vector field Z in $\boldsymbol{R}^{n+2}$ and let Z^T and Z^N be the tangential and normal components of Z to S^{n+1}, respectively, so that $Z=Z^T+Z^N$. Let $\bar{N}$ be the unit normal vector field to S^{n+1} in $\boldsymbol{R}^{n+2}$, and put $f=\langle Z^n, \bar{N}\rangle$. It then follows that

$$\bar{\nabla}_X Z^T=-fX \quad \text{and} \quad Xf=g(X, Z^T), \quad X\in\mathfrak{X}(M), \tag{2.3}$$

$\mathfrak{X}(M)$ being the Lie algebra of vector fields on M as a hypersurface of S^{n+1}.

Finally we define a smooth function $\rho: M\to\boldsymbol{R}$ by setting $Z^T=t+\rho N$, $t\in\mathfrak{X}(M)$. As pointed out above, ρ is the relative support function of the hypersurface M with respect to the parallel unit vector field Z on $\boldsymbol{R}^{n+2}$.

Using the equations (2.1), (2.2) and (2.3), we arrive at

$$\nabla_X t=-fX+\rho AX, \quad X\rho=g(AX, t), \quad \text{and} \quad Xf=g(X, t), \quad X\in\mathfrak{X}(M), \tag{2.4}$$

$$\operatorname{grad}\rho=-At, \qquad \operatorname{grad} f=t. \tag{2.5}$$

Gauss' equation gives the Ricci curvature tensor of M as

$$\operatorname{Ric}(X, Y)=(n-1)g(X, Y)+n\alpha g(AX, Y)-g(AX, AY), \tag{2.6}$$

where $\alpha=(1/n)\sum_1^n g(Ae_i, e_i)$, $\{e_i\}$ being a local orthonormal frame in M, is the mean curvature of M (see [1]). On the other hand, Codazzi's equation gives

$$(\nabla_X A)Y=(\nabla_Y A)X, \qquad X, Y\in\mathfrak{X}(M). \tag{2.7}$$

Using the symmetry of A in this equation, we conclude that

$$g((\nabla_X A)Y, Z)=g((\nabla_X A)Z, Y), \qquad X, Y, Z\in\mathfrak{X}(M). \tag{2.8}$$

Let Δ be the Laplacian operator acting on the smooth functions on M.

Lemma 2.1.
(a) $\Delta f=n(-f+\rho\alpha)$,
(b) $(1/2)\Delta f^2=n(-f^2+\alpha f\rho)+\|t\|^2$.

Proof. (a) follows immediately from equation (2.4), and (b) follows from (a), (2.5) and the equality $\Delta f^2=2f\Delta f+2\|\operatorname{grad} f\|^2$.

3. Hypersurfaces of constant mean curvature

For hypersurfaces of constant mean curvature in S^{n+1}, Nomizu and Smyth (see [5], Theorem 2, p. 490) proved that, if the Gauss image lies in a closed hemisphere of S^{n+1}, then the hypersurface is necessarily a hypersphere in S^{n+1}. Here we shall prove

Theorem 3.1. *Let M be a compact, connected and orientable hypersurface of constant mean curvature in S^{n+1}. If some relative support function of M with respect to a parallel unit vector field Z in $\boldsymbol{R}^{n+2}$ is nowhere zero on M, then M is a hypersphere in S^{n+1}.*

Proof. Using the equation (2.4), we obtain the following expression for the Hessian H_ρ of the function ρ

$$H_\rho(X, Y)=-g((\nabla_X A)Y, t)+fg(AX, Y)-\rho g(AX, AY).$$

This, together with (2.8), implies

$$\Delta\rho=-nt\alpha+nf\alpha-\rho\operatorname{tr}A^2. \tag{3.1}$$

From (2.4) it follows that $\operatorname{div} t=n(-f+\rho\alpha)$ and hence $\operatorname{div}(\alpha t)=t\alpha+n\alpha(-f+\rho\alpha)$. Using this last equation in (3.1), we arrive at

$$\Delta\rho=-n(n-1)f\alpha+n^2\alpha^2\rho-\rho\operatorname{tr}A^2-\operatorname{div}(n\alpha t). \tag{3.2}$$

If α is a constant, then this equation, combined with Lemma 2.1 (a), yields

$$\Delta(\rho-(n-1)\alpha f)=\rho(n\alpha^2-\operatorname{tr}A^2)-\operatorname{div}(n\alpha t).$$

By integrating over M we conclude that

$$\int_M \rho(n\alpha^2-\operatorname{tr}A^2)dv=0.$$

From the Schwarz inequality we have $n\alpha^2-\operatorname{tr}A^2\geqq 0$, where the equality holds if and only if M is totally umbilic. If $\rho\neq 0$ on M and M is connected, then $n\alpha^2=\operatorname{tr}A^2$, i.e. M is totally umbilic. M being compact, this implies that M

is a hypersphere of S^{n+1}.

4. Proof of the main theorem

Let S be the scalar curvature of M, which is given by $S=n(n-1)+n^2\alpha^2-\operatorname{tr}A^2$. The equation (3.2) then gives

$$\rho\Delta\rho=\rho(S-n(n-1))-n(n-1)f\rho\alpha-\rho\operatorname{div}(n\alpha t). \tag{4.1}$$

Since

$$\operatorname{div}(n\alpha\rho t)=n\alpha t\rho+\rho\operatorname{div}(n\alpha t),$$

the second equation in (2.4) implies

$$-\rho\operatorname{div}(n\alpha t)=n\alpha g(At, t)-\operatorname{div}(n\alpha\rho t).$$

Thus the equation (4.1) becomes

$$\rho\Delta\rho=\rho(S-n(n-1))-n(n-1)f\rho\alpha-n\alpha g(At, t)-\operatorname{div}(n\alpha\rho t).$$

Using the identity $(1/2)\Delta\rho^2=\rho\Delta\rho+\|\operatorname{grad}\rho\|^2$ and the equation (2.5), we therefore obtain

$$\frac{1}{2}\Delta\rho^2=\rho^2(S-n(n-1))-f\rho\alpha n(n-1)-[\operatorname{Ric}(t, t)-(n-1)\|t\|^2]-\operatorname{div}(n\alpha\rho t).$$

Now Lemma 2.1 (b) gives

$$\frac{1}{2}\Delta(\rho^2+(n-1)f^2)+\operatorname{div}(n\alpha\rho t)=\rho^2S-n(n-1)(\rho^2+f^2+\|t\|^2)$$
$$+(n-1)(n+2)\|t\|^2-\operatorname{Ric}(t, t).$$

Since $Z=t+\rho N+f\bar{N}$ is a unit vector field, we have $\|t\|^2+\rho^2+f^2=1$. Thus, integrating the above equation over M, we get

$$\int_M [\|t\|^2(\operatorname{Ric}(\hat{t}, \hat{t})-(n-1)(n+2))+(n(n-1)-\rho^2S)]dv=0,$$

where $\hat{t}$ is the unit vector field defined by $t/\|t\|$ on the open subset of M where t is non-zero. Now, using the hypothesis of the theorem, we get $t=0$ and $\rho^2S=n(n-1)$. From the equation (2.4) we see that ρ is a non-zero constant, $f|_M$ is a constant, and that $A=(f/\rho)I$. Consequently $M=S^n(1/\rho^2)$ by Gauss' equation.

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