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HADAMARD PRODUCTS AND NEIGHBOURHOODS OF SPIRALLIKE FUNCTIONS

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Summary. In this paper, we find Hadamard products and neighbourhoods of spirallike functions. Using convolution properties, we find a sufficient condition guaranteeing a function to be in a subclass of λ-spirallike functions.

1. Introduction

Let A denote the family of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n} \tag{1.1}$$

that are analytic in the open unit disk $\Delta(|z|<1)$, and suppose S is a subfamily of A consisting of univalent functions in Δ. Given λ, $-\pi/2<\lambda<\pi/2$, let H^{λ} be the class of λ-spiral-like functions in Δ. An analytic function f of the form (1.1) is in H^{λ} if and only if $\mathrm{Re}(e^{i\lambda}zf'(z)/f(z))>0$, $z\in\Delta$. This class was introduced by Špaček [15], who showed that $H^{\lambda}\subset S$.

If f in A is of the form (1.1) and $\delta\geqq 0$ we define the δ-neighbourhood of f by

$$N_{\delta}(f)=\{t(z)=z+\sum_{k=2}^{\infty}c_{k}z^{k}\in A:\ \sum_{k=2}^{\infty}k|a_{k}-c_{k}|\leqq\delta\}.$$

Goodman [4] proved that if $f_{0}(z)\equiv z$, the identity function in A, then $N_{1}(f_{0})\subset S^{*}$, the class of starlike functions in A. Ruscheweyh [9] extended this result and proved that if $f\in A$ satisfies the condition that $(f(z)+\varepsilon z)/(1+\varepsilon)$ is λ-spirallike for all $\varepsilon\in C$ with $|\varepsilon|<\delta$, then $N_{\delta\cos\lambda}(f)\subset H^{\lambda}$, $|\lambda|<\pi/2$. Analogous results for δ-neighbourhoods involving other subclasses of A can be found in [1], [3], [11], and others. We shall find δ-neighbourhood information for certain subclasses of H^{λ} defined as follows.

For $-1\leqq B<A\leqq 1$ and $-\pi/2<\lambda<\pi/2$, let

$$S^{\lambda}(A, B)=\left\{f\in A:\ e^{i\lambda}\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz}\cos\lambda+i\sin\lambda \text{ for } z\in\Delta\right\},$$

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where $\prec$ denotes subordination. This family was introduced in [2]. Since $S^{\lambda}(A, B) \subseteq S^{\lambda}(1, -1) \equiv H^{\lambda}$, the functions in $S^{\lambda}(A, B)$ are spirallike in Δ. By taking $A=1-2\alpha$, $B=-1$, the family $S^{\lambda}(A, B)$ coincides with the class $H^{\lambda}(\alpha)$, $0 \leqq \alpha < 1$, introduced by Libera [6]. For $\lambda=0$, the class $S^*(A, B) \equiv S^0(A, B)$ was defined by Janowski [5]. Note that $S^0(1-2\alpha, 1) \equiv S^*(\alpha)$, the family of starlike functions of order α, $0 \leqq \alpha < 1$.

A function g of the form

$$g(z) \equiv g_{\gamma}(z) = z\left(\frac{f(z)}{z}\right)^{\gamma} \tag{1.2}$$

is said to be in $S_{\gamma}^{\lambda}(A, B)$ if f belongs to $S^{\lambda}(A, B)$ for some fixed real γ and for all $z \in \Delta$. Note that for $\gamma=1$, $S_{\gamma}^{\lambda}(A, B)$ coincides with $S^{\lambda}(A, B)$. For $\gamma=-1$, we may write $g(z)=z/(1+\sum_{k=1}^{\infty} b_k z^k)$ with $a_k = b_{k-1}$. Recently, Silverman [12] has found some properties of the class $S_{\gamma}^{0}(1-2\alpha, -1)$.

In the present paper, we first find convolution properties of the operator (1.2) when f is in $S^{\lambda}(A, B)$. Using convolution properties we then determine δ-neighbourhoods information concerning the classes $S_{\gamma}^{\lambda}(A, B)$ and $S^{\lambda}(A, B)$. Finally, convolution properties are used to obtain a sufficient condition guaranteeing a function of the form (1.2) to be in $S^{\lambda}(A, B)$.

2. Convolution properties

For $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ and $g(z)=z+\sum_{n=2}^{\infty} b_n z^n$ in A, the Hadamard or convolution product is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in \Delta).$$

The Hadamard products offer a useful characterization for memberships in classes. For example, the convolution characterizations allow us to benefit from the distributive property of the Hadamard product.

Theorem 1. *Let* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, $g(z)=z(f(z)/z)^{\gamma}$, $z \in \Delta$. *Then function* $g \in S^{\lambda}(A, B)$ *if and only if*

$$\frac{1}{z}\left(f(z) * \frac{z+cz^2}{(1-z)^2}\right) \neq 0 \qquad (z \in \Delta),$$

where

$$c = \frac{\gamma e^{i\lambda}(x-B)-(A-B)\cos\lambda}{(A-B)\cos\lambda} \qquad (|x|=1). \tag{2.1}$$

Proof. Since

$$\gamma e^{i\lambda}\frac{zf'(z)}{f(z)} + e^{i\lambda}(1-\gamma) = e^{i\lambda}\frac{zg'(z)}{g(z)},$$

it follows that $g \in S^{\lambda}(A, B)$ if and only if

$$\gamma e^{i\lambda}\frac{zf'(z)}{f(z)}+e^{i\lambda}(1-\gamma) \prec \frac{1+Az}{1+Bz}\cos\lambda+i\sin\lambda \qquad (z\in\Delta). \tag{2.2}$$

The function

$$\frac{1+Az}{1+Bz}\cos\lambda+i\sin\lambda \tag{2.3}$$

maps the unit circle $|z|=1$ onto the boundary of the circle centered at $((1-AB)/(1-B^2))\cos\lambda+i\sin\lambda$ with radius $((A-B)/(1-B^2))\cos\lambda$. Note that $zf'(z)/f(z)=1$ at $z=0$. The function (2.3) is analytic and hence maps the regions onto regions. Thus the image of every point in the interior of Δ goes over to an interior point of the image of the function (2.3). Consequently, (2.2) is equivalent to

$$\gamma e^{i\lambda}\frac{zf'(z)}{f(z)}+e^{i\lambda}(1-\gamma)\neq\frac{1+Ax}{1+Bx}\cos\lambda+i\sin\lambda \quad (|x|=1,\ Bx\neq-1),$$

which simplifies to

$$\begin{aligned}\gamma e^{i\lambda}(1+Bx)zf'(z)+e^{i\lambda}(1-\gamma)(1+Bx)f(z)&\\ -(1+Ax)f(z)\cos\lambda-i(1+Bx)f(z)\sin\lambda\neq 0&.\end{aligned} \tag{2.4}$$

Since $zf'(z)=f(z)*(z/(1-z)^2)$, $f(z)=f(z)*(z/(1-z))$, we notice that (2.4) is equivalent to

$$\frac{1}{z}\left[f(z)*\left\{\frac{(B-A)xz\cos\lambda+((A-B)x\cos\lambda+\gamma e^{i\lambda}(1+Bx))z^2}{(1-z)^2}\right\}\right]\neq 0. \tag{2.5}$$

By some simple computations and writing $-\bar{x}$ as x, (2.5) may be written as $(f(z)*h_1(z))/z\neq 0$, where

$$h_1(z)=\frac{z+\left(\dfrac{\gamma e^{i\lambda}(x-B)-(A-B)\cos\lambda}{(A-B)\cos\lambda}\right)z^2}{(1-z)^2}.$$

We thus obtain the desired conclusion.

We now obtain a characterization of $S_{\gamma}^{\lambda}(A, B)$ in terms of convolution.

Theorem 2. *With f and g as defined in Theorem 1, the function $g\in S_{\gamma}^{\lambda}(A, B)$ if and only if*

$$\frac{1}{z}\left(g(z)*\frac{z+dz^2}{(1-z)^2}\right)\neq 0 \qquad (z\in\Delta),$$

where

$$d=\frac{e^{i\lambda}(x-B)-(A-B)\gamma\cos\lambda}{(A-B)\gamma\cos\lambda}\qquad(|x|=1). \tag{2.6}$$

Proof. Since

$$\frac{zg'(z)}{g(z)}=(1-\gamma)+\gamma\frac{zf'(z)}{f(z)}\qquad(z\in\Delta) \tag{2.7}$$

and $f\in S^{\lambda}(A, B)$, it follows that $g\in S_{\gamma}^{\lambda}(A, B)$ if and only if

$$e^{i\lambda}\frac{zg'(z)}{g(z)}-e^{i\lambda}(1-\gamma)\prec\frac{1+Az}{1+Bz}\gamma\cos\lambda+i\gamma\sin\lambda. \tag{2.8}$$

Using arguments similar to the lines of the proof of Theorem 1, we find that $g\in S_{\gamma}^{\lambda}(A, B)$ if and only if

$$\frac{1}{z}\left[g(z)*\left\{\frac{(A-B)\gamma xz\cos\lambda-((A-B)\gamma x\cos\lambda+(Bx+1)e^{i\lambda})z^2}{(1-z)^2}\right\}\right]\neq 0,$$

which is equivalent to

$$\frac{1}{z}(g(z)*h_2(z))\neq 0\qquad(z\in\Delta),$$

where

$$h_2(z)=\frac{z+\left(\dfrac{e^{i\lambda}(x-B)-(A-B)\gamma\cos\lambda}{(A-B)\gamma\cos\lambda}\right)z^2}{(1-z)^2},$$

and the proof is complete.

Remark. For $\gamma=1$, the results in Theorems 1 and 2 coincide.

Given a normal family $F\subset A$, the dual of F is defined as the family

$$F'=\{h\in A:\ h*f\neq 0 \text{ for all } f\in F,\ 0<|z|<1\}.$$

The concept of duality was introduced by Ruscheweyh in [10]. In view of the definition of duality and Theorem 2, the class $S_{\gamma}^{\lambda}(A, B)$ is the dual of the family

$$(S_{\gamma}^{\lambda}(A, B))'=\left\{\frac{z+\left(\dfrac{e^{i\lambda}(x-B)-(A-B)\gamma\cos\lambda}{(A-B)\gamma\cos\lambda}\right)z^2}{(1-z)^2}:\ |x|=1\right\}. \tag{2.9}$$

Fixing $\gamma=1$ and choosing suitable values of the parameters λ $(-\pi/2<\lambda<\pi/2)$, A, and B $(-1\leq B<A\leq 1)$, we may obtain the dual sets of several subclasses of H^{λ} and S^*.

Corollary 1. *For a fixed real number* λ, $-\pi/2<\lambda<\pi/2$, *the family* H^{λ} *is* *the dual of*

$$(H^{\lambda})'=\left\{\frac{z+\dfrac{x-e^{-2i\lambda}}{1+e^{-2i\lambda}}z^2}{(1-z)^2}: |x|=1\right\}.$$

Proof. The result follows from (2.9) by fixing $\gamma=1$, $A=1$, $B=-1$ and noting from (2.6) that

$$d=\frac{(x+1)e^{i\lambda}-2\cos\lambda}{2\cos\lambda}$$
$$=\frac{xe^{i\lambda}-e^{-i\lambda}}{e^{i\lambda}+e^{-i\lambda}}=\frac{x-e^{-2i\lambda}}{1+e^{-2i\lambda}}.$$

Corollary 2. *For a fixed real* α $(0\leqq\alpha<1)$, *the family* $S^*(\alpha)$ *is the dual of*

$$(S^*(\alpha))'=\left\{\frac{z+\dfrac{x+2\alpha-1}{2-2\alpha}z^2}{(1-z)^2}: |x|=1\right\}.$$

Proof. Let $\gamma=1$, $\lambda=0$, $A=1-2\alpha$, $B=-1$ in (2.9) and the result follows.

Remarks. The results in Corollaries 1 and 2 were contained in [14]. The case $\lambda=0$, $\gamma=1$ in Theorem 2 was established in [13].

Corollary 3. *The function* $g(z)=z/(1+\sum_{k=1}^{\infty}b_k z^k)$ *is starlike of order* α, $0\leqq\alpha<1$, *if and only if*

$$\frac{1}{z}\left[g(z)*\left(\frac{z-\dfrac{x+3-2\alpha}{2-2\alpha}z^2}{(1-z)^2}\right)\right]\neq 0$$

for all $z\in\Delta$ *and* $|x|=1$.

Proof. Set $\gamma=-1$, $\lambda=0$, $A=1-2\alpha$, $B=-1$, and $a_n=b_{n-1}$ in Theorem 2.

3. Neighbourhoods of spiral-like functions

Theorem 3. *With* f *and* g *as defined in Theorem* 1, *let* $g(z)=z+\sum_{k=2}^{\infty}b_k z^k$ *be in* A *and suppose* $0<|\gamma|\cos\lambda\leqq 1$. *If for all complex* ε, $|\varepsilon|<\delta$, assume $(g(z)+\varepsilon z)/(1+\varepsilon)\in S_{\gamma}^{\lambda}(A, B)$ $(z\in\Delta)$, *then*

$$N_{\rho}(g)\subset S_{\gamma}^{\lambda}(A, B),$$

where $\rho=(\delta|\gamma|(A-B)\cos\lambda)/(1+|B|)$.

Proof. In view of (2.9), we first observe that

$$t\in S_{\gamma}^{\lambda}(A, B)\Longleftrightarrow (t*h)(z)\neq 0 \tag{3.1}$$

for all $h\in(S_\gamma^\lambda(A, B))'$ and for all $0<|z|<1$. Since $(g(z)+\varepsilon z)/(1+\varepsilon)\in S_\gamma^\lambda(A, B)$, we have

$$\frac{g(z)+\varepsilon z}{1+\varepsilon}*h(z)\neq 0 \qquad (0<|z|<1) \tag{3.2}$$

for all $h\in(S_\gamma^\lambda(A, B))'$. We observe that (3.2) simplifies to

$$\frac{1}{z}(g*h)(z)\neq -\varepsilon \qquad (z\in\Delta)$$

for all ε such that $|\varepsilon|<\delta$, which is equivalent to

$$\left|\frac{(g*h)(z)}{z}\right|\geqq\delta \qquad (z\in\Delta). \tag{3.3}$$

We next notice that if $h\in(S_\gamma^\lambda(A, B))'$ and $h(z)=z+\sum_{k=2}^\infty h_k z^k$, then

$$h_k=\frac{(k-1)xe^{i\lambda}+(A-B)\gamma\cos\lambda-(k-1)Be^{i\lambda}}{(A-B)\gamma\cos\lambda} \qquad (k\geqq 2). \tag{3.4}$$

Assume $t(z)=z+\sum_{k=2}^\infty c_k z^k\in N_\rho(g)$. Then

$$\begin{aligned}\left|\frac{1}{z}(t*h)(z)\right| &= \left|\frac{(g*h)(z)}{z}+\frac{(t-g)(z)*h(z)}{z}\right| \\ &\geqq \left|\frac{(g*h)(z)}{z}\right|-\left|\frac{(t-g)(z)*h(z)}{z}\right| \\ &\geqq \delta-\left|\sum_{k=2}^\infty(c_k-b_k)h_k z^k\right| \\ &\geqq \delta-|z|\sum_{k=2}^\infty|b_k-c_k||h_k|.\end{aligned}$$

Since from (3.4)

$$\begin{aligned}|h_k| &\leqq \frac{(k-1)+(A+|B|)|\gamma|\cos\lambda+(k-1)|B|}{|\gamma|(A-B)\cos\lambda} \\ &\leqq \frac{(k-1+|\gamma|\cos\lambda)(1+|B|)}{|\gamma|(A-B)\cos\lambda} \\ &\leqq \frac{k(1+|B|)}{|\gamma|(A-B)\cos\lambda},\end{aligned}$$

we have

$$\begin{aligned}\left|\frac{1}{z}(t*h)(z)\right| &\geqq \delta-\frac{|z|(1+|B|)}{|\gamma|(A-B)\cos\lambda}\sum_{k=2}^\infty k|b_k-c_k| \\ &\geqq \delta-\frac{|z|(1+|B|)\rho}{|\gamma|(A-B)\cos\lambda} \\ &=\delta(1-|z|)>0.\end{aligned}$$

This proves that

$$\frac{1}{z}(t*h)(z)\neq 0 \qquad (z\in\Delta),$$

and, therefore, it follows from (3.1) that $t\in S_\gamma^\lambda(A, B)$. This completes the proof of the theorem.

Corollary. *Let $f\in A$ be of the form* (1.1), *and λ a fixed real number with $|\lambda|<\pi/2$. For all complex ε, $|\varepsilon|<\delta$, assume $(f(z)+\varepsilon z)/(1+\varepsilon)\in S^\lambda(A, B)$ $(z\in\Delta)$, then*

$$N_{\delta'}(f)\subset S^\lambda(A, B),$$

where $\delta'=(\delta(A-B)\cos\lambda)/(1+|B|)$.

Remarks. The special cases of (i) $A=1$, $B=-1$, (ii) $\lambda=0$, $A=1$, $B=-1$ in Corollary were proved by Ruscheweyh [9].

4. Sufficient coefficient conditions

We shall now use the properties of Hadamard products in Section 2 to determine a sufficient condition guaranteeing a function of the form (1.3) to be in $S^\lambda(A, B)$.

Lemma. *With f and g as defined in Theorem* 1, *the function $g\in S^\lambda(A, B)$ if and only if*

$$1+\sum_{n=2}^\infty (n+(n-1)c)a_n z^{n-1}\neq 0$$

for all $z\in\Delta$ and $|x|=1$, where c is given in (2.1).

Proof. Since

$$h_1(z)=\frac{z+cz^2}{(1-z)^2}=z+\sum_{n=2}^\infty (n+(n-1)c)z^n,$$

we may write

$$\frac{1}{z}(f*h_1)(z)=1+\sum_{n=2}^\infty (n+(n-1)c)a_n z^{n-1}$$

and the result follows from Theorem 1.

Theorem 4. *With f and g as defined in Theorem* 1, *the function $g\in S^\lambda(A, B)$ if*

$$\sum_{n=2}^\infty D_n(\lambda, \gamma, A, B)|a_n|\leqq 1$$

for

$$D_n(\lambda, \gamma, A, B)=\frac{(n-1)|\gamma|+\sqrt{(A-B)(A-B-2B\gamma(n-1))\cos^2\lambda+B^2\gamma^2(n-1)^2}}{(A-B)\cos\lambda}. \tag{4.1}$$

Proof. With c defined in (2.1) and $z\in\Delta$, we notice that

$$|1+\sum_{n=2}^{\infty}(n+(n-1)c)a_nz^{n-1}|\geqq 1-\sum_{n=2}^{\infty}|n+(n-1)c||a_n||z|^{n-1}$$
$$\geqq 1-\sum_{n=2}^{\infty}|n+(n-1)c||a_n|,$$

and

$$\begin{aligned}|n+(n-1)c| &= \frac{|(n-1)\gamma e^{i\lambda}x+(A-B)\cos\lambda-(n-1)B\gamma e^{i\lambda}|}{(A-B)\cos\lambda}\\ &\leqq \frac{(n-1)|\gamma|+|(A-B)\cos\lambda-(n-1)B\gamma e^{i\lambda}|}{(A-B)\cos\lambda}\\ &= \frac{(n-1)|\gamma|+|(A-B-B\gamma(n-1))\cos\lambda-iB\gamma(n-1)\sin\lambda|}{(A-B)\cos\lambda}\\ &= D_n(\lambda, \gamma, A, B).\end{aligned}$$

We thus conclude from Lemma that $\sum_{n=2}^{\infty}D_n(\lambda, \gamma, A, B)|a_n|\leqq 1$ is a sufficient condition for g to be in $S^{\lambda}(A, B)$.

Corollary 1. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, and let α, λ be constants, $0\leqq\alpha<1$, $-\pi/2<\lambda<\pi/2$. If the coefficients $\{a_n\}$ satisfy*

$$\sum_{n=2}^{\infty}[(n-1)+((n-1)^2+4(n-\alpha)(1-\alpha)\cos^2\alpha)^{1/2}]|a_n|\leqq 2(1-\alpha)\cos\lambda,$$

then f is in $H^{\lambda}(\alpha)$, the class of λ-spiral functions of order α.

Proof. By setting $A=1-2\alpha$, $B=-1$, $\gamma=1$ in Theorem 4, the result follows.

Corollary 2. *Let $g(z)=z/(1+\sum_{k=1}^{\infty}b_k)$, and let α be constant, $0\leqq\alpha<1$. If the coefficients $\{b_k\}$ satisfy*

$$\sum_{k=2}^{\infty}(k-1+\alpha)|b_k|\leqq\begin{cases}(1-\alpha)-(1-\alpha)|b_1|, & 0\leqq\alpha\leqq 1/2,\\ (1-\alpha)-\alpha|b_1|, & 1/2\leqq\alpha<1,\end{cases}$$

then the function g is starlike of order α in the disk Δ.

Proof. Let $\gamma=-1$, $A=1-2\alpha$, $B=-1$, and $a_n=b_{n-1}$ in Theorem 4 and the result follows.

Remarks. By using a different technique, the results in the preceding corollaries were proven, respectively, in [7] and [8].

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